Workshop Mathematical methods for spectral problems:
Applications to waveguides, periodic media and metamaterials

A bilaplacian problem with a sign-changing coefficient

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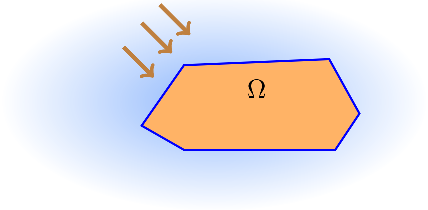
Introduction: the ITEP

- Scattering in time-harmonic regime by a penetrable inclusion Ω (coefficient n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.



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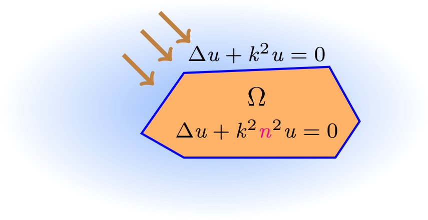

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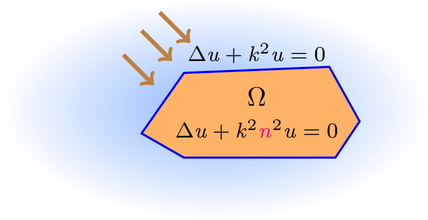

- This leads to study the Interior Transmission Eigenvalue Problem:
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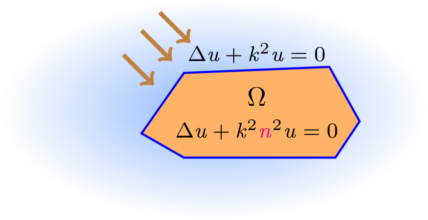

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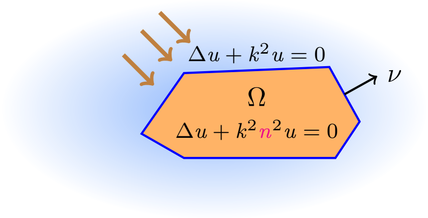

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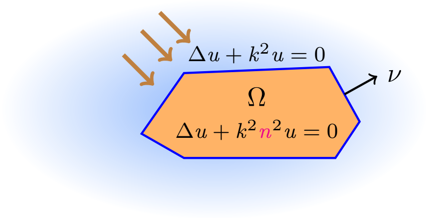

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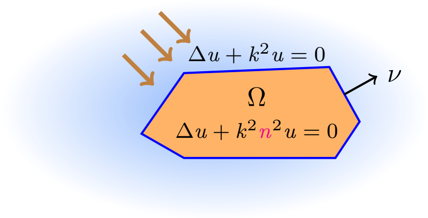

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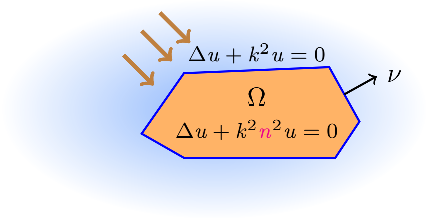

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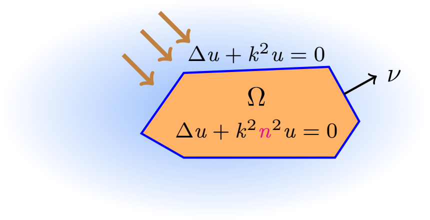

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DEFINITION. Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution (u, w) are called transmission eigenvalues.

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- We deduce $\Delta v + k^2 n^2 v = k^2(1 - n^2)w$ in Ω.
- This implies

$$\left| \begin{array}{ll} (\Delta + k^2)\left(\frac{1}{1-n^2}(\Delta v + k^2 n^2 v)\right) = 0 & \text{in } \Omega \\ v = \nu \cdot \nabla v = 0 & \text{on } \partial\Omega. \end{array} \right.$$

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- Introducing $v = u - w$ the scattered field inside Ω, we can write an equivalent formulation:

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What happens when $1 - n^2$ changes sign?

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

$$\left| \begin{array}{l} \text{Find } v \in \mathrm{H}_0^2(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma \Delta v \Delta v' = \langle f, v' \rangle_\Omega, \quad \forall v' \in \mathrm{H}_0^2(\Omega). \end{array} \right.$$

Outline of the talk

- ... and more generally, we study the problem:

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We study $(\mathscr{P})$ with $\mathbf{X} = \mathbf{H}^1_0(\boldsymbol{\Delta}) := \{v \in \mathbf{H}^1_0(\Omega) \,|\, \boldsymbol{\Delta} v \in \mathbf{L}^2(\Omega)\}$.

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3. A bilaplacian problem with Dirichlet boundary conditions

   We study $(\mathscr{P})$ with $\mathbf{X} = \mathbf{H}^2_0(\boldsymbol{\Omega})$.

Reminder: properties of div $(\sigma\nabla\cdot)$

- In the fields of plasmonic and negative metamaterials, we study:

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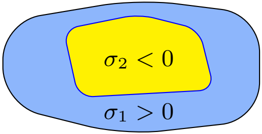
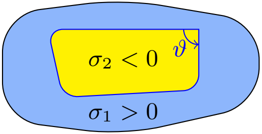
Smooth interface



✔ $(\mathscr{F})$ well-posed in the Fredholm sense iff $\kappa_\sigma = \sigma_2/\sigma_1 \neq -1$.

Interface with a corner



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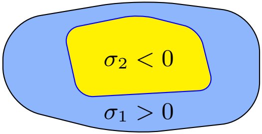

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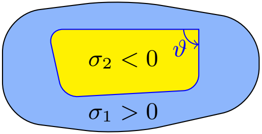

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Well-posedness depends on the smoothness of the interface and on σ (c.f. talks given by X. Claeys and A.-S. Bonnet-Ben Dhia).

Mixed Boundary Conditions I

Let $\mathtt{T}$ be an isomorphism of X.

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In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}^{\mathtt{T}})$ (and so $(\mathscr{P})$) well-posed.

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In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}^{\mathtt{T}})$ (and so $(\mathscr{P})$) well-posed.

In this section, $\mathrm{X} = \mathrm{H}^1_0(\Delta)$.

1. Define $\mathtt{T}u \in \mathrm{H}^1_0(\Omega)$ the function such that $\Delta(\mathtt{T}u) = \sigma^{-1}\Delta u$.
2. $\mathtt{T}$ is an isomorphism of $\mathrm{H}^1_0(\Delta)$.
3. One obtains $a(u, \mathtt{T}u) = \displaystyle\int_\Omega \sigma\, \Delta u \Delta(\mathtt{T}u) = \|\Delta u\|^2_\Omega$.

Theorem. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Then, the operator $A : \mathrm{H}^1_0(\Delta) \to \mathrm{H}^1_0(\Delta)$ associated with $(\mathscr{P})$ is an isomorphism.

Mixed Boundary Conditions I

Let $\mathtt{T}$ be an isomorphism of X.

$(\mathscr{P}) \;\Leftrightarrow (\mathscr{P}^{\mathtt{T}})$ | Find $u \in \mathrm{X}$ such that:
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The change of sign of σ is not a problem!

Theorem. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Then, the operator $A : \mathrm{H}^1_0(\Delta) \to \mathrm{H}^1_0(\Delta)$ associated with $(\mathscr{P})$ is an isomorphism.

Mixed Boundary Conditions II

In this section, $X = H_0^1(\Omega) \cap H^2(\Omega)$.

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \cap H^2(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma \, \Delta u \Delta v = l(v), \; \forall v \in H_0^1(\Omega) \cap H^2(\Omega). \end{array} \right.$$

Mixed Boundary Conditions II

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2. Assume that Ω is convex or of class $\mathscr{C}^2$.

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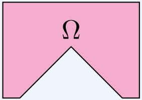
What happens if Ω has a reentrant corner ?

Polygonal $\partial\Omega$ with one reentrant corner



Reminder

i) The space of functions $\psi \in \mathrm{L}^2(\Omega)$ s.t $\Delta\psi = 0$ in Ω and $\psi = 0$ on $\partial\Omega$, is of dimension 1, spanned by some ζ.

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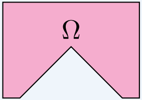

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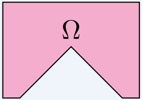

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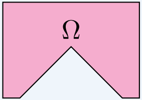

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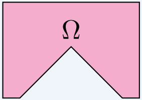

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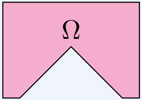
Polygonal $\partial\Omega$ with one reentrant corner



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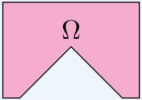

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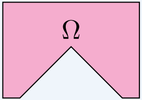

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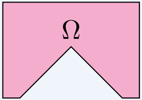

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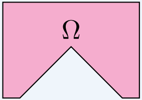

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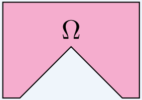

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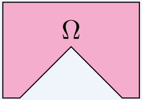
2. One can prove that $\mathtt{T}$ is an isomorphism of $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$.

Polygonal $\partial\Omega$ with one reentrant corner

REMINDER



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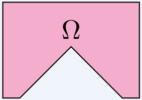

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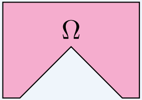

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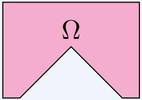

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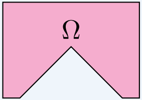
THEOREM. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Introduce $A : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ the operator associated with $(\mathscr{P})$.

- If $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$, then A is an isomorphism.

Polygonal $\partial\Omega$ with one reentrant corner

Reminder



i) The space of functions $\psi \in L^2(\Omega)$ s.t $\Delta\psi = 0$ in Ω and $\psi = 0$ on $\partial\Omega$, is of dimension 1, spanned by some ζ.

ii) $\varphi \in H^1_0(\Omega)$ s.t. $\Delta\varphi \in L^2(\Omega)$ is in $H^2(\Omega)$ iff $(\Delta\varphi, \zeta)_\Omega = 0$.

1. Define $\mathtt{T}u \in H^1_0(\Omega)$ the function such that $\Delta(\mathtt{T}u) = \sigma^{-1}(\Delta u - a\zeta)$ with $a = (\sigma^{-1}\Delta u, \zeta)_\Omega / (\sigma^{-1}\zeta, \zeta)_\Omega$. (c.f. Sapongyan paradox S.A. Nazarov, G. Sweers)

2. One can prove that $\mathtt{T}$ is an isomorphism of $H^1_0(\Omega) \cap H^2(\Omega)$.

3. One obtains $a(u, \mathtt{T}u) = \int_\Omega \sigma\, \Delta u \Delta(\mathtt{T}u) = \int_\Omega \Delta u(\Delta u - a\zeta) = \|\Delta u\|^2_\Omega$.

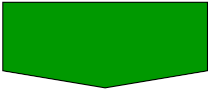
Theorem. Assume that $\sigma \in L^\infty(\Omega)$ is such that $\sigma^{-1} \in L^\infty(\Omega)$. Introduce $A : H^1_0(\Omega) \cap H^2(\Omega) \to H^1_0(\Omega) \cap H^2(\Omega)$ the operator associated with $(\mathscr{P})$.

- If $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$, then A is an isomorphism.
- If $(\sigma^{-1}\zeta, \zeta)_\Omega = 0$, then A is Fredholm of index zero and $\dim\ker A = 1$.

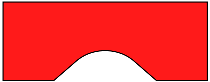
Summary of the results when $\mathrm{X} = \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma \, \Delta u \Delta v = l(v), \; \forall v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega). \end{array} \right.$$

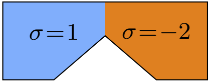
- We introduce the operator $A : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ such that $(\Delta(Au), \Delta v)_\Omega = (\sigma \Delta u, \Delta v)_\Omega$ for all $u,\ v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$.



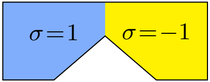
✔ A is an isomorphism.



✔ A is an isomorphism.



✔ A is an isomorphism because $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$.



✔ A is a Fredholm operator of index 0 and $\dim \ker A = 1$ because $(\sigma^{-1}\zeta, \zeta)_\Omega = 0$.

A bilaplacian problem with Dirichlet boundary conditions

- In this section, $\mathrm{X} = \mathrm{H}^2_0(\Omega)$.

$$
(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma\, \Delta u \Delta v = l(v), \quad \forall v \in \mathrm{H}^2_0(\Omega). \end{array} \right.
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Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\operatorname{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

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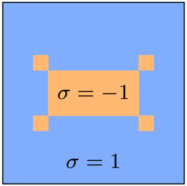
Theorem. The problem $(\mathscr{P})$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



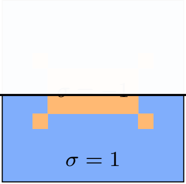
A bilaplacian problem with Dirichlet boundary conditions

Ideas of the proof: We have

$$a(v,v) = (\sigma \Delta v, \Delta v)_\Omega.$$

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v$

so that $$a(v, \mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$



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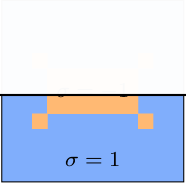

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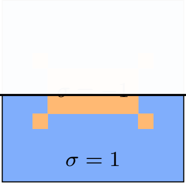

A bilaplacian problem with Dirichlet boundary conditions

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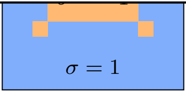

A bilaplacian problem with Dirichlet boundary conditions

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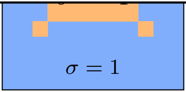
3. We find $a(v, \mathtt{T}v) = ([\zeta + \sigma(1-\zeta)]\Delta v, \Delta v)_\Omega + (Kv, v)_{\mathrm{H}^2_0(\Omega)}$

   where $K : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is compact.



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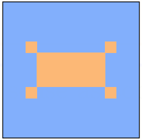

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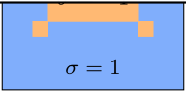
Fredholm



A bilaplacian problem with Dirichlet boundary conditions

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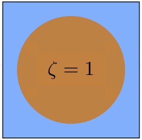
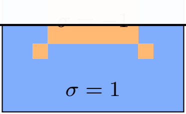




A bilaplacian problem with Dirichlet boundary conditions

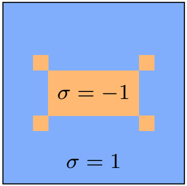
- In this section, $\mathrm{X} = \mathrm{H}^2_0(\Omega)$.

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma\, \Delta u \Delta v = l(v), \quad \forall v \in \mathrm{H}^2_0(\Omega). \end{array} \right.$$

Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\mathrm{div}\,(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

THEOREM. The problem $(\mathscr{P})$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



A bilaplacian problem with Dirichlet boundary conditions

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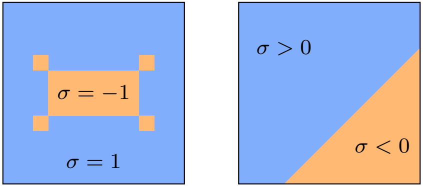
Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\mathrm{div}\,(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

... but $(\mathscr{P})$ can be ill-posed (not Fredholm) when σ changes sign "on $\partial\Omega$"
$\Rightarrow$ work with `J. Firozaly`.

Fredholm

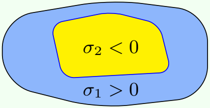


Not always Fredholm

Conclusion

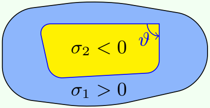
> Find $v \in \mathrm{H}^1_0(\Omega)$ s.t., $\forall v' \in \mathrm{H}^1_0(\Omega)$,
> $$\int_\Omega \sigma \nabla v \cdot \nabla v' = \ell(v').$$

♠ Smooth interface



Well-posed in the Fredholm sense iff $\kappa_\sigma = \sigma_2/\sigma_1 \neq -1$.

♠ Interface with a corner



Well-posed in the Fredholm sense iff $\kappa_\sigma \notin [-I; -1/I]$, $I = (2\pi - \vartheta)/\vartheta$.

> Find $v \in \mathrm{X}$ s.t., $\forall v' \in \mathrm{X}$,
> $$\int_\Omega \sigma \Delta v \Delta v' = \ell(v').$$

We assume $\sigma \in \mathrm{L}^\infty(\Omega)$, $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$.

♠ If $\mathrm{X} = \mathrm{H}^1_0(\Delta)$: Well-posed.

♠ If $\mathrm{X} = \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$:

- Well-posed when Ω is convex or of class $\mathscr{C}^2$.
- When Ω has one reentrant corner, it can occur a kernel of dimension 1.

♠ If $\mathrm{X} = \mathrm{H}^2_0(\Omega)$:

- Well-posed in the Fredholm sense when σ does not change sign on a neighbourhood of $\partial\Omega$.
- When σ changes sign on $\partial\Omega$, Fredholmness can be lost.

Thank you for your attention!!!