The Singular Complement Method for solving Maxwell equations

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Time-dependent Maxwell equations

- *In vacuum, over the time interval* $]0,T[$, $T>0$.

$$\begin{cases} \text{Find } (\mathcal{E}(t),\mathcal{H}(t)) \in \mathbf{L}^2(\cdot)\times\mathbf{L}^2(\cdot) \text{ such that} \\ \varepsilon_0 \dfrac{\partial \mathcal{E}}{\partial t} - \mathbf{curl}\,\mathcal{H} = -\mathcal{J}\ ; \\ \mu_0 \dfrac{\partial \mathcal{H}}{\partial t} + \mathbf{curl}\,\mathcal{E} = 0\ ; \\ \operatorname{div}(\varepsilon_0\mathcal{E}) = \rho\ ; \\ \operatorname{div}(\mu_0\mathcal{H}) = 0\ ; \\ \mathcal{E}(0) = \mathcal{E}_0\ ,\ \mathcal{H}(0) = \mathcal{H}_0\ . \end{cases}$$

$$\Big(\frac{\partial \mathcal{J}}{\partial t} \in L^2(0,T;\mathbf{L}^2(\cdot)),\ \rho \in \mathcal{C}^0(0,T;L^2(\cdot))\ ;\ \frac{\partial \rho}{\partial t} + \operatorname{div}\mathcal{J} = 0.$$

$$\mathcal{E}_0 \in \mathbf{H}(\mathbf{curl}\,,\cdot),\ \operatorname{div}\mathcal{E}_0 = \frac{1}{\varepsilon_0}\rho(0)\ ;\ \mathcal{H}_0 \in \mathbf{H}(\mathbf{curl}\,,\cdot),\ \operatorname{div}\mathcal{H}_0 = 0. \Big)$$

- Goal: compute the EM field around a perfect conducting body $\mathcal{O}$, with Lipschitz polyhedral boundary.

Time-dependent Maxwell equations (2)

But... Consider a *bounded* computational domain Ω, with Lipschitz polyhedral boundary.

Its boundary $\partial\Omega$ is split as $\partial\Omega = \bar{\Gamma}_C \cup \bar{\Gamma}_A$, with $\bar{\Gamma}_C = \partial\mathcal{O} \cap \partial\Omega$.
A *Silver-Müller* boundary condition is imposed on the artificial boundary Γ_A: incoming plane waves ($\mathbf{e}^\star \neq 0$), or 1st order absorbing condition ($\mathbf{e}^\star = 0$).

- *Boundary conditions*

$$\begin{cases} \mathcal{E} \times \mathbf{n} = 0 \text{ on } \Gamma_C \;; \\ (\mathcal{E} - \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\mathcal{H} \times \mathbf{n}) \times \mathbf{n} = \vec{\mathbf{e}}^\star \times \mathbf{n} \text{ on } \Gamma_A . \end{cases}$$

$$\left(\frac{\partial \vec{\mathbf{e}}^\star}{\partial t} \in L^2(0,T;\mathbf{L}^2(\Gamma_A)). \right)$$

- Consequences: some *"additional" boundary conditions*

$$\mathcal{H} \cdot \mathbf{n} = \mathcal{H}_0 \cdot \mathbf{n} \;;\; (\mathbf{curl}\,\mathcal{H}) \times \mathbf{n} = \mathcal{J} \times \mathbf{n} \text{ on } \Gamma_C .$$

$$\left.\begin{aligned} (\mathbf{curl}\,\mathcal{E}) \times \mathbf{n} &= \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{e}_T^\star) - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{E}_T \\ (\mathbf{curl}\,\mathcal{H}) \times \mathbf{n} &= \mathcal{J} \times \mathbf{n} + \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{e}^\star \times \mathbf{n}) - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{H}_T \end{aligned}\right\} \text{ on } \Gamma_A .$$

Time-dependent Maxwell equations (3)

2nd order in time, electric field $\mathcal{E}$...

- Equation

$$\left\{ \begin{array}{l} \dfrac{\partial^2 \mathcal{E}}{\partial t^2} + c^2 \,\mathbf{curl}\,\mathbf{curl}\,\mathcal{E} = -\dfrac{1}{\varepsilon_0}\dfrac{\partial \mathcal{J}}{\partial t} \;;\; \dfrac{\partial \mathcal{E}}{\partial t}(0) = \mathcal{E}_1 \\ \left(\mathcal{E}_1 := \dfrac{1}{\varepsilon_0}\left(\mathbf{curl}\,\mathcal{H}_0 - \mathcal{J}(0)\right)\right) \end{array} \right. .$$

- Functional space

$$\mathcal{T}^{0,\Gamma_C} := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}\,,\Omega) \;:\; \mathbf{v} \times \mathbf{n}_{|\partial\Omega} \in \mathbf{L}^2_t(\partial\Omega),\; \mathbf{v} \times \mathbf{n}_{|\Gamma_C} = 0\}.$$

- Variational Formulation
  Find $\mathcal{E} \in \mathcal{T}^{0,\Gamma_C}$ such that

$$\begin{array}{rcl} \dfrac{d^2}{dt^2}(\mathcal{E}, \mathbf{v})_0 & + & c^2(\mathbf{curl}\,\mathcal{E}, \mathbf{curl}\,\mathbf{v})_0 + c\dfrac{d}{dt}(\mathcal{E}_T, \mathbf{v}_T)_{0,\Gamma_A} \\ & = & -\dfrac{1}{\varepsilon_0}\dfrac{d}{dt}(\mathcal{J}, \mathbf{v})_0 + c\dfrac{d}{dt}(\vec{\mathbf{e}}^\star_T, \mathbf{v}_T)_{0,\Gamma_A},\; \forall \mathbf{v} \in \mathcal{T}^{0,\Gamma_C}. \end{array} \quad (1)$$

Continuous and discrete formulations (1)

- Putting *scalar potentials in (1):* $q \in H^1_0(\Omega)$, $\mathbf{v} = \nabla q \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$

$$-\frac{d^2}{dt^2}(\mathcal{E}, \nabla q)_0 = -\frac{1}{\varepsilon_0}\frac{d}{dt}(\operatorname{div}\mathcal{J}, q)_0 \overset{cce}{=} \frac{1}{\varepsilon_0}\frac{d^2}{dt^2}(\rho, q)_0, \ \forall q \in H^1_0(\Omega).$$

- Discretization: *Edge FE (Nédélec's 1st family)* + P_1 Lagrange FE for potentials

$$-\frac{d^2}{dt^2}(\mathcal{E}_h, \nabla q_h)_0 = \frac{1}{\varepsilon_0}\frac{d^2}{dt^2}(\rho, q_h)_0, \ \forall q_h.$$

  The divergence constraint is *weakly* enforced in the discrete case.

- But:
  - An $\mathbf{H}(\mathbf{curl}\,)$-conforming FEM yields a discontinuous approximation of the field, whereas it is smooth (except at interfaces between different materials).
  - Implicit schemes for the discretization in time are expensive (cf. [Cohen'02,Lacoste'04].)

Continuous and discrete formulations (2)

Consider an approximation of the field, *via* an $\mathbf{H}(\mathbf{curl}\,,\mathrm{div}\,)$-conforming discretization, using the P_k Lagrange FE (cf. [Assous et al'93]). Some *a priori* remarks:

- A continuous approximation.
- Mass lumping is possible (and optimal!), so the discretization in time is no longer an issue.
- But, the divergence constraint does not appear to be enforced...

- Define

$$\mathcal{X}^{0,\Gamma_C} := \mathcal{T}^{0,\Gamma_C} \cap \mathbf{H}(\mathrm{div}\,,\Omega),$$

$$(\mathbf{v},\mathbf{w})_X := (\mathbf{curl}\,\mathbf{v},\mathbf{curl}\,\mathbf{w})_0 + (\mathrm{div}\,\mathbf{v},\mathrm{div}\,\mathbf{w})_0 + (\mathbf{v}_T,\mathbf{w}_T)_{0,\partial\Omega};$$

$$\mathcal{X}^0 := \mathbf{H}_0(\mathbf{curl}\,,\Omega) \cap \mathbf{H}(\mathrm{div}\,,\Omega),$$

$$(\mathbf{v},\mathbf{w})_{X^0} := (\mathbf{curl}\,\mathbf{v},\mathbf{curl}\,\mathbf{w})_0 + (\mathrm{div}\,\mathbf{v},\mathrm{div}\,\mathbf{w})_0.$$

- Hypothesis: the semi-norms associated to the scalar products above define norms on $\mathcal{X}^{0,\Gamma_C}$ and $\mathcal{X}^0$, which are *equivalent* to the full norm.
  From [Weber'80], [Fernandes-Gilardi'97], [Amrouche-Bernardi-Dauge-Girault'98]:
  Assume for instance that $\partial\Omega$ *is connected*.

Continuous and discrete formulations (3)

- Then, $\mathcal{E}$ is the solution to the Mixed, Augmented Variational Formulation:
  find $(\mathcal{E}, p) \in \mathcal{X}^{0,\Gamma_C} \times L^2(\Omega)$ *s.t.*

$$\frac{d^2}{dt^2}(\mathcal{E}, \mathbf{v})_0 + c\frac{d}{dt}(\mathcal{E}_T, \mathbf{v}_T)_{0,\Gamma_A} + c^2(\mathcal{E}, \mathbf{v})_{X^0} + (p, \operatorname{div} \mathbf{v})_0$$

$$= -\frac{1}{\varepsilon_0}\frac{d}{dt}(\mathcal{J}, \mathbf{v})_0 + \frac{c^2}{\varepsilon_0}(\rho, \operatorname{div} \mathbf{v})_0 + c\frac{d}{dt}(\vec{\mathbf{e}}_T^\star, \mathbf{v}_T)_{0,\Gamma_A}, \ \forall \mathbf{v} \in \mathcal{X}^{0,\Gamma_C},$$

$$(\operatorname{div} \mathcal{E}, q)_0 = \frac{1}{\varepsilon_0}(\rho, q)_0, \ \forall q \in L^2(\Omega).$$

- The constraint on Γ_C is usually enforced numerically, which means that the discrete field satisfies $\mathcal{E}_h \times \mathbf{n}_{|\Gamma_C} = 0$...

$\Rightarrow$ This plain discretization leads to trouble, when the polyhedral domain is not convex. (That is, when *strong electromagnetic fields* appear.)

- From now on, it is assumed that $\Gamma_C = \partial\Omega$, i. e. the electric field belongs to $\mathcal{X}^0$.

Strong/singular electric fields

- By construction, the discrete field $\mathcal{E}_h$ belongs to $\mathcal{X}^0 \cap \mathbf{H}^1(\Omega) := \mathcal{X}^0_R$.
- But, $\mathcal{X}^0_R$ is a strict, closed subspace of $\mathcal{X}^0$ when Ω is not convex (cf. [Grisvard'85], [Birman-Solomyak'87].) One can thus write

$$\mathcal{X}^0 = \mathcal{X}^0_R \overset{\perp}{\oplus} \mathcal{X}^0_S, \text{ with } \mathcal{X}^0_S \text{ the subspace of } \textit{singular electric fields}.$$

- Consequently, if one splits $\mathcal{E}$ as $\mathcal{E} = \mathcal{E}_R + \mathcal{E}_S$, one finds $\|\mathcal{E} - \mathcal{E}_h\|_{X^0} \geq \|\mathcal{E}_S\|_{X^0}$: a strong electric field cannot be approximated in $\mathcal{X}^0$ by the discrete field only...
- Consider $\Phi := \{\phi \in H^1_0(\Omega) \;:\; \Delta\phi \in L^2(\Omega)\}$.
  If ones defines the *orthogonal complement* Φ_S of $H^2(\Omega) \cap H^1_0(\Omega)$ in Φ, there holds (cf. [Bonnet-Hazard-Lohrengel'99])

$$\forall \mathbf{x}_S \in \mathcal{X}^0_S,\ \exists!(\tilde{\mathbf{x}}, \phi_S) \in \mathcal{X}^0_R \times \Phi_S, \quad \mathbf{x}_S = \tilde{\mathbf{x}} + \nabla\phi_S.$$

- Idea: study the singular potentials and ways to *compute* them, to derive numerical techniques for computing the singular electric fields.

The SCM for scalar fields: theory

- Let $(\Gamma_f)_{1\le f\le F}$ denote the set of faces of the boundary $\partial\Omega$.

  According to [Assous-Jr'97], one can write

$$L^2(\Omega) = \Delta(H^2(\Omega)\cap H^1_0(\Omega)) \overset{\perp}{\oplus} S_D, \text{ with}$$

$$S_D := \{s \in L^2(\Omega) \ : \ \Delta s = 0,\ s_{|\Gamma_f} = 0 \text{ in } \left(H^{1/2}_{00}(\Gamma_f)\right)',\ 1\le f\le F\}.$$

- From now on, assume ω is a polygon, with K reentrant corners on its boundary. Let (r_k,θ_k) denote the local polar coordinates, with incoming edges described locally by $\theta_k = 0$ or $\theta_k = \pi/\alpha_k$. There holds [Grisvard'92]
  - $dim(S_D) = K$: let $(s_{D,k})_k$ be a basis of S_D;
  - $s_{D,k}$ can be chosen as $s_{D,k} = r_k^{-\alpha_k}\sin(\alpha_k\theta_k) + \tilde{s}_{D,k}$, with $\tilde{s}_{D,k} \in H^1(\omega)$;
  - given $f\in L^2(\omega)$, the solution u to the problem *find* $u\in H^1_0(\omega)$ *s.t.* $-\Delta u = f$ can be written as

$$u = \sum_{k=1}^{K} \lambda_k r_k^{\alpha_k}\sin(\alpha_k\theta_k) + \tilde{u}, \text{ with } \lambda_k = \frac{1}{\pi}(f, s_{D,k})_0,\ \tilde{u}\in H^2(\omega).$$

The SCM for scalar fields: numerical analysis

- Many Refs:
  (i) The *Dual Singular Function Method*: [Blum-Dobrowolski'81], [Dobrowolski'81] and many, many others... With a cut-off function, and a different representation formula for λ_k.
  (ii) Or [Moussaoui'84] and others... No cut-off, but a non-homogeneous boundary condition, see the previous page.

- Here, we follow (ii) and set $\alpha = min_k \alpha_k$.
  Given a regular triangulation T_h of ω with meshsize h, define V_h the space of continuous and piecewise linear functions on T_h.
- Write $s^h_{D,k} = r_k^{-\alpha_k} \sin(\alpha_k \theta_k) + \tilde{s}^h_{D,k}$, with $\tilde{s}^h_{D,k} \in V_h$.
  Proposition: $\exists C > 0$ such that $||s_{D,k} - s^h_{D,k}||_0 \leq C h^{2\alpha}$.
- Write $u^h = \sum_{k=1}^{K} \lambda_k^h r_k^{\alpha_k} \sin(\alpha_k \theta_k) + \tilde{u}^h$, with $\lambda_k^h = \frac{1}{\pi}(f, s^h_{D,k})_0$.
  Proposition: $\exists C > 0$ such that $||u - u^h||_1 \leq C h$.

The SCM for electric fields: theory

- We follow here [Assous-Jr-Garcia'00], [Hazard-Lohrengel'02], [Jamelot'04].

- $dim(\mathbf{X}_S^0) = K$: let $(\mathbf{x}_{S,k})_k$ be a basis of $\mathbf{X}_S^0$.

  The electric field can be split as $\mathbf{E} = \mathbf{E}_R + \sum_{k=1}^{K} c_k \mathbf{x}_{S,k}$, $\mathbf{E}_R \in \mathbf{X}_R^0$.

(i) $(\mathbf{x}_{S,k})_k$ can be chosen as the solution to:
find $\mathbf{x}_{S,k} \in \mathbf{X}^0$ *s.t.* $\mathbf{curl}\,\mathbf{x}_{S,k} = s_{N,k}$, $\mathrm{div}\,\mathbf{x}_{S,k} = s_{D,k}$.

(ii) Letting $\mathbf{x}_{P,k} = -\alpha_k r_k^{\alpha_k - 1} \begin{pmatrix} \sin(\alpha_k \theta_k) \\ \cos(\alpha_k \theta_k) \end{pmatrix}$, $(\mathbf{x}_{S,k})_k$ can be chosen as

$$\mathbf{x}_{S,k} = \frac{1}{\pi} \left(\|s_{D,k}\|_0^2 + \|s_{N,k}\|_0^2 \right) \mathbf{x}_{P,k} + \tilde{\mathbf{x}}_k,$$ with $\tilde{\mathbf{x}}_k \in H^1(\omega)^2$ ensuring orthogonality.

- The two choices yield the same basis.

The SCM for electric fields: numer. anal.

- To compute the additional basis vectors, which approximate the singular fields of $\mathbf{X}^0_S$, no cut-off is required, but a non-homogeneous boundary condition is used again.

  $\mathbf{x}^h_{S,k} = \frac{1}{\pi}\left(\|s^h_{D,k}\|_0^2 + \|s^h_{N,k}\|_0^2\right)\mathbf{x}_{P,k} + \tilde{\mathbf{x}}^h_k$, with $\tilde{\mathbf{x}}^h_k \in V^2_h$.

  Proposition:

  - $\forall\varepsilon > 0$, $\exists C_\varepsilon > 0$ such that $\|\mathbf{x}_{S,k} - \mathbf{x}^h_{S,k}\|_X \leq C_\varepsilon h^{2\alpha-1-\varepsilon}$.
  - $\forall\varepsilon > 0$, $\exists C_\varepsilon > 0$ such that $\|\mathbf{x}_{S,k} - \mathbf{x}^h_{S,k}\|_0 \leq C_\varepsilon h^{4\alpha-2-\varepsilon}$.

- To approximate the electric field, one uses elements of $\mathbf{X}^0 \cap V^2_h$ and $(\mathbf{x}^h_{S,k})_k$, to get

  $\mathbf{E}^h = \mathbf{E}^h_R + \sum_{k=1}^{K} c^h_k \mathbf{x}^h_{S,k}$, and similarly for test fields.

  Proposition:

  - $\forall\varepsilon > 0$, $\exists C_\varepsilon > 0$ such that $\|\mathbf{E} - \mathbf{E}^h\|_X \leq C_\varepsilon h^{2\alpha-1-\varepsilon}$.
  - $\forall\varepsilon > 0$, $\exists C_\varepsilon > 0$ such that $\|\mathbf{E} - \mathbf{E}^h\|_0 \leq C_\varepsilon h^{4\alpha-2-\varepsilon}$.

Extensions: finite dimensional $\mathcal{X}_S^0$

(1) Case of the 3D axisymmetric Maxwell equations:

- In a domain Ω, which is invariant by rotation;
- given data, which is also invariant by rotation.

One gets that the dimension of the singular space of electric fields behaves like

$$dim(\mathcal{X}_S^0) = K_e + K_c,$$

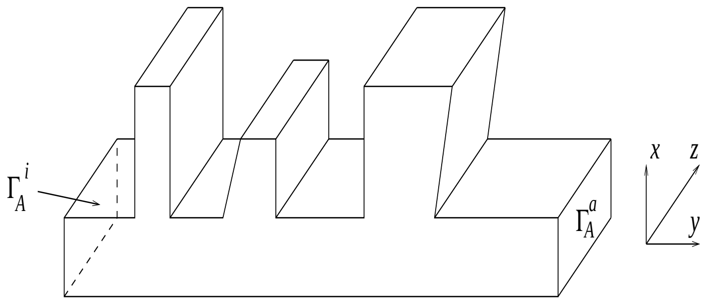
with K_e the number of circular reentrant edges, and K_c the number of sharp conical vertices (see [Labrunie et al'99-'00-'02-'03]...)

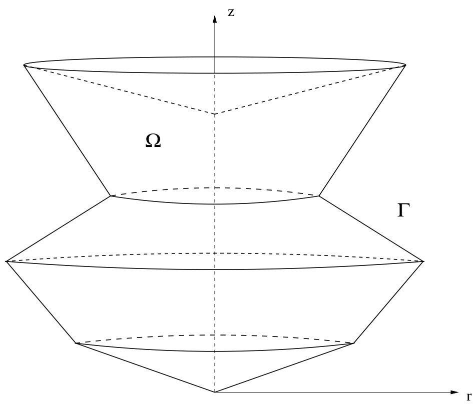
(2) Case of the 3D Maxwell equations, set in a domain Ω, whose only geometrical singularities are smooth-based, sharp conical vertices.
One gets that the dimension of the singular space behaves like

$$dim(\mathcal{X}_S^0) = K_c.$$

(see [Garcia'02], [Labrunie'05].)

Extensions: the Fourier SCM





(3) The 3D Laplace problem in a prismatic or axisymmetric domain (see [Kaddouri-Jung et al'04a,b,c].)

(4) The 3D Maxwell equations in an axisymmetric domain (see [Labrunie'05].)

(5) The 3D Maxwell equations in a prismatic domain (see [Jr-Garcia-Zou'04].)

(x) An *open problem*: the 3D Maxwell equations in a general domain... The main difficulty is that edge and vertex singularities are linked (see [Costabel-Dauge'00].)

Alternative methods (1)

- One can choose to include explicitly the vanishing boundary condition on Γ_C in the Variational Formulation, i.e.

  replace $(\mathcal{E}_T, \mathbf{v}_T)_{0,\Gamma_A}$ by $(\mathcal{E}_T, \mathbf{v}_T)_{0,\partial\Omega}$.

  It is thus handled as a natural boundary condition, with the electric and test fields in

$$\mathcal{X} := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}\,, \Omega) \cap \mathbf{H}(\mathrm{div}\,, \Omega)\ :\ \mathbf{v} \times \mathbf{n}_{|\partial\Omega} \in \mathbf{L}^2_t(\partial\Omega)\}.$$

- According to [Jr-Hazard-Lohrengel'98], [Costabel-Dauge'98]:
  the subspace of $\mathbf{H}^1$-regular fields is *dense* in $\mathcal{X}$. No need for a Singular Complement!

- One can also add a Lagrange multiplier for the boundary condition, see [Jr'05].

Alternative methods (2)

- Solving the electric problem in a weighted Sobolev space...
- Introduce:
  - The set E of reentrant edges of $\partial\Omega$, and the distance $d_0(\mathbf{x}) = d(\mathbf{x}, E)$.
  - The sets ($\gamma \in [0,1]$)

$$L^2_\gamma(\Omega) := \{g \,:\, g \in \mathcal{D}'(\Omega),\ {d_0}^\gamma g \in L^2(\Omega)\}, \text{ with norm } \|g\|_{0,\gamma} = \|{d_0}^\gamma g\|_0 \,;$$
$$\mathcal{X}^0_{E,\gamma} := \{\mathbf{v} \,:\, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega),\ \mathrm{div}\,\mathbf{v} \in L^2_\gamma(\Omega)\}.$$

- Theorem [Costabel-Dauge'02]: $\exists\gamma_0 \in]0,1/2[$, such that
  (i) $\forall\gamma \in]\gamma_0, 1]$, the regular subspace $\mathbf{H}^1(\Omega) \cap \mathcal{X}^0_{E,\gamma}$ is *dense* in $\mathcal{X}^0_{E,\gamma}$.
  (ii) $\forall\gamma \in]\gamma_0, 1[$, the semi-norm associated to

$$(\cdot,\cdot)_{X^0_\gamma} \,:\, (\mathbf{u},\mathbf{v}) \mapsto (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_{0,\gamma}$$

  is a norm in $\mathcal{X}^0_{E,\gamma}$, which is *equivalent* to the full norm.