Solving Maxwell's equations with the Weighted Regularization Method and a Lagrange multiplier

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Time-dependent Maxwell equations

- *In vacuum, over the time interval* $]0,T[$, $T>0$.
- *Goal*: compute the EM field in a domain Ω (with Lipschitz polyhedral boundary) encased in a perfect conductor.
  Find $(\mathcal{E}(t),\mathcal{H}(t))$ such that

$$\begin{cases} \varepsilon_0\partial_t\mathcal{E} - \mathbf{curl}\,\mathcal{H} = -\mathcal{J} & \text{in } \Omega,\ 0<t<T\ ;\\ \mu_0\partial_t\mathcal{H} + \mathbf{curl}\,\mathcal{E} = 0 & \text{in } \Omega,\ 0<t<T\ ;\\ \mathrm{div}\,(\varepsilon_0\mathcal{E}) = \rho & \text{in } \Omega,\ 0<t<T\ ;\\ \mathrm{div}\,(\mu_0\mathcal{H}) = 0 & \text{in } \Omega,\ 0<t<T\ ;\\ \mathcal{E}\times\mathbf{n} = 0 & \text{on } \partial\Omega,\ 0<t<T\ ;\\ \mathcal{E}(0)=\mathcal{E}_0\ ,\ \mathcal{H}(0)=\mathcal{H}_0 & \text{in } \Omega. \end{cases}$$

(Charge conservation equation: $\partial_t\rho + \mathrm{div}\,\mathcal{J} = 0$.

Initial conditions: $\mathrm{div}\,\mathcal{E}_0 = \frac{1}{\varepsilon_0}\rho(0)\ ;\ \mathrm{div}\,\mathcal{H}_0 = 0$.

$\mathbf{n}$ is the unit outward normal to $\partial\Omega$.)

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Related systems of equations

Second order (in time) wave equations...

- *In the electric field $\mathcal{E}$*

  Equivalent system : Find $\mathcal{E}(t)$ such that

$$
\begin{cases}
\partial_{tt}^2 \mathcal{E} + c^2\, \mathbf{curl}\, \mathbf{curl}\, \mathcal{E} = -\dfrac{1}{\varepsilon_0} \partial_t \mathcal{J} & \text{in } \Omega,\ 0 < t < T\ ; \\
\mathrm{div}\, (\varepsilon_0 \mathcal{E}) = \rho & \text{in } \Omega,\ 0 < t < T\ ; \\
\mathcal{E} \times \mathbf{n} = 0 & \text{on } \partial\Omega,\ 0 < t < T\ ; \\
\mathcal{E}(0) = \mathcal{E}_0\ ,\ \partial_t \mathcal{E}(0) = \mathcal{E}_1 & \text{in } \Omega.
\end{cases}
$$

$$
\left(\mathcal{E}_1 := \frac{1}{\varepsilon_0} \Big(\mathbf{curl}\, \mathcal{H}_0 - \mathcal{J}(0) \Big). \right)
$$

Or ...

Related systems of equations

Eigenmode computations in a resonator cavity...

- Assume the time-dependence writes $\exp(-\imath\omega t)$.
  $\Big(\omega > 0$ is the pulsation.$\Big)$
- *In the electric field* $\mathcal{E}$

  Equivalent system: Find $(\mathcal{E}, \omega)$ such that

$$\begin{cases} c^2\,\mathbf{curl}\,\mathbf{curl}\,\mathcal{E} = \omega^2\mathcal{E} & \text{in } \Omega\ ; \\ \operatorname{div}\mathcal{E} = 0 & \text{in } \Omega\ ; \\ \mathcal{E}\times\mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

  Or ...

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Related systems of equations

(Magnetic) quasi-static computations...

- Assume that the electric displacement current $\varepsilon_0 \partial_t \mathcal{E}$ is negligible.
- *In the electric field $\mathcal{E}$*

  Find $\mathcal{E}$ such that

$$\begin{cases} \mathbf{curl}\, \mathcal{E} = -\mu_0 \partial_t \mathcal{H} & \text{in } \Omega,\ 0 < t < T\ ; \\ \mathrm{div}\,(\varepsilon_0 \mathcal{E}) = \rho & \text{in } \Omega,\ 0 < t < T\ ; \\ \mathcal{E} \times \mathbf{n} = 0 & \text{on } \partial\Omega,\ 0 < t < T. \end{cases}$$

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Functional space: possible choices

- Which *functional space* to measure the electric field?

First choice:

$$\mathcal{H}_0(\mathbf{curl}\,,\Omega) := \{\mathcal{F} \in L^2(\Omega)^3 \mid \mathbf{curl}\,\mathcal{F} \in L^2(\Omega)^3,\ \mathcal{F} \times \mathbf{n}_{|\partial\Omega} = 0\}\,.$$

(cf. [Kikuchi'87/'89], [Demkowicz et al'9x], [Boffi et al'9x/'0x], ...)

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(cf. [Kikuchi'87/'89], [Demkowicz et al'9x], [Boffi et al'9x/'0x], ...)

Scalar product: $(u,v)_{\mathcal{H}(\mathbf{curl}\,,\Omega)} := (u,v)_0 + (\mathbf{curl}\,u, \mathbf{curl}\,v)_0.$

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- Which *functional space* to measure the electric field?

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$$\mathcal{X}_0 := \{\mathcal{F} \in \mathcal{H}_0(\mathbf{curl}\,, \Omega) \,|\, \operatorname{div} \mathcal{F} \in L^2(\Omega)\}\,.$$

OK in a convex domain Ω

(cf. [Assous-Degond-Heintzé-Raviart-Segré'93].)

OK in a 2D or 2D1/2 non-convex domain Ω (Singular Complement Method)

(cf. [Assous-Jr et al'98/'00/'03], [Bonnet-Hazard-Lohrengel'99/'02].)

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Functional space: possible choices

- Which *functional space* to measure the electric field?

Third choice:

$$\mathcal{X}_\gamma := \{\mathcal{F} \in \mathcal{H}_0(\mathbf{curl}\,,\Omega) \,|\, \mathrm{div}\,\mathcal{F} \in L^2_\gamma(\Omega)\}\,.$$

$$\left(L^2_\gamma(\Omega) := \{v \in L^2_{\mathrm{loc}}(\Omega) \,|\, w_\gamma\, v \in L^2(\Omega)\},\ ||v||_{0,\gamma} := ||w_\gamma\, v||_0.\right.$$

The weight w_γ is a function of the distance r to the reentrant edges (called E):

$$w_\gamma(r) \approx r^\gamma \text{ for small } r,$$

with a suitable $\gamma \in]\gamma_{min}, 1[$, $0 < \gamma_{min} < \frac{1}{2}$, cf. [Costabel-Dauge'02/'03].)

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This is the so-called Weighted Regularization Method: *our choice from now on...*

The constraint on the divergence

- What happens if one wants to take into account the *constraint* on the divergence of the electric field *explicitly*? $\left(\operatorname{div}(\varepsilon_0 \mathcal{E}) = \rho\right.$ or $\left.\operatorname{div} \mathcal{E} = 0.\right)$

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  Motivations:

  - Improve the quality of the divergence of the discrete fields.
    (For instance, for the computed eigenmodes.)
  - Resolve numerical problems related to the discrete charge conservation equation.
    (Solve the Vlasov-Maxwell system to compute the motion of charged particles.)

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Solution:

Introduce a *Lagrange multiplier*.

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Mixed Variational Formulations

- The *eigenproblem* to be solved writes equivalently ($\lambda = \omega^2/c^2$)
  Find $(\mathcal{E}, \lambda) \in \mathcal{K}_\gamma \times \mathbb{R}^+$ such that

  $$(\mathbf{curl}\,\mathcal{E}, \mathbf{curl}\,\mathcal{F})_0 = \lambda(\mathcal{E}, \mathcal{F})_0,\ \forall \mathcal{F} \in \mathcal{K}_\gamma,$$

  with $\mathcal{K}_\gamma := \{\mathcal{F} \in \mathcal{X}_\gamma \,|\, \mathrm{div}\,\mathcal{F} = 0\}$.

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Mixed Variational Formulations

- The *eigenproblem* to be solved writes
  Find $(\mathcal{E}, \lambda) \in \mathcal{K}_\gamma \times \mathbb{R}^+$ such that

$$(\mathcal{E}, \mathcal{F})_{\mathcal{X}_\gamma} = \lambda(\mathcal{E}, \mathcal{F})_0, \ \forall \mathcal{F} \in \mathcal{K}_\gamma,$$

  with $\mathcal{K}_\gamma := \{\mathcal{F} \in \mathcal{X}_\gamma \,|\, \mathrm{div}\, \mathcal{F} = 0\}$.

Mixed Variational Formulations

- The *mixed eigenproblem* to be solved writes
  Find $(\mathcal{E}, p, \lambda) \in \mathcal{X}_\gamma \times L^2_{-\gamma}(\Omega) \times \mathbb{R}^+$ such that

$$\begin{cases} (\mathcal{E}, \mathcal{F})_{\mathcal{X}_\gamma} + {}_{L^2_{-\gamma}}\langle p, \operatorname{div} \mathcal{F} \rangle_{L^2_\gamma} = \lambda(\mathcal{E}, \mathcal{F})_0, \ \forall \mathcal{F} \in \mathcal{X}_\gamma \\ {}_{L^2_{-\gamma}}\langle q, \operatorname{div} \mathcal{E} \rangle_{L^2_\gamma} = 0, \ \forall q \in L^2_{-\gamma}(\Omega). \end{cases}$$

  It is equivalent to the original eigenproblem ($p = 0$, see the Annex of [Jr'05].)

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- A *discrete approximation* is $((\mathcal{X}_h)_h \subset \mathcal{X}_\gamma, \ (M_h)_h \subset L^2_{-\gamma}(\Omega))$
  Find $(\mathcal{E}_h, p_h, \lambda_h) \in \mathcal{X}_h \times M_h \times \mathbb{R}^+$ such that

$$\begin{cases} (\mathcal{E}_h, \mathcal{F}_h)_{\mathcal{X}_\gamma} + {}_{L^2_{-\gamma}}\langle p_h, \operatorname{div} \mathcal{F}_h \rangle_{L^2_\gamma} = \lambda_h(\mathcal{E}_h, \mathcal{F}_h)_0, \ \forall \mathcal{F}_h \in \mathcal{X}_h \\ {}_{L^2_{-\gamma}}\langle q_h, \operatorname{div} \mathcal{E}_h \rangle_{L^2_\gamma} = 0, \ \forall q_h \in M_h. \end{cases}$$

  Abstract convergence theory, see [Boffi-Brezzi-Gastaldi'97], [Boffi'06].
  Uses *strong approximability* of solutions $\mathcal{E}$, *weak approximability* of solutions p
  $\Big($ with $(\mathcal{E}, p)$ solutions to the *plain* mixed problem... $\Big)$

Elements of convergence theory

- A *desired property* is the uniform discrete inf-sup condition

$$\exists\beta > 0,\ \forall h,\ \inf_{q_h\in M_h}\ \sup_{\mathcal{F}_h\in\mathcal{X}_h} \frac{{}_{L^2_{-\gamma}}\langle q_h, \operatorname{div}\mathcal{F}_h\rangle_{L^2_\gamma}}{\|\mathcal{F}_h\|_{\mathcal{X}_\gamma}\,\|q_h\|_{0,-\gamma}} \geq \beta.$$

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- Formulation with $\mathcal{E}$ set in $\mathcal{X}_0$ (Ω convex or SCM in 2D, 2D1/2 domains).
  - With the $P_2 - iso - P_1$ Taylor-Hood finite element, as in [Assous-Degond-Heintzé-Raviart-Segré'93]. The *udisc* is satisfied, cf. [Girault-Jr'02].
  - With the $P_{k+1} - P_k$ Taylor-Hood finite elements, the *udisc* is satisfied, cf. [Stenberg'84], [Boffi'97].

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- Formulation in $\mathcal{E}$ set in $\mathcal{X}_\gamma$ (WRM).
  - With the $P_{k+1} - P_k$ Taylor-Hood finite elements
    The *udisc* is not satisfied anymore!

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Why?

A negative result

- In order to check the discrete inf-sup condition, let

$$\beta_h = \inf_{q_h \in M_h} \sup_{\mathcal{F}_h \in \mathcal{X}_h} \frac{{}_{L^2_{-\gamma}}\langle q_h, \operatorname{div} \mathcal{F}_h \rangle_{L^2_\gamma}}{\|\mathcal{F}_h\|_{\mathcal{X}_\gamma} \|q_h\|_{0,-\gamma}}.$$

How can one estimate $(\beta_h)_h$?

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  How can one estimate $(\beta_h)_h$?

- Introduce the *plain* mixed Variational Formulation (rhs f, g).
  Find $(\mathcal{E}_h, p_h)$ such that

$$\left\{ \begin{array}{l} a(\mathcal{E}_h, \mathcal{F}_h) + b(p_h, \mathcal{F}_h) = f(\mathcal{F}_h), \ \forall \mathcal{F}_h \in \mathcal{X}_h \\ b(q_h, \mathcal{E}_h) = g(q_h), \ \forall q_h \in M_h. \end{array} \right.$$

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  How can one estimate $(\beta_h)_h$?

- Introduce the *matrix version* of the plain mixed Variational Formulation.
  Find $(\vec{\mathcal{E}}, \vec{p})$ such that

$$\left\{ \begin{array}{l} \mathbb{A}\,\vec{\mathcal{E}} + \mathbb{B}^T \vec{p} = \vec{f} \\ \mathbb{B}\,\vec{\mathcal{E}} = \vec{g}. \end{array} \right.$$

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- Proposition (e. g. [Jamelot'05]): *Define* $\mathbb{M}$ *by* $(\mathbb{M}\vec{q} \,|\, \vec{q}) = \|q_h\|^2_{M_h}$*. There holds*

$$\kappa(\mathbb{M}^{-1}(\mathbb{B}\,\mathbb{A}^{-1}\mathbb{B}^T)) \le \left(\frac{\|b\|}{\beta_h} \right)^2.$$

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  How can one estimate $(\beta_h)_h$?

- Practical experiments with the $P_2 - P_1$ Taylor-Hood finite element.
  - In the unit cube (see [Hechme-Jr'07a])

Meshsize	h'	$h'/2$	$h'/4$
κ	2.9	2.8	2.8

$\Rightarrow$ *Consistent* with the fact that $(\beta_h)_h$ is independent of h...

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  How can one estimate $(\beta_h)_h$?

- Practical experiments with the $P_2 - P_1$ Taylor-Hood finite element.
  - The WRM in a 2D L-shape domain (see [Hechme-Jr'07a])

Meshsize	h	$h/2$	$h/4$	$h/8$
κ	29	69	161	364

$\Rightarrow$ $(\beta_h)_h$ decreases sharply when h decreases...

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A new family of finite elements for the WRM

- Consider a family of triangular/tetrahedral meshes $(\mathcal{T}_h)_h$ of $\Omega \subset \mathbb{R}^d$.

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- Consider a family of triangular/tetrahedral meshes $(\mathcal{T}_h)_h$ of $\Omega \subset \mathbb{R}^d$.
- Standard family of $P_{k+1} - P_k$ Taylor-Hood finite elements:

$$\begin{aligned} \mathcal{X}_h &= \{\mathcal{F}_h \in C^0(\bar{\Omega})^d \mid \mathcal{F}_{h|T} \in P_{k+1}(T)^d, \forall T \in \mathcal{T}_h, \text{and } \mathcal{F}_h \times \mathbf{n}_{|\partial\Omega} = 0\}, \\ M_h &= \{q_h \in C^0(\bar{\Omega}) \mid q_{h|T} \in P_k(T), \forall T \in \mathcal{T}_h\}. \end{aligned}$$

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with E_h a neighborhood of the reentrant corners and/or edges:

$$E_h = \cup_{T \in \mathcal{T}_h \; s.t. \; T \cap E \neq \emptyset} T.$$

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with E_h a neighborhood of the reentrant corners and/or edges:

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$\Longrightarrow$ **Zero Near Singularity $P_{k+1} - P_k$ finite elements**.

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Remarks

- In the variational formulations, at the discrete level, one has $_{L^2_{-\gamma}}\langle \bar{q}_h, \operatorname{div} \mathcal{F}_h \rangle_{L^2_\gamma} = (\bar{q}_h, \operatorname{div} \mathcal{F}_h)_0$, $\forall (\mathcal{F}_h, q_h) \in \mathcal{X}_h \times \bar{M}_h$...

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As a consequence, the quantity of interest is

$$\bar{\beta}_h = \inf_{\bar{q}_h \in \bar{M}_h} \sup_{\mathcal{F}_h \in \mathcal{X}_h} \frac{(\bar{q}_h, \operatorname{div} \mathcal{F}_h)_0}{\|\mathcal{F}_h\|_{\mathcal{X}_\gamma} \, \|\bar{q}_h\|_{0,-\gamma}}.$$

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  - $\rightarrow$ Follows (more or less!) the series of lemmas of [Stenberg'84].
  - $\rightarrow$ Difficulties:
    - – Presence of *weights* in $\|\mathcal{F}_h\|_{\mathcal{X}_\gamma}$ and $\|\bar{q}_h\|_{0,-\gamma}$.
    - – *Local estimates* (near the reentrant edges).
    - – Existence of *gradients* in $\mathcal{X}_h$.
    - – Non-zero mean value Lagrange multipliers.

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- The final result on $(\bar{\beta}_h)_h$:
  Measuring the quality of the *regular* family of triangulations $(\mathcal{T}_h)_h$.

$$\exists \sigma > 1,\ \forall h,\ \forall T \in \mathcal{T}_h,\ h_T \leq \sigma\, \rho_T\ ;$$

$$\exists \eta > 1,\ \forall h,\ \forall T, T' \in \mathcal{T}_h,\ T \cap T' \neq \emptyset \implies \rho_T \leq \eta\, \rho_{T'}.$$

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$\Longrightarrow$ **Zero Near Singularity finite elements satisfy the udisc**.

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Computing eigenvalues and eigenvectors

- Find $(\mathcal{E}_h, \bar{p}_h, \lambda_h) \in \mathcal{X}_h \times \bar{M}_h \times \mathbb{R}^+$ such that

$$\begin{cases} (\mathcal{E}_h, \mathcal{F}_h)_{\mathcal{X}_\gamma} + (\bar{p}_h, \operatorname{div} \mathcal{F}_h)_0 = \lambda_h (\mathcal{E}_h, \mathcal{F}_h)_0, \ \forall \mathcal{F}_h \in \mathcal{X}_h \\ (\bar{q}_h, \operatorname{div} \mathcal{E}_h)_0 = 0, \ \forall \bar{q}_h \in \bar{M}_h. \end{cases}$$

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- Convergence results on n smallest eigenvalues ($n \in \mathbb{N}$.)
  - $(E_\lambda)_{\lambda \leq \lambda_n}$ the corresponding *eigenspaces*.
  - Approximation error $\varepsilon_\lambda(h) = \sup_{v \in E_\lambda, \|v\|_{\mathcal{X}_\gamma}=1} \inf_{\mathcal{F}_h \in \mathcal{X}_h} \|v - \mathcal{F}_h\|_{\mathcal{X}_\gamma}$ (*worst case*: $\varepsilon_\lambda(h) \leq C_\varepsilon h^{\gamma - \gamma_{min} - \varepsilon}$.)
  - Error on eigenvalues: $|\lambda - \lambda_h| < C_n \, \varepsilon_\lambda(h)^2$.
  - Gap between exact and discrete eigenspaces: $\hat{\delta}(E_\lambda, E_{\lambda_h}) < C_n \, \varepsilon_\lambda(h)$.

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$\Longrightarrow$ **No more spurious eigenmodes**.

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Numerical experiments in 2D

On a 'practical' example, taken from Monique Dauge's benchmark.

- 2D, L-shaped, domain, straight sides, corners in (0,0), (1,0), (1,1), (-1,1), (-1,-1), (0,-1).
- First five eigenvalues (with repetition), up to six digits:
  - $\lambda_1 = 1.47562$, eigenmode has the strong unbounded singularity ;
  - $\lambda_2 = 3.53403$; $\lambda_3 = \lambda_4 = 9.86960$; $\lambda_5 = 11.3895$.
- The weight is implemented with $\gamma = 0.95$ (NB. $\gamma_{min} = 1/3$.)
- Experiments (cf. [Buffa-Jamelot-Jr'07]):
  - on a series of *quasi-uniform meshes* ;
  - relative errors $r_{k,h} = |\lambda_{k,h} - \lambda_k|/\lambda_k$, $1 \leq k \leq 5$ are reported.

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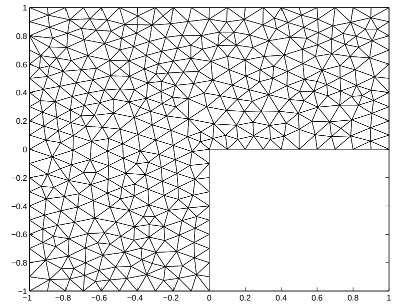
Numerical experiments in 2D



- Three meshes with respectively
  - 738, 2952 and 11808 triangles ;
  - 410, 1557 and 6065 vertices ;
- Results for the Zero Near Singularity finite elements:

$mesh$	$r_{1,h}$	$r_{2,h}$	$r_{3,h}$	$r_{4,h}$	$r_{5,h}$
$uniform1$	$1.3e-2$	$3.3e-4$	$9.4e-5$	$1.1e-4$	$9.9e-3$
$uniform2$	$8.0e-3$	$6.2e-5$	$2.3e-5$	$2.5e-5$	$1.3e-5$
$uniform3$	$4.4e-3$	$1.2e-5$	$5.5e-6$	$6.2e-6$	$5.3e-6$

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Comparisons in 3D

On a second 'practical' example, taken from Monique Dauge's benchmark.

- 3D, thick L-shaped, domain $(]-1,1[^2\backslash[-1,0]^2)\times]0,1[$.
- First nine eigenvalues (with repetition), up to six digits:
  - $\lambda_1 = 9.6397$; $\lambda_2 = 11.3452$; $\lambda_3 = 13.4036$; $\lambda_4 = 15.1972$;
  - $\lambda_5 = 19.5093$; $\lambda_6 = \lambda_7 = \lambda_8 = 19.7392$; $\lambda_9 = 21.2591$.
- The weight is implemented with $\gamma = 0.95$ (NB. $\gamma_{min} = 1/3$.)
- Experiments (cf. [Hechme-Jr'07b]):
  - on a *graded mesh* (grading towards the reentrant edge) ;
  - relative errors $r_{k,h} = |\lambda_{k,h} - \lambda_k|/\lambda_k$, $1 \leq k \leq 9$ are reported ;

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- Experiments (cf. [Hechme-Jr'07b]):
  - comparison of the mixed approach with
    - the *parameterized approach* [Costabel-Dauge'02] with parameter $s = \imath$.
      Find $(\mathcal{E}'_h, \lambda'_h) \in \mathcal{X}_h \times \mathbb{C}$ such that

$$(\mathbf{curl}\,\mathcal{E}'_h, \mathbf{curl}\,\mathcal{F}_h)_0 + \imath(\mathrm{div}\,\mathcal{E}'_h, \mathrm{div}\,\mathcal{F}_h)_{0,\gamma} = \lambda'_h(\mathcal{E}'_h, \mathcal{F}_h)_0,\ \forall \mathcal{F}_h \in \mathcal{X}_h.$$

Spurious (curl-free) eigenvalues are filtered out by comparing

$$Re(\lambda'_h) \text{ to } Im(\lambda'_h).$$

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- Experiments (cf. [Hechme-Jr'07b]):
  - comparison of the mixed approach with
    - the *filter approach* [Costabel-Dauge'03], [Hechme-Jr'07b].
      Find $(\mathcal{E}_h, \lambda_h) \in \mathcal{X}_h \times \mathbb{R}^+$ such that

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Spurious (curl-free) eigenvalues are filtered out by evaluating the *filter ratio*

$$\frac{\|\mathrm{div}\, \mathcal{E}_h\|_{0,\gamma}}{\|\mathbf{curl}\, \mathcal{E}_h\|_0}.$$

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Comparisons in 3D

- A mesh with

  4032 tetrahedra ; 1010 vertices.

- Number of d.o.f.

  15818 for the parameterized and filter approaches ; 18162 for the mixed approach.

- Results:

Method	Filter	Parameterized	Mixed
r_1	6.1×10^{-4}	6.1×10^{-4}	6.2×10^{-4}
r_2	6.5×10^{-3}	1.1×10^{-2}	8.5×10^{-3}
r_3	8.1×10^{-4}	7.4×10^{-4}	8.4×10^{-4}
r_4	1.1×10^{-4}	1.0×10^{-4}	1.1×10^{-4}
r_5	2.0×10^{-3}	4.7×10^{-3}	6.9×10^{-3}
r_6	1.8×10^{-4}	1.8×10^{-4}	1.8×10^{-4}
r_7	1.2×10^{-3}	1.1×10^{-3}	1.2×10^{-3}
r_8	1.2×10^{-3}	1.1×10^{-3}	1.3×10^{-3}
r_9	1.3×10^{-3}	1.1×10^{-3}	1.1×10^{-2}

Focusing on eigenvalues or eigenvectors?

On a last 'practical' example, taken from Monique Dauge's benchmark.

- 3D, Fichera corner, domain $(]-1,1[^3 \backslash [-1,0]^3)$.
- The weight is implemented with $\gamma = 0.95$ (NB. $\gamma_{min} = 1/3$.)
- A graded mesh with

  2688 tetrahedra ; 665 vertices.
- Experiments on the first eight eigenpairs (cf. [Hechme-Jr'07b]):

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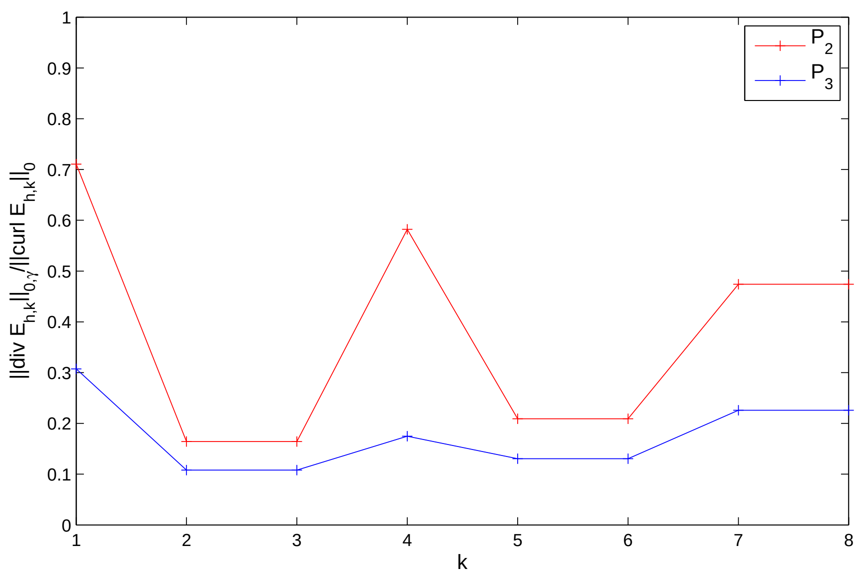
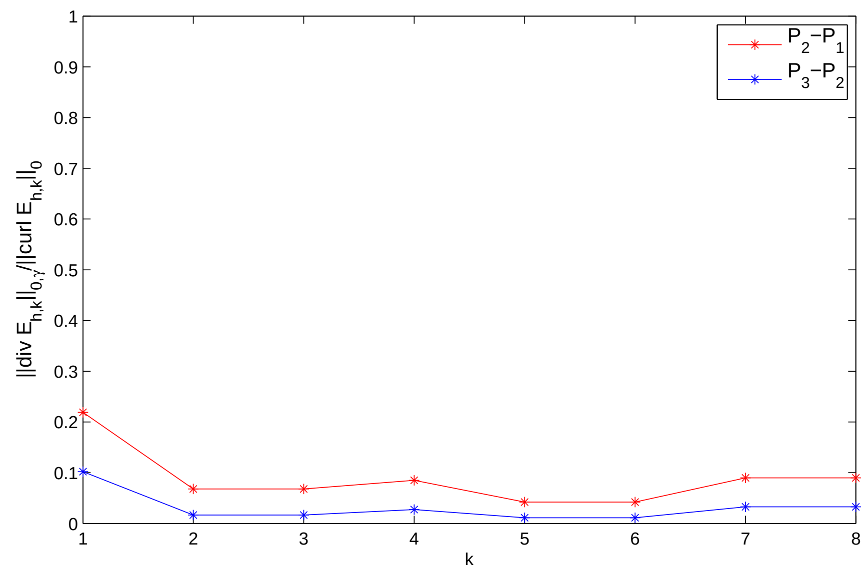
  Filter ratios for the filter (left) and mixed (right) methods





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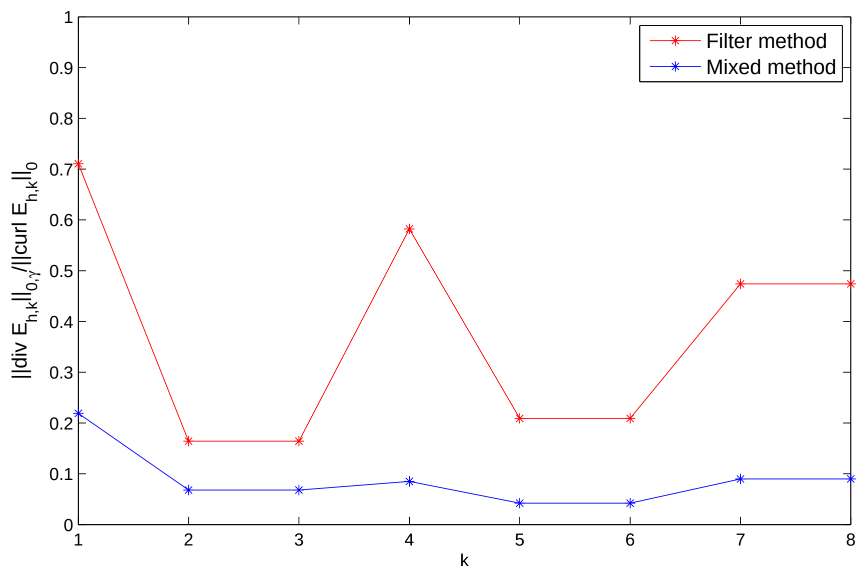

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Concluding remarks

- Implementing the mixed method with the WRM turned out to be a challenging problem!

- The classical $P_{k+1} - P_k$ Taylor-Hood finite elements fail to verify the udisc.
  The Zero Near Singularity $P_{k+1} - P_k$ finite elements provide an adequate answer.
  (with G. Hechme.)

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  No more spurious eigenmodes.
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  (with E. Jamelot, A. Buffa, G. Hechme.)
- Application to the time-dependent problem (Vlasov-Maxwell) has been completed.
  (with S. Labrunie.)
- Extension to materials (ε, μ piecewise constant) is possible.
  (with F. Lefèvre, S. Lohrengel, S. Nicaise.)

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