The Singular Complement Method (Part II)

How to improve the convergence rate?

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I. Introduction

Basics

Consider a non-smooth and non-convex domain ω:

- polygon (2d)
- polyhedron (3d).

Aims of the Singular Complement Method:

- *Enable* convergence of the P_1 Lagrange FEM.

  (static or time-dependent Maxwell equations, fluid problems, etc.)
- *Improve* the convergence rate of the P_1 Lagrange FEM.

  (Poisson-like problems, wave equation, etc.)

How to improve the convergence rate

A number of techniques have been developed...

- Mesh refinement

  (Raugel'78 ; Apel-Nicaise'99, etc.)
- The Singular Function Method

  (Strang-Fix'73, etc.)
- The Dual Singular Function Method

  (Dobrowolski'80 ; Amara-Moussaoui'90 ; Grisvard'92, etc.)
- Other techniques

  (Brenner'99 ...)

(Potential) Drawbacks & Advantages

- Mesh refinement

  mesh *generation* ; troublesome for *time-dependent* problems ;

  works in 3d for Poisson-like problems.

- The Singular Function Method

  does not converge in practice.

- The Dual Singular Function Method

  converges slowly in practice.

II. The framework

The problem and its discretization

- Consider a 2d polygon ω.
- Aim: solve numerically

  Given $f \in L^2(\omega)$

  Find $u \in H_0^1(\omega)$ such that $-\Delta u = f$ in ω.
- Tools: the P_1 Lagrange FEM.

  $\mathcal{T}_h$ a regular triangulation.

  $V_h^0 = \{v_h \in \mathcal{C}^0(\omega) \;:\; v_h|_T \in P_1(T),\, \forall T \in \mathcal{T}_h,\, v_{h|_{\partial\omega}} = 0\}$.

$$\text{Find } x_h \in V_h^0 \text{ such that } \int_\omega \nabla x_h \cdot \nabla v_h \, d\omega = \int_\omega f\, v_h \, d\omega,\ \forall v_h \in V_h^0,$$

or

$$\mathbb{K}\underline{x} = \underline{f}.$$

Convergence results

- If ω is *convex*:

  $\exists C > 0$ such that $\|u - x_h\|_1 \leq C\, h$.

- If ω is *non-convex*:

  $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ such that $\|u - x_h\|_1 \leq C_\varepsilon\, h^{\alpha - \varepsilon}$

  (α is geometry dependent; $\alpha \in]\frac{1}{2}, 1[$).

How to bring the convergence rate back to h in all cases?

A few words about the (D)SFM

- Start from the FE space V_h^0

  Add *singular test-functions*

  $\eta(r_k){r_k}^{\alpha_k}\sin(\alpha_k\theta_k)$

  (one for each reentrant corner, with $\alpha_k = \pi/\Theta_k \in]\frac{1}{2}, 1[$.)

- Approximate u by

$$x_h' + \sum_k \lambda_k \eta(r_k){r_k}^{\alpha_k}\sin(\alpha_k\theta_k).$$

  Compute the coefficients $(\lambda_k)_k$:

  directly (SFM);

  via a scalar product $\lambda_k = \int_\omega f\, g_s^k\, d\omega$ (DSFM).

Problem: the truncation function η.

The SCM: elements of theory

- Idea...

  Trade the truncation function for a non-zero boundary condition,

  while keeping the dual approach.

- How?

  Use an orthogonal decomposition of the space $L^2(\omega)$:

$$\begin{aligned} L^2(\omega) &= \Delta(H^2(\omega) \cap H_0^1(\omega)) \overset{\perp}{\oplus} N, \text{ with} \\ N &= \{p \in L^2(\omega) \, : \, \Delta p = 0, \; p_{|\gamma_k} = 0 \text{ in } (H_{00}^{1/2}(\gamma_k))', \; 1 \leq k \leq n_e\}; \end{aligned}$$

  map it back (*via* Δ^{-1}) to the space of solutions.

NB. dim(N)= number of reentrant corners (*one* from now on...)

III. Algorithms & Numerical Analysis

Algorithms (1)

- How to isolate the singular part (in general)?

(1) Find a basis of N: p_s.

(2) Find $\phi_s \in H_0^1(\omega)$ such that $-\Delta\phi_s = p_s$.

- How to compute the solution of the problem with right-hand side f?

  Write $u = \tilde{u} + c\,\phi_s$, with $\tilde{u} \in H^2(\omega) \cap H_0^1(\omega)$.

(3) Compute $c = \dfrac{\displaystyle\int_\omega f p_s \, d\omega}{\|p_s\|_0^2}$.

(4) Find $\tilde{u} \in H_0^1(\omega)$ such that $-\Delta\tilde{u} = f - c\,p_s$.

Algorithms (2)

(1) How to compute p_s?

$p_s = \tilde{p} + p_P$, with

regular part $\tilde{p} \in H^1(\omega)$ $\quad$ ($\Delta\tilde{p} = 0$, $\tilde{p}_{|\gamma_c} = 0$);

principal part $p_P = r^{-\alpha}\sin(\alpha\theta)$.

(2) How to compute ϕ_s?

$\phi_s = \tilde{\phi} + \beta\,\phi_P$, with

regular part $\tilde{\phi} \in H^2(\omega)$ $\quad$ ($-\Delta\tilde{\phi} = p_s$, $\tilde{\phi}_{|\gamma_c} = 0$);

$\beta \in \mathbb{R}$;

principal part $\phi_P = r^{\alpha}\sin(\alpha\theta)$.

Algorithms (3)

- To get an explicit expression of β...

$$
\begin{aligned}
\|p_s\|_0^2 &= -\int_\omega \Delta\phi_s\, p_s\, d\omega \\
\Big(\quad \phi_s &= (\tilde{\phi} + \beta(1-\eta)\phi_P) + \beta\eta\phi_P \\
&\quad \in H^2(\omega) \cap H_0^1(\omega) \quad \Big) \\
&= -\beta \int_\omega \Delta(\eta\phi_P)\, p_s\, d\omega \\
&= -\beta \left\{ \int_\omega \Delta(\eta\phi_P)\, \tilde{p}\, d\omega + \int_\omega \Delta(\eta\phi_P)\, p_P\, d\omega \right\} \\
&= -\beta \{ \qquad 0 \qquad - \qquad \pi \qquad \}
\end{aligned}
$$

... Therefore $\beta = \frac{1}{\pi}\|p_s\|_0^2$.

Algorithms (4)

- Another approach...

$$
\begin{aligned}
u &= \tilde{u} + c\,\phi_s \\
&= \tilde{u} + c\,\tilde{\phi} + c\beta\,\phi_P \\
&= \tilde{u}' + \lambda\,\phi_P, \text{ with } \tilde{u}' \in H^2(\omega).
\end{aligned}
$$

(3b) Compute $\lambda = \dfrac{1}{\pi}\displaystyle\int_\omega f p_s \, d\omega$.

(4b) Find $\tilde{u}' \in H^1(\omega)$ such that $-\Delta\tilde{u}' = f$ and $\tilde{u}'_{|\partial\omega} = -\lambda\phi_{P\,|\partial\omega}$.

NB. This corresponds to the method described by Moussaoui'84...

Numerical Analysis (1)

- Numerical approximation of the *dual singular function* p_s.

(1) $p_s^h = \tilde{p}_h + p_P$, with $\tilde{p}_h = p_h^* - q_h$ defined by

$$q_h = Q_h(p_P), \text{ and } Q_h(g) = \sum_{M_i \in \partial\omega} g(M_i)\phi_i;$$

$$p_h^* \in V_h^0 \text{ such that } \int_\omega \nabla p_h^* \cdot \nabla v_h \, d\omega = \int_\omega \nabla q_h \cdot \nabla v_h \, d\omega, \ \forall v_h \in V_h^0$$

$$\Big(\text{or, in matrix form, } \mathbb{K}\underline{p}^* = \underline{q}.\Big)$$

- Convergence results...

(1) $\forall \varepsilon > 0, \exists C_\varepsilon > 0$ such that $\|p_s - p_s^h\|_0 \le C_\varepsilon \, h^{2\alpha-\varepsilon}$.

Numerical Analysis (2)

- Numerical approximation of the *singular function* ϕ_s.

(2) $\phi_s^h = \tilde{\phi}_h + \beta_h \, \phi_P$, with β_h and $\tilde{\phi}_h = \phi_h^* - q_h'$ defined by

$$\beta_h = \frac{1}{\pi} \|p_s^h\|_0^2,$$

$$q_h' = Q_h(\phi_P) \text{ and}$$

$$\phi_h^* \in V_h^0 \text{ such that } \int_\omega \nabla \phi_h^* \cdot \nabla v_h \, d\omega = \int_\omega p_s^h v_h \, d\omega + \beta_h \int_\omega \nabla q_h' \cdot \nabla v_h \, d\omega, \; \forall v_h \in V_h^0$$

$$\left(\text{or, in matrix form, } \begin{pmatrix} \mathbb{K} & -\underline{q}' \\ 0 & \pi \end{pmatrix} \begin{pmatrix} \underline{\phi}^* \\ \beta_h \end{pmatrix} = \begin{pmatrix} \underline{l} \\ p \end{pmatrix} . \right)$$

- Convergence results...

(2) $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ such that $|\beta - \beta_h| \leq C_\varepsilon \, h^{2\alpha - \varepsilon}$

(2) $\exists C > 0$ such that $\|\phi_s - \phi_s^h\|_1 \leq C \, h$.

Numerical Analysis (3)

- Numerical approximation of the *solution* u.

  Write $u_h = \tilde{u}_h + c_h\,\phi_s^h$, with

(3) $c_h = \dfrac{\displaystyle\int_\omega f p_s^h \, d\omega}{\|p_s^h\|_0^2}$;

(4) $\tilde{u}_h$ is directly computable...

$$\tilde{u}_h \in V_h^0 \text{ such that } \int_\omega \nabla\tilde{u}_h \cdot \nabla v_h \, d\omega = \int_\omega f v_h \, d\omega - c_h \int_\omega p_s^h v_h \, d\omega, \ \forall v_h \in V_h^0$$

$$\left(\text{or, in matrix form, } \begin{pmatrix} \mathbb{K} & \underline{l} \\ 0 & p \end{pmatrix} \begin{pmatrix} \underline{\tilde{u}} \\ c_h \end{pmatrix} = \begin{pmatrix} \underline{f}_0 \\ f \end{pmatrix}.\right)$$

- Convergence results...

  (3) $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ such that $|c - c_h| \leq C_\varepsilon \, h^{2\alpha - \varepsilon}$

  (4) $\exists C > 0$ such that $\|\tilde{u} - \tilde{u}_h\|_1 \leq C\,h$.

Numerical Analysis (4)

Overall convergence rate and computational cost...

- *Convergence rate* $\exists C > 0$ such that $\|u - u_h\|_1 \leq C\,h$.
- *Total cost* three linear systems solves...

  Step (1) requires solving one linear system of order $N_i \times N_i$.

  Step (2) requires solving one linear system of order $(N_i + 1) \times (N_i + 1)$.

  Step (4) requires solving one linear system of order $(N_i + 1) \times (N_i + 1)$.

Numerical Analysis (5)

- Another approach...

  $u_h = \tilde{u}'_h + \lambda_h \, \phi_P$.

  (3b) $\lambda_h = \dfrac{1}{\pi} \displaystyle\int_\omega f p_s^h \, d\omega$;

  (4b) $\tilde{u}'_h = u_h^* - \lambda_h \, q'_h$, where $u_h^* = \tilde{u}_h + c_h \, \phi_h^*$...

  In matrix form,

$$\begin{pmatrix} \mathbb{K} & -\underline{q}' \\ 0 & \pi \end{pmatrix} \begin{pmatrix} \underline{u}^* \\ \lambda_h \end{pmatrix} = \begin{pmatrix} \underline{f}_0 \\ f \end{pmatrix}.$$

- *Convergence rate* $\exists C > 0$ such that $\|u - u_h\|_1 \leq C \, h$.
- *Total cost* two linear systems solves...

  Step (1) requires solving one linear system of order $N_i \times N_i$.

  Step (4b) requires solving one linear system of order $(N_i + 1) \times (N_i + 1)$.

IV. Numerical Experiments

With Jiwen He (University of Houston)

1. Cheese Case

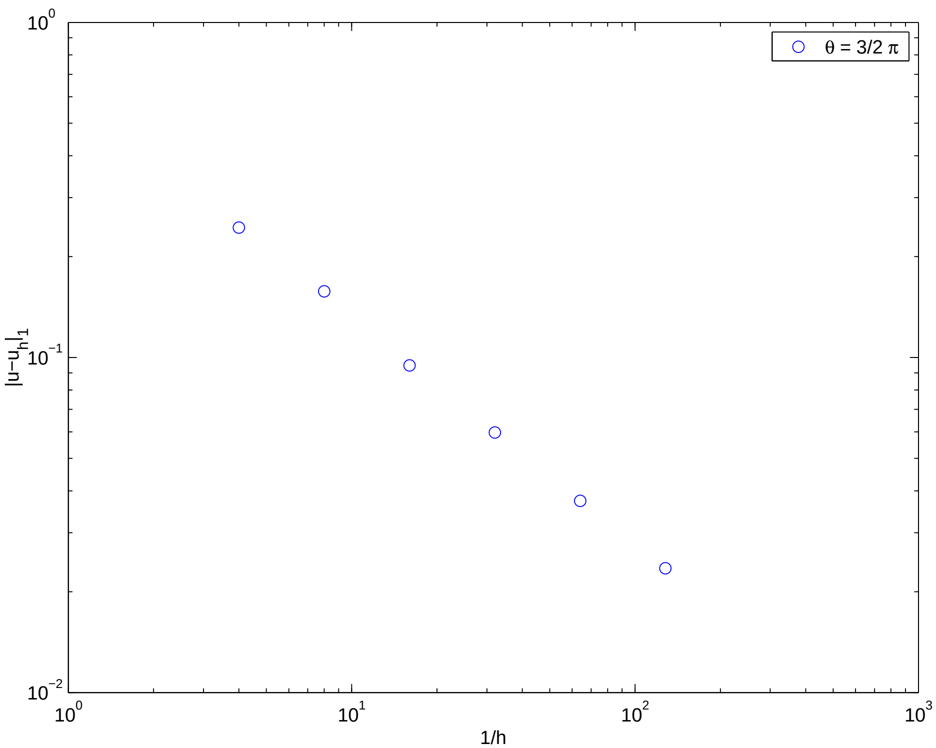
$$\omega_\alpha = \{(r,\theta)\ :\ r \in]0,1[,\ \theta \in]0,\Theta_\alpha[\}$$

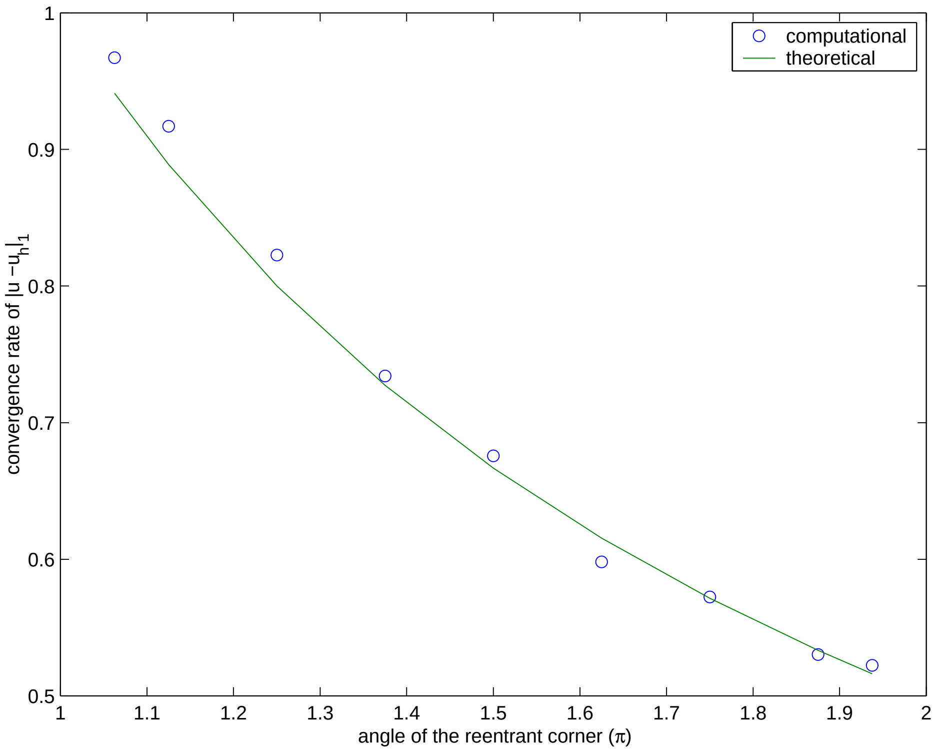
Exact solution $u = r^\alpha \sin(\alpha\theta)$ in ω_α

α vary in $]\frac{1}{2},1[$

$\alpha = \frac{2}{3}$ ($\Theta_\alpha = \frac{3}{2}\pi$)

$\|u - x_h\|_1 < C_\epsilon h^{\alpha-\epsilon}$ (no SCM)

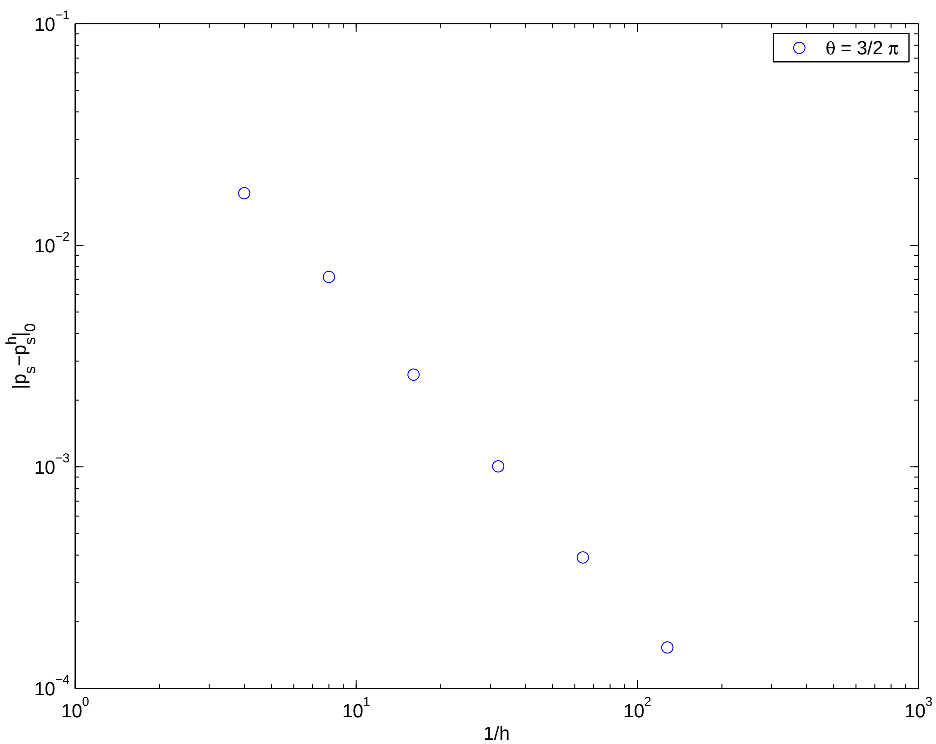


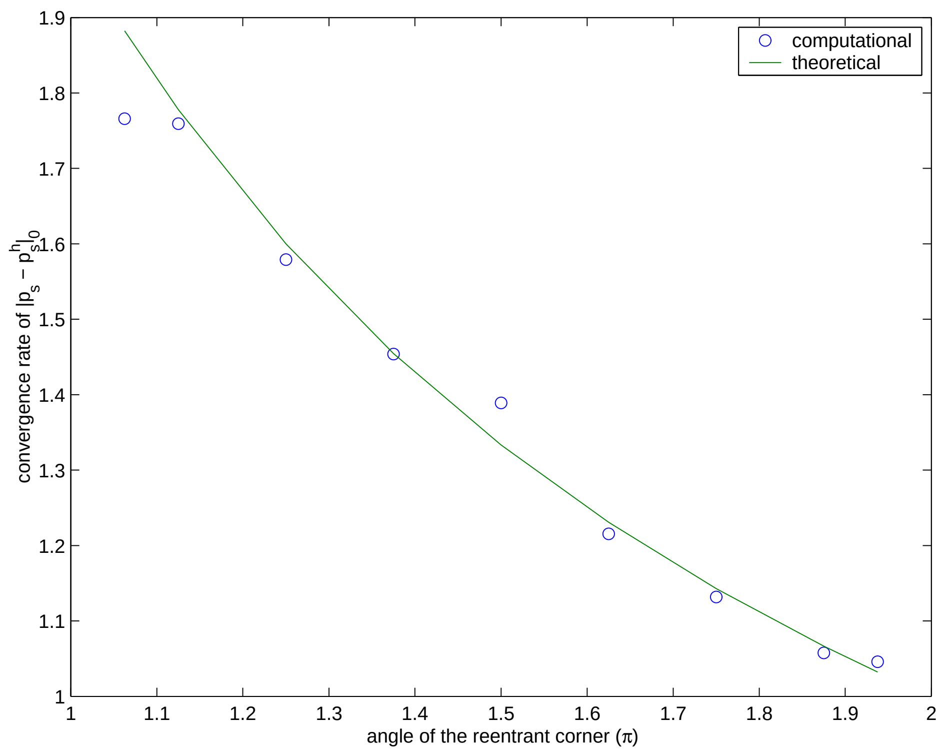


$\Theta_\alpha = 3/2\pi$

$\Theta_\alpha \in]\pi, 2\pi[$

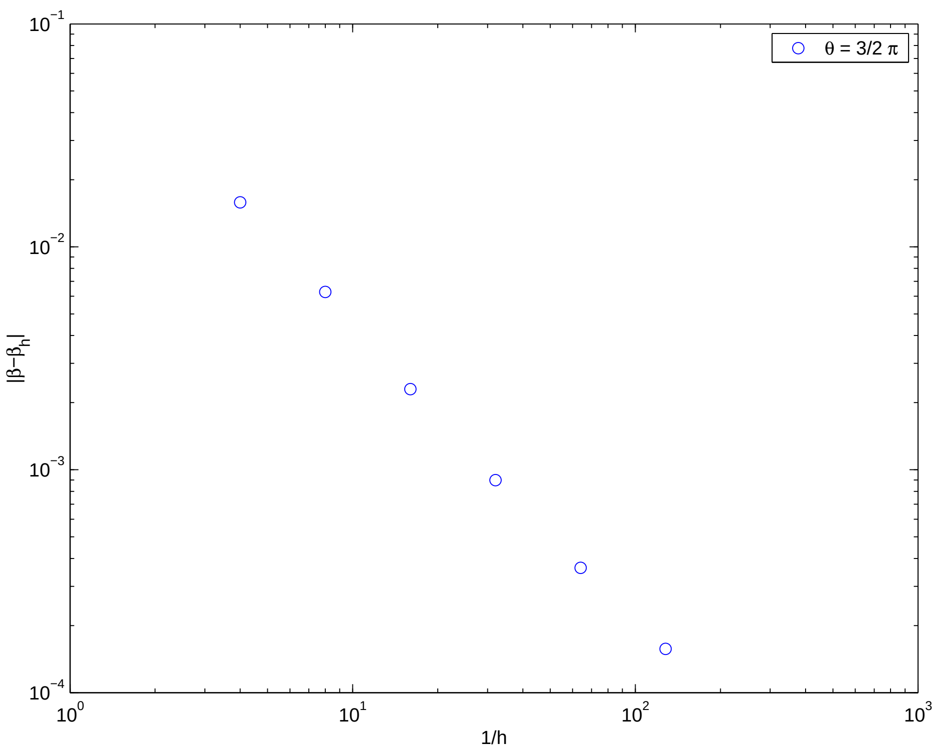
$$\|p_s - p_s^h\|_0 < C_\epsilon h^{2\alpha-\epsilon}$$

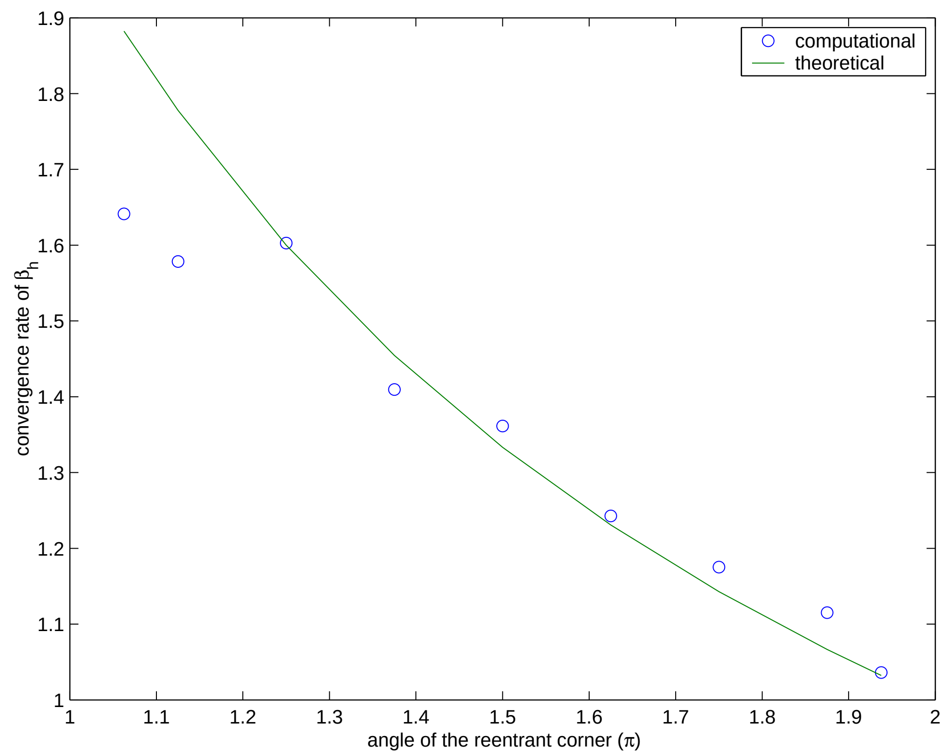




$$\Theta_\alpha = 3/2\pi \qquad \Theta_\alpha \in]\pi, 2\pi[$$

$$|\beta - \beta_h| < C_\epsilon h^{2\alpha - \epsilon}$$

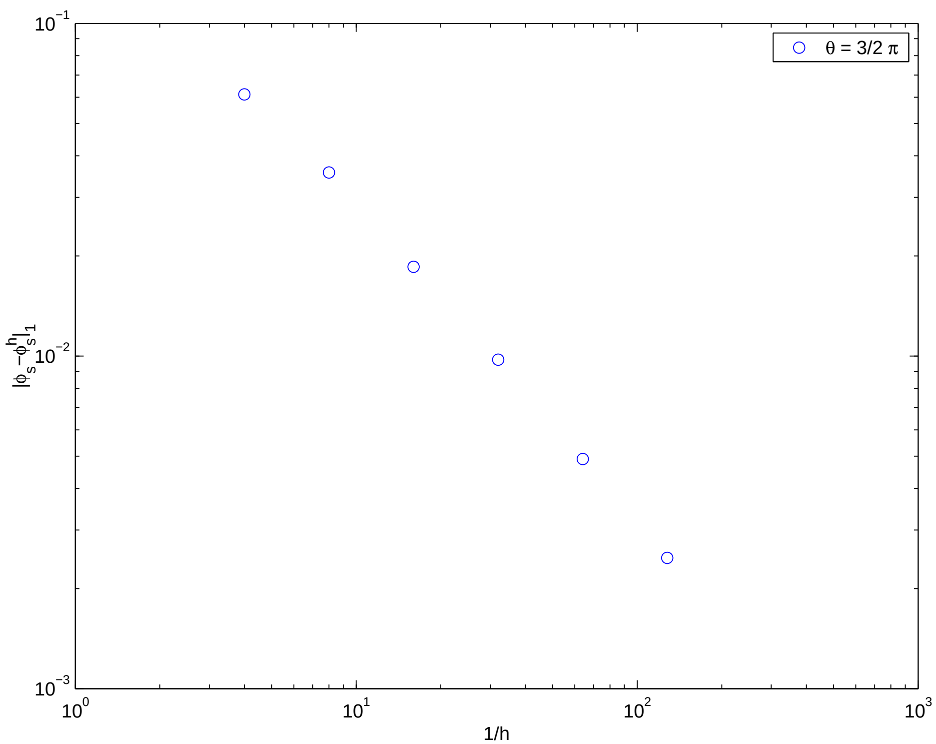


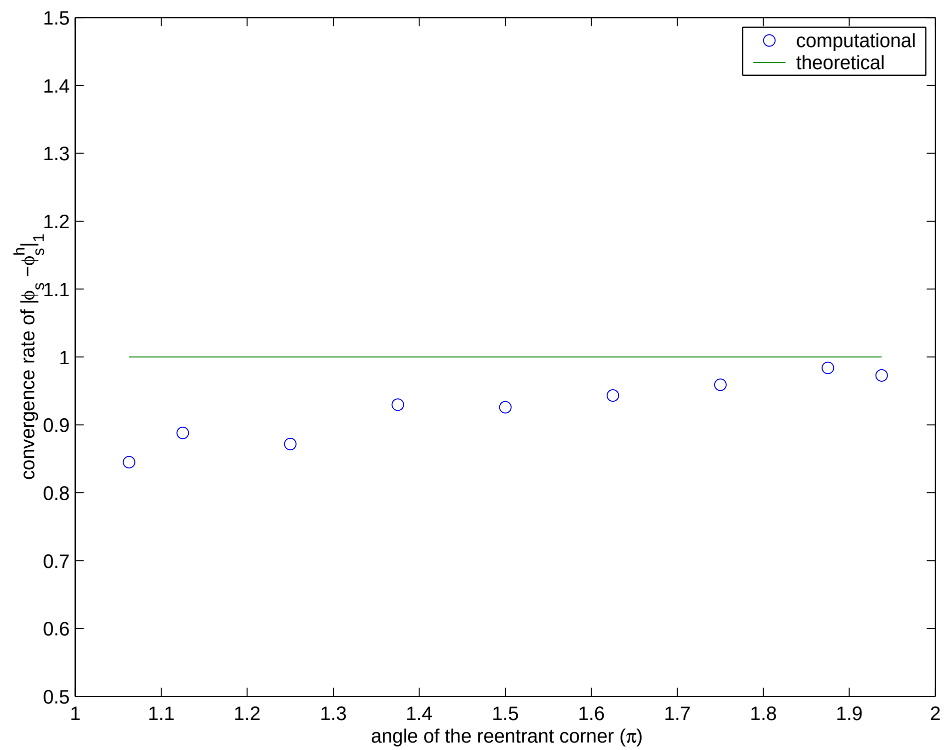


$$\Theta_\alpha = 3/2\pi$$

$$\Theta_\alpha \in]\pi, 2\pi[$$

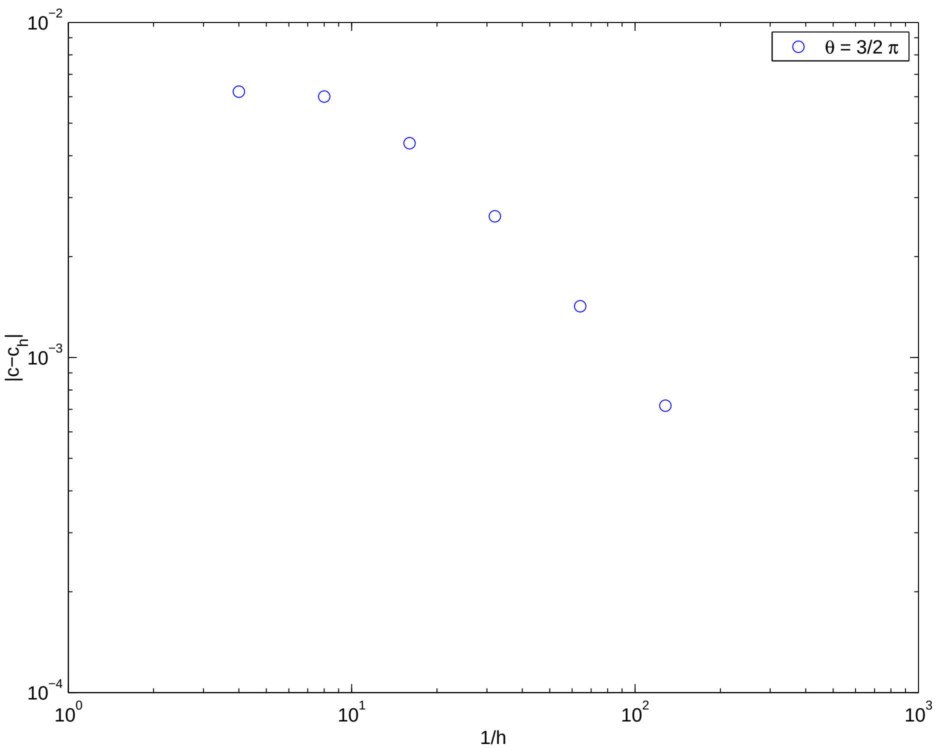
$$\|\phi_s - \phi_s^h\|_1 < Ch$$

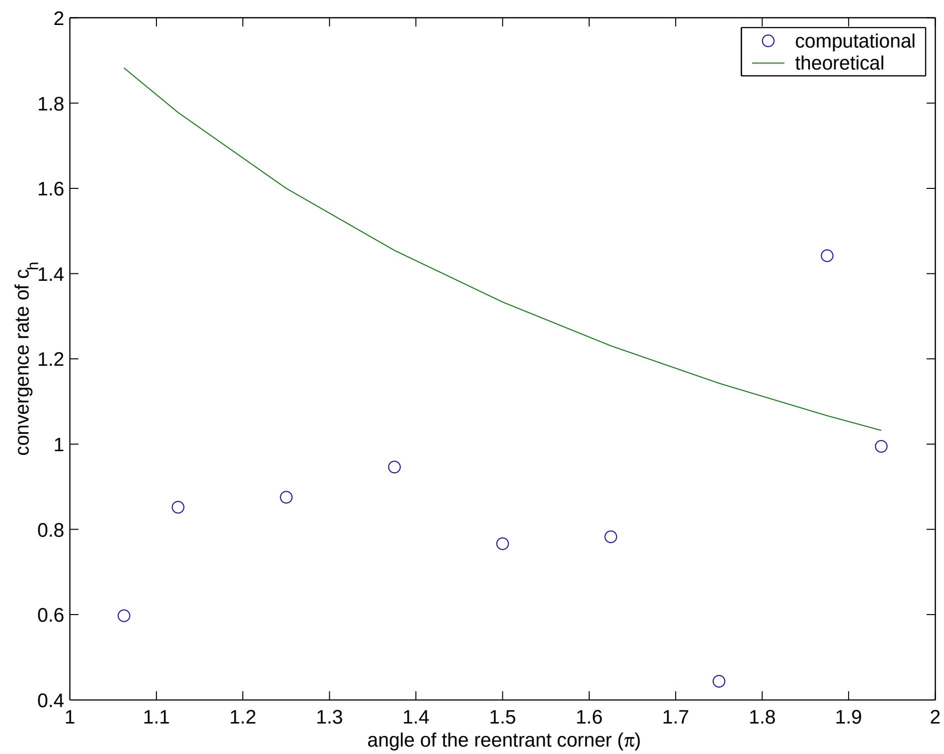




$$\Theta_\alpha = 3/2\pi \qquad \Theta_\alpha \in]\pi, 2\pi[$$

$$|c - c_h| < C_\epsilon h^{2\alpha - \epsilon}$$

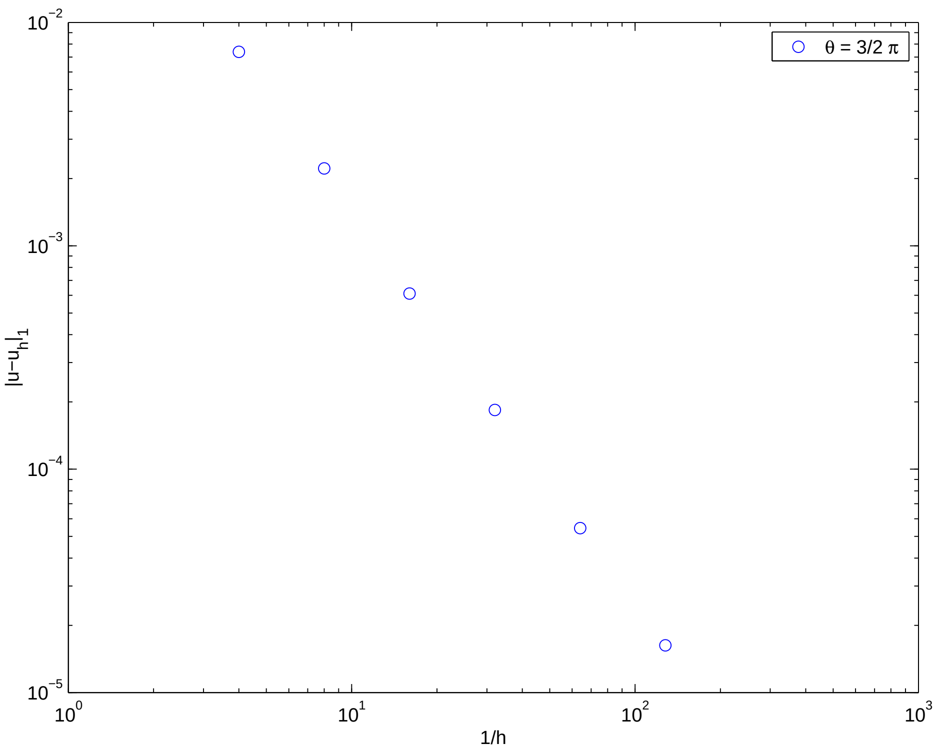


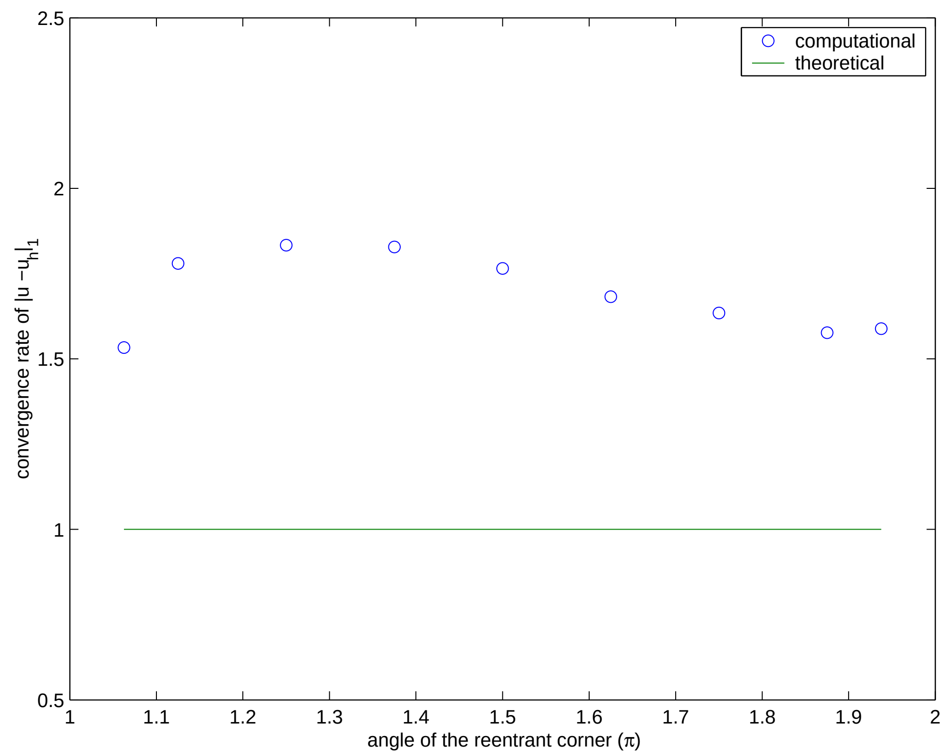


$$\Theta_\alpha = 3/2\pi$$

$$\Theta_\alpha \in]\pi, 2\pi[$$

$$\|u - u_h\|_1 < Ch$$

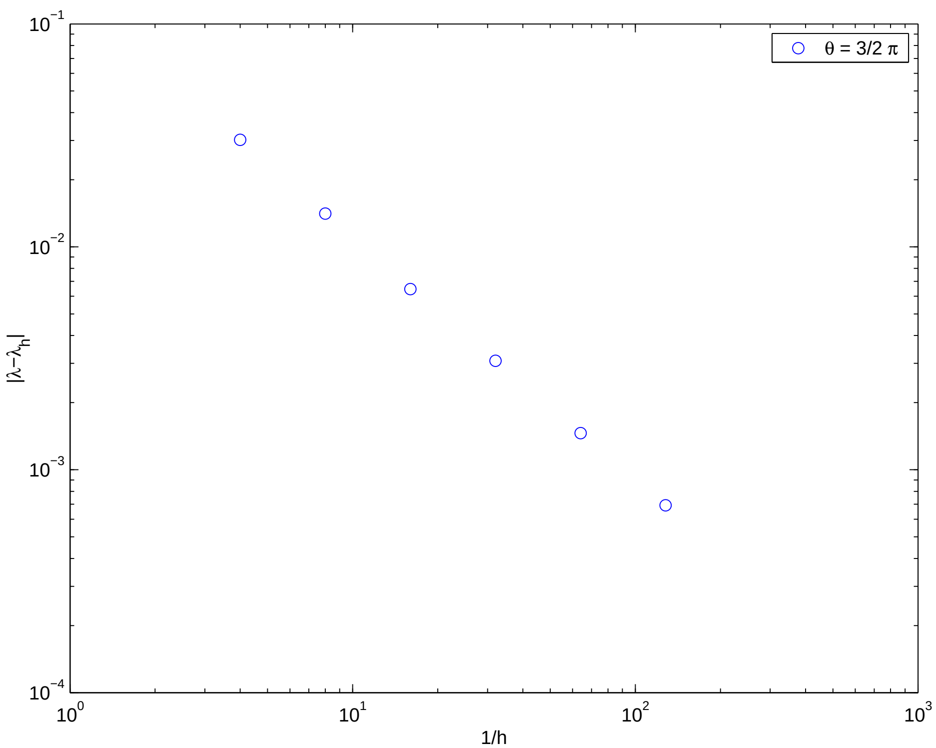


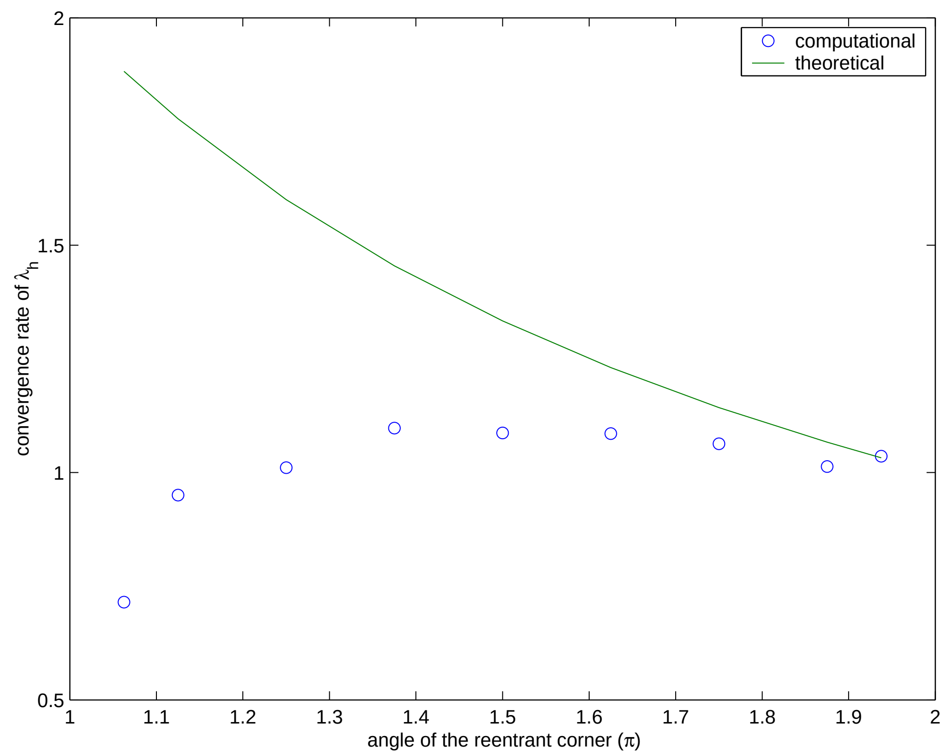


$\Theta_\alpha = 3/2\pi$

$\Theta_\alpha \in]\pi, 2\pi[$

$$|\lambda - \lambda_h| < C_\epsilon h^{2\alpha-\epsilon}$$

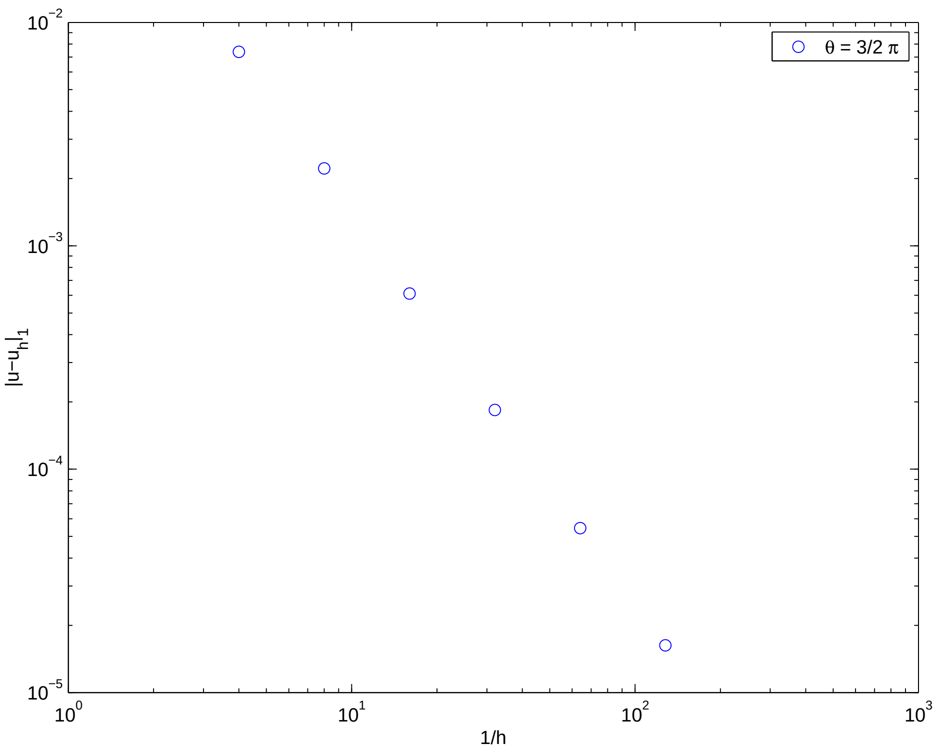


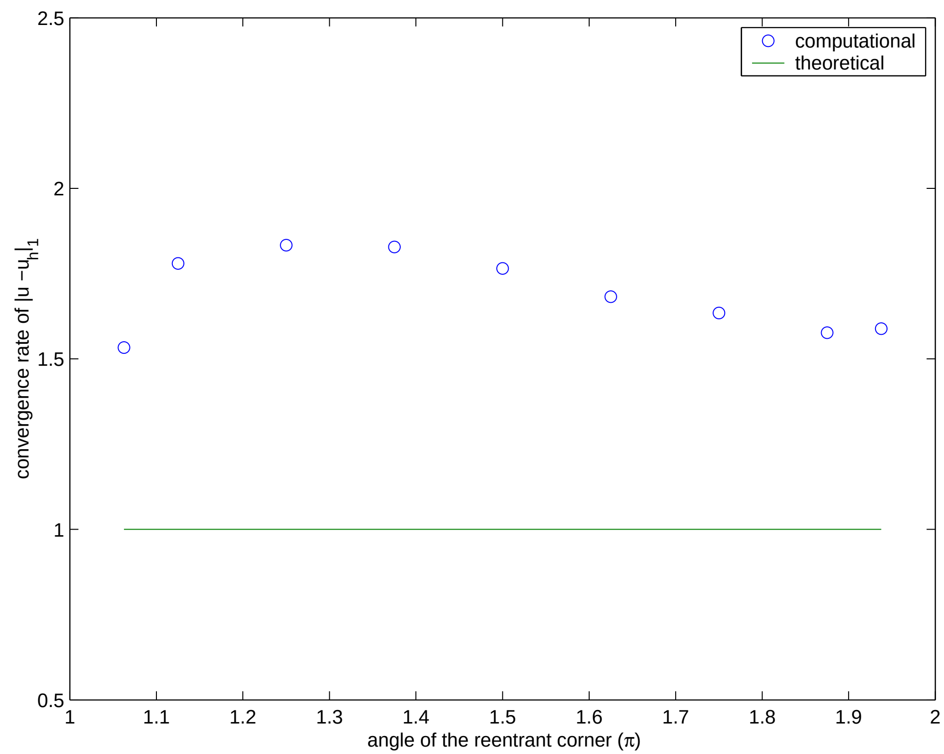


$\Theta_\alpha = 3/2\pi$

$\Theta_\alpha \in]\pi, 2\pi[$

$\|u - u_h\|_1 < Ch$ (λ approach)

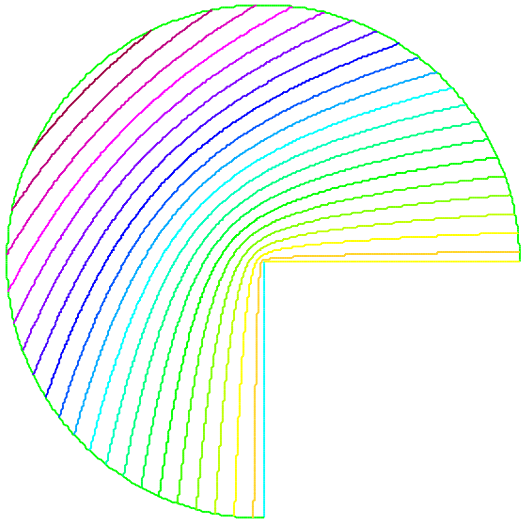




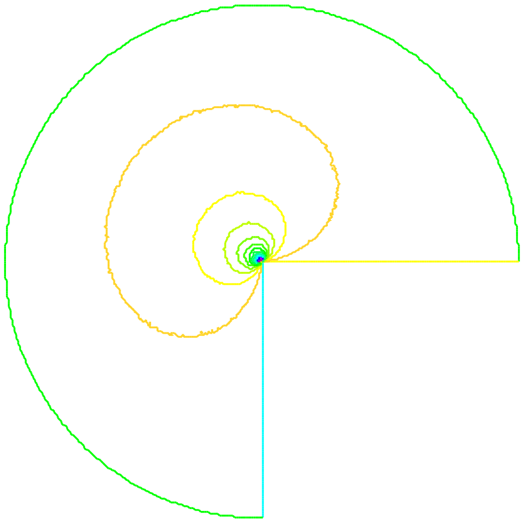
$\Theta_\alpha = 3/2\pi$

$\Theta_\alpha \in]\pi, 2\pi[$

$u_h - x_h$ (Differences...)

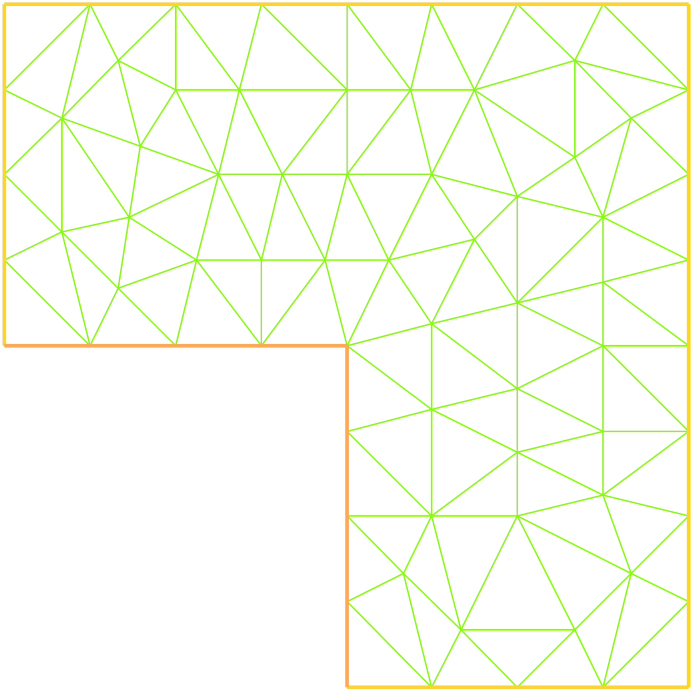


u_h



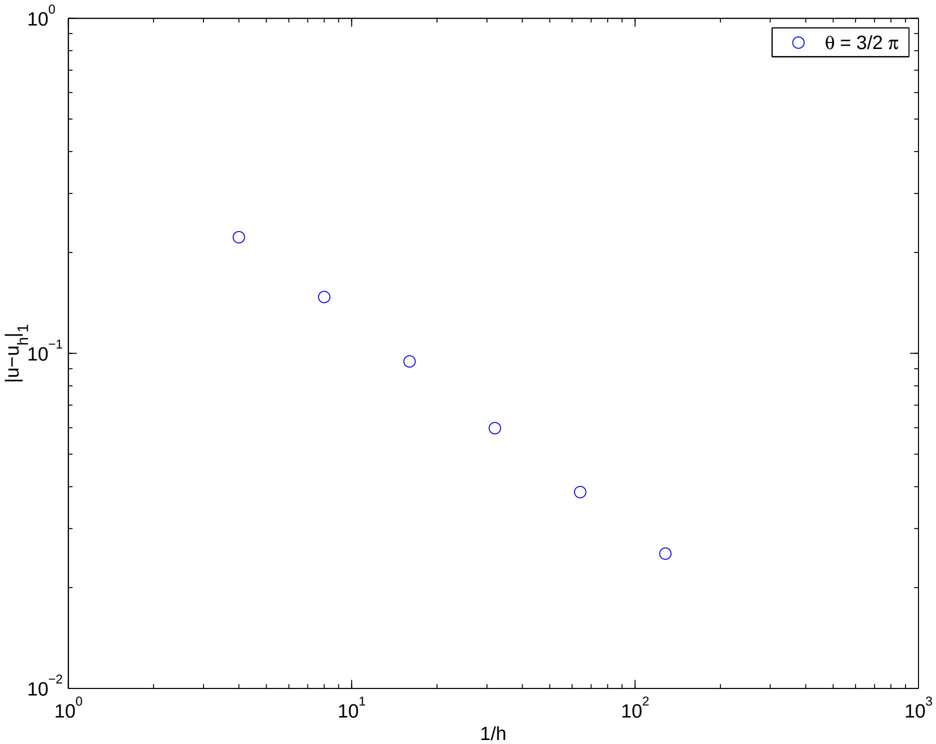
$x_h - u_h$

2. L-shaped domain (analytical solution)

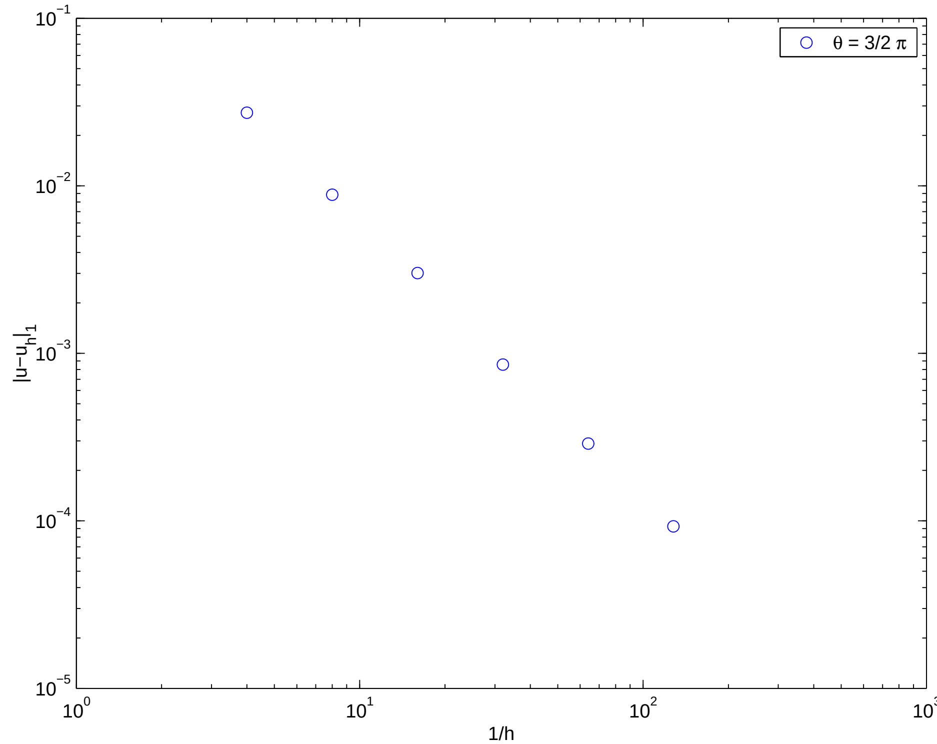


Exact solution

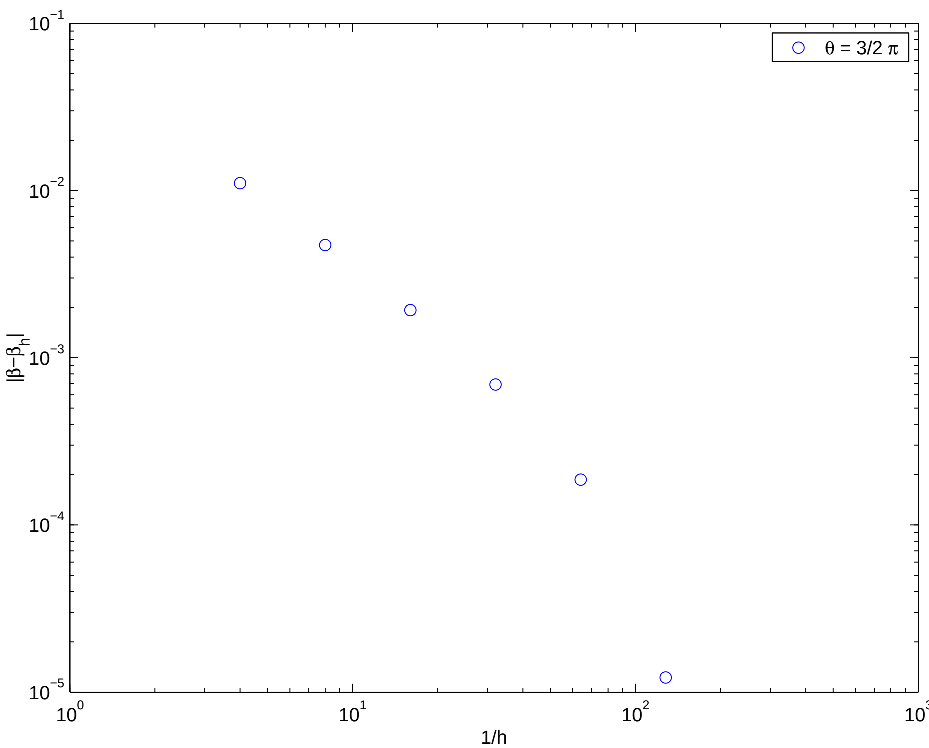
$u = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$ in ω



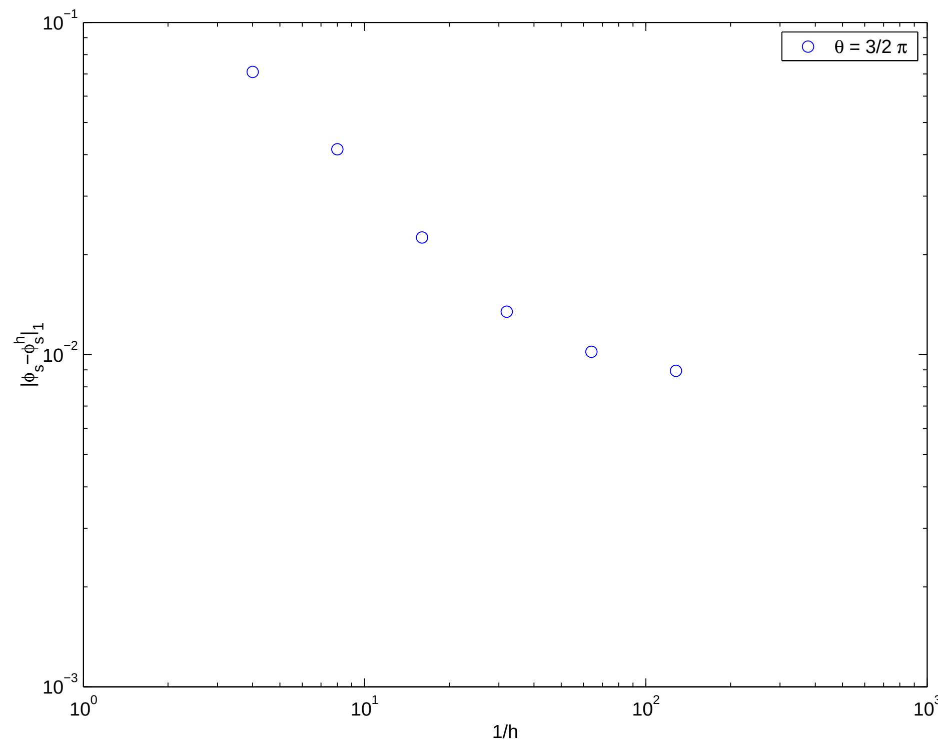
$$\|u - x_h\|_1$$



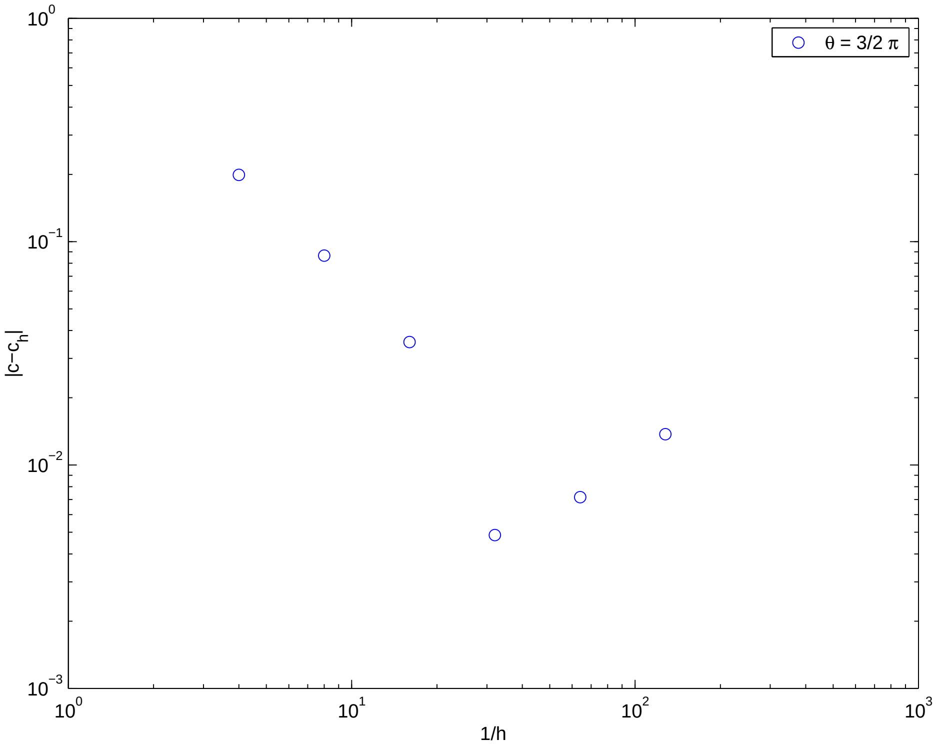
$$\|u - u_h\|_1$$

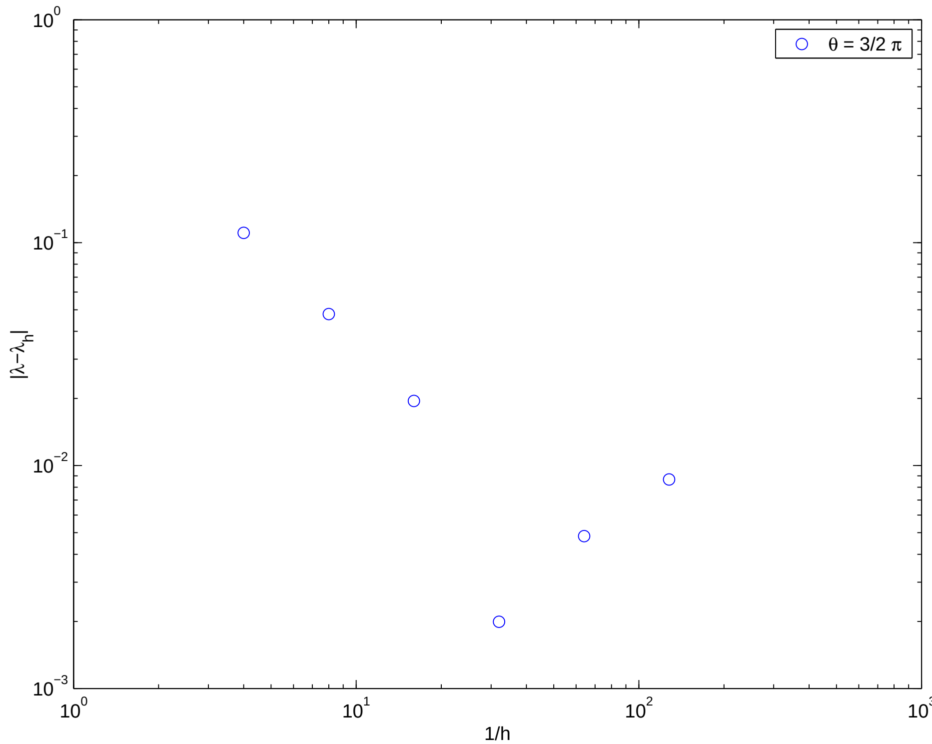


$$|\beta - \beta_h|$$



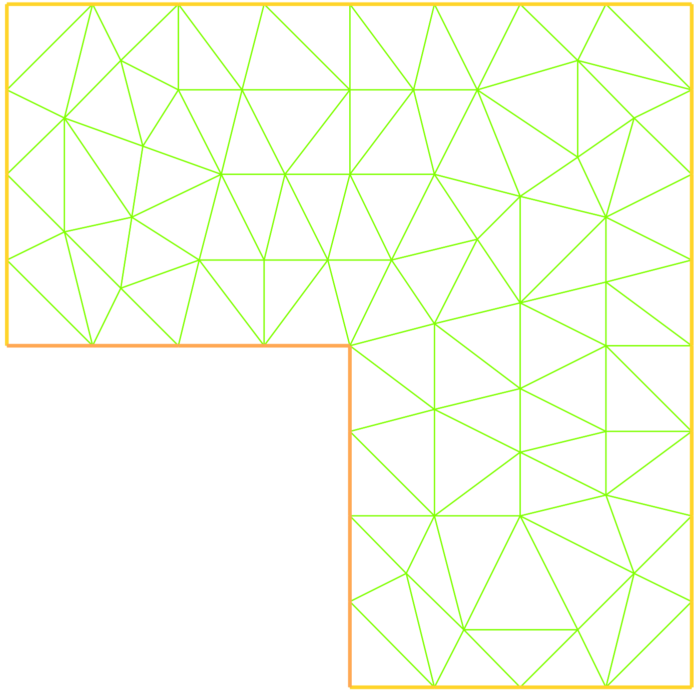
$$\|\phi_s - \phi_s^h\|_1$$





$|c - c_h|$ $|\lambda - \lambda_h|$

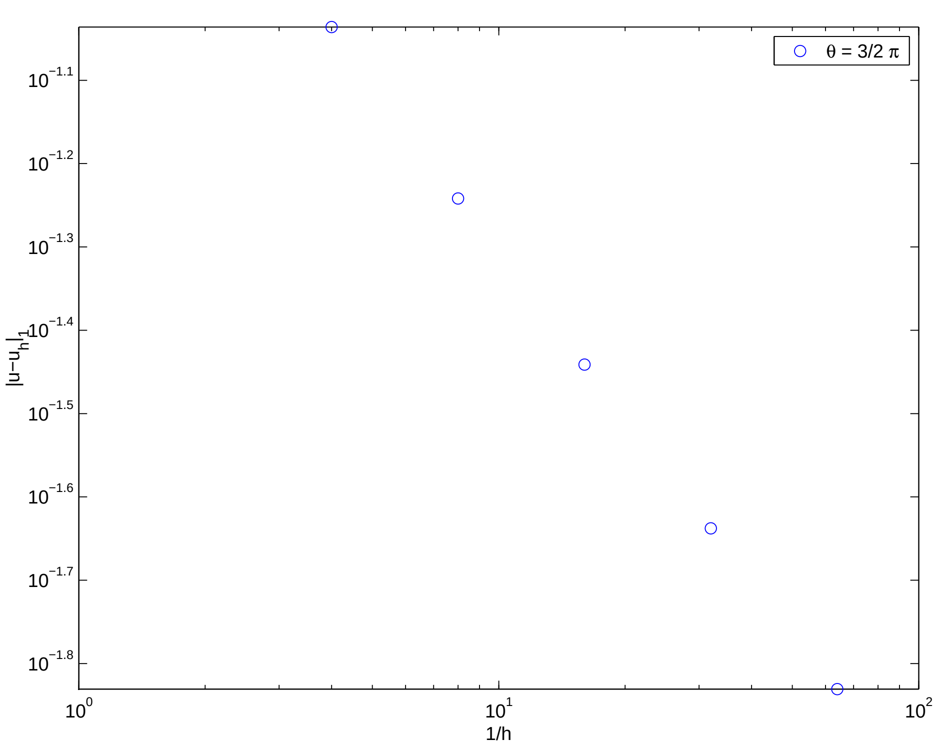
2. L-shaped domain (general)



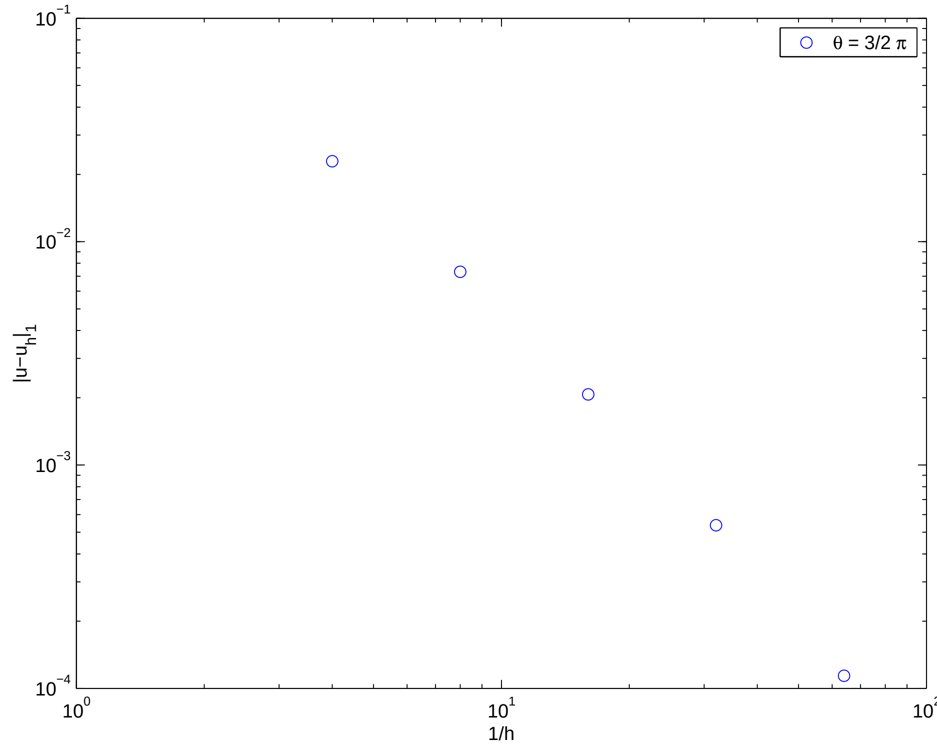
Unknown solution

$$-\Delta u = 1 \text{ in } \omega$$

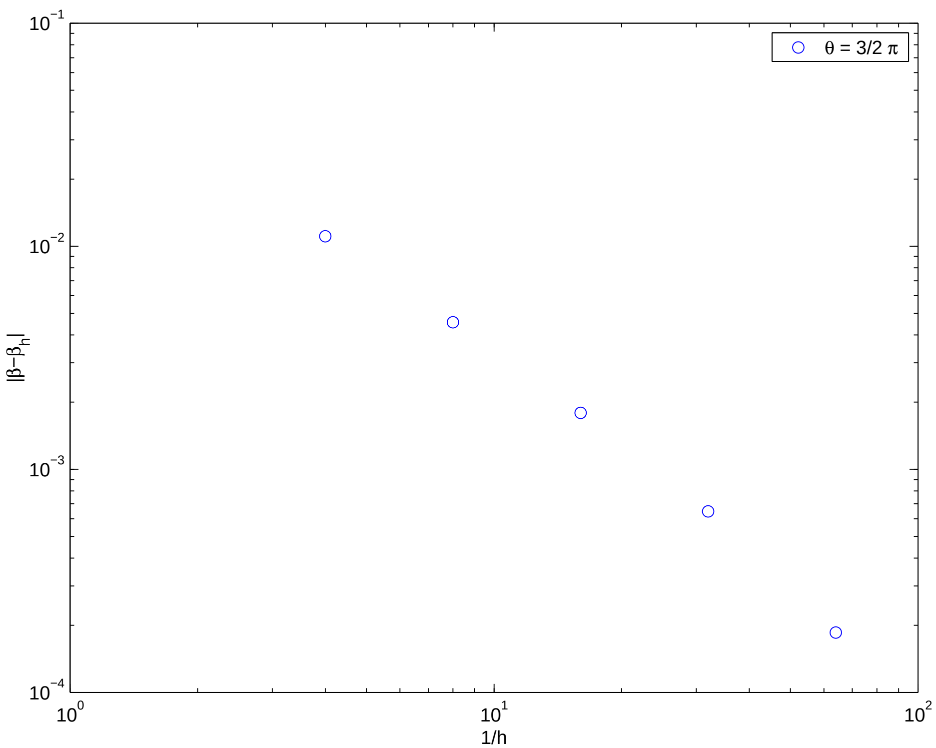
$$u_{|\partial\omega} = 0$$



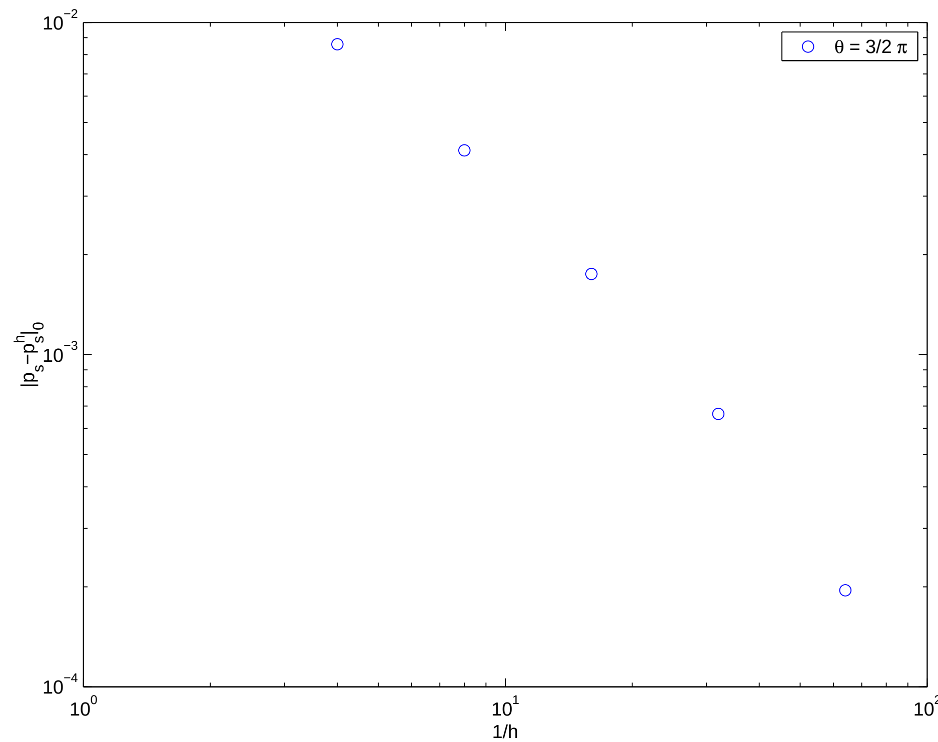
$$\|u^\dagger - x_h\|_1$$



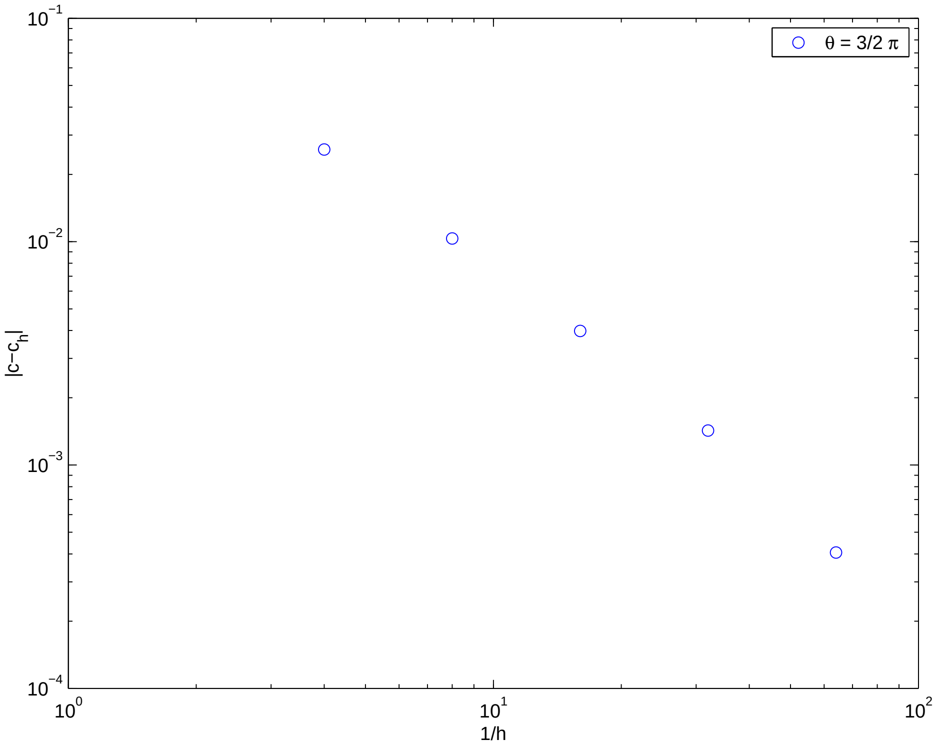
$$\|u^\dagger - u_h\|_1$$



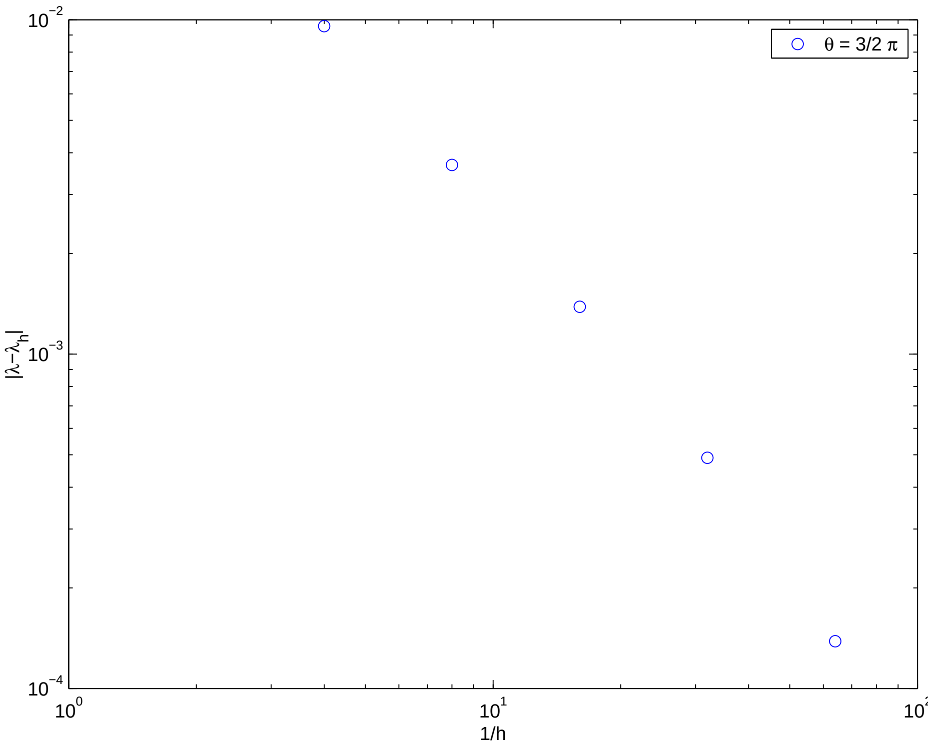
$$|\beta^{\dagger} - \beta_h|$$



$$\|\phi_s^{\dagger} - \phi_s^h\|_1$$

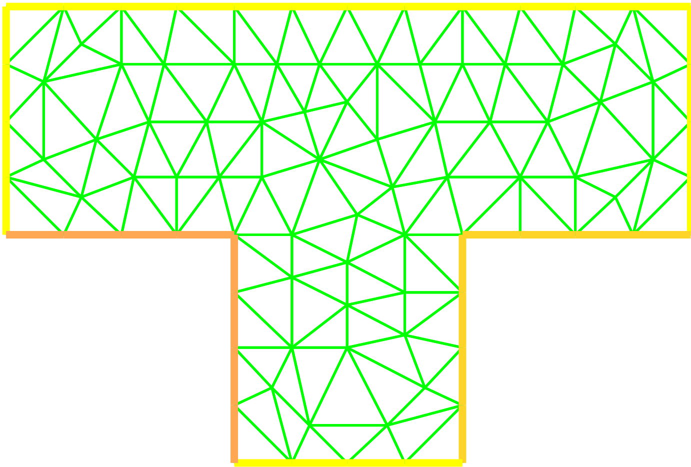


$$|c^\dagger - c_h|$$



$$|\lambda^\dagger - \lambda_h|$$

3. T-shaped domain (general)

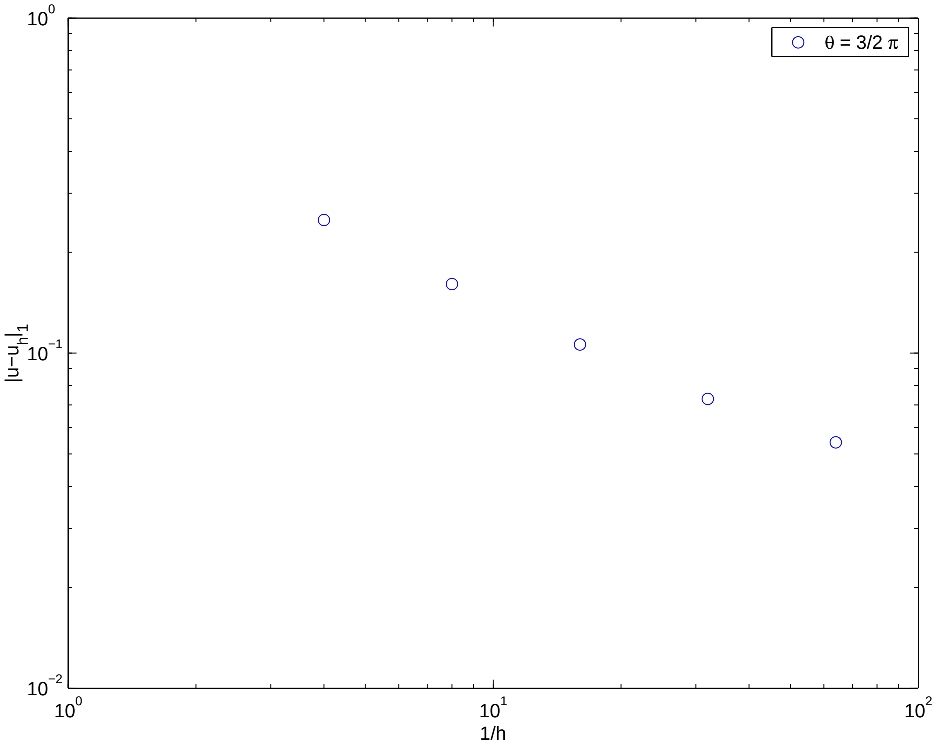


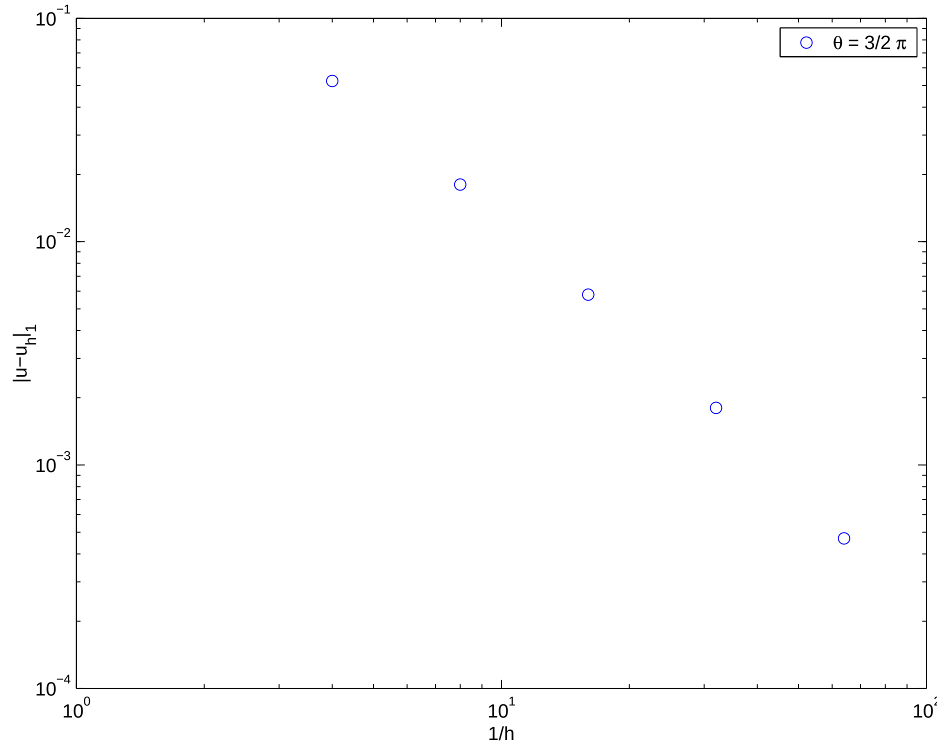
Unknown solution

$$-\Delta u = 1 \text{ in } \omega$$

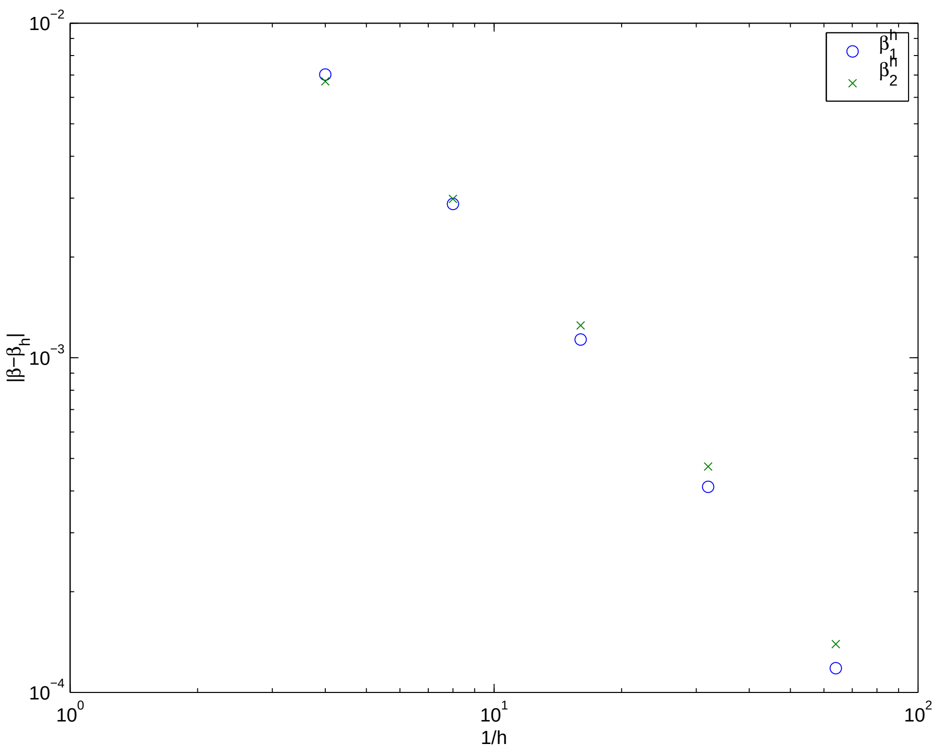
$$u_{|\partial\omega} = 0$$

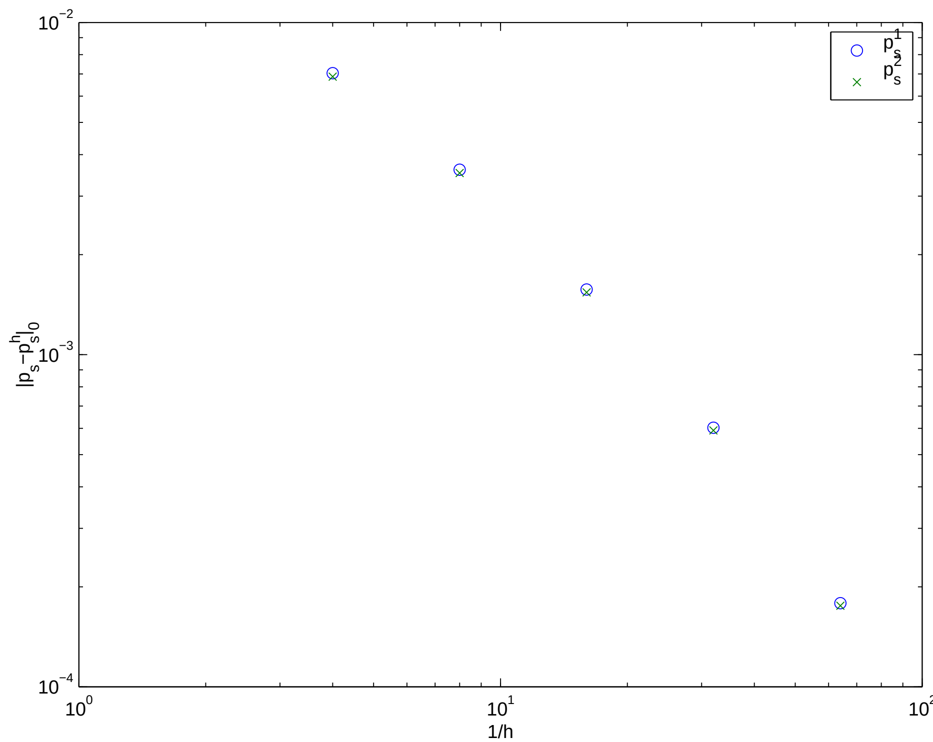
$$\alpha_1 = \alpha_2 = \frac{2}{3}$$



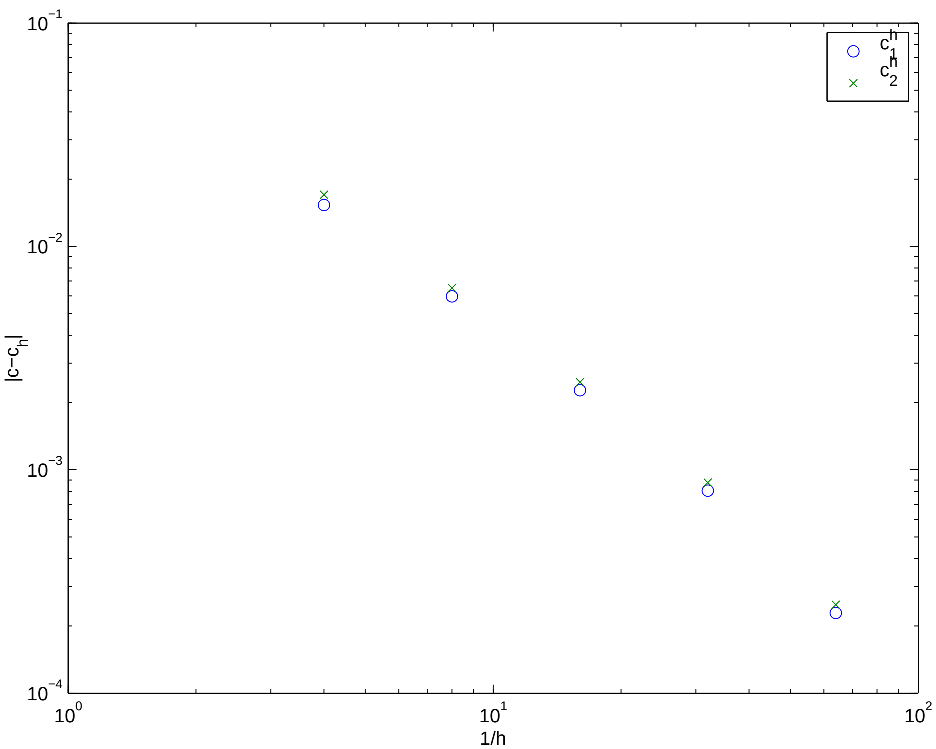


$$\|u^\dagger - x_h\|_1 \qquad \|u^\dagger - u_h\|_1$$

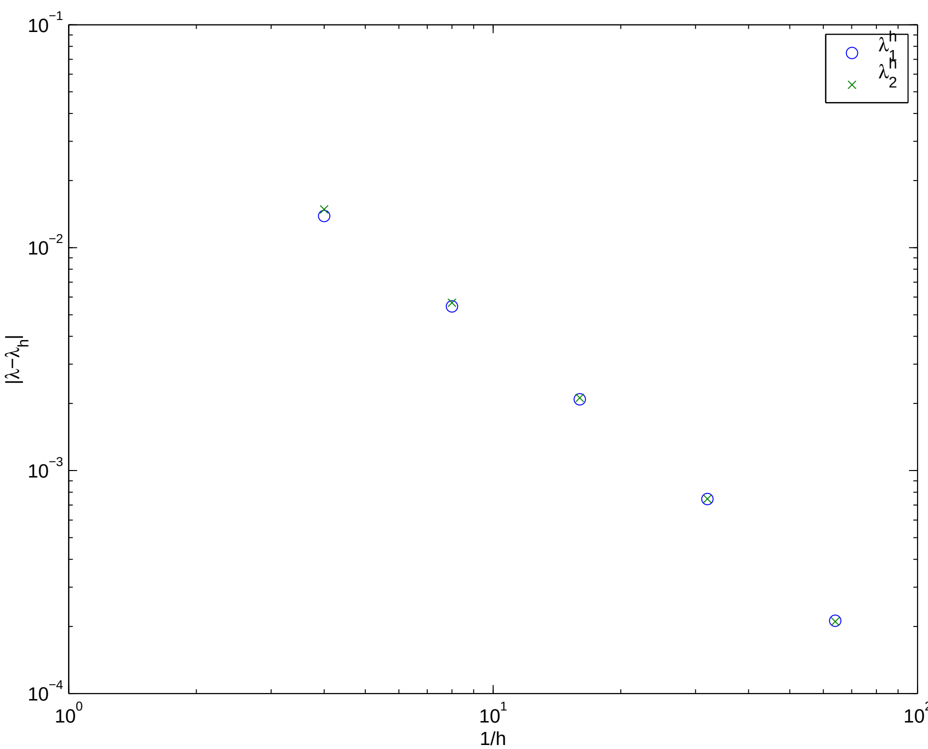




$$|\beta^{j,\dagger} - \beta_h^j| \ (j = 1, 2) \qquad \|\phi_s^{j,\dagger} - \phi_s^{j,h}\|_1 \ (j = 1, 2)$$



$$|c^{j,\dagger} - c_h^j| \; (j = 1, 2)$$



$$|\lambda^{j,\dagger} - \lambda_h^j| \; (j = 1, 2)$$

V. Conclusion

Extensions & Perspectives

- (Algorithmic) applications to the vector problems.

- Neumann problem.
- Heterogeneous b.c.
- Several corners.
- Problems with jumps in the coefficients.
- Wave equation.

- 3d domains with conical points.
- 3d prismatic or axisymmetric domains (work in progress).