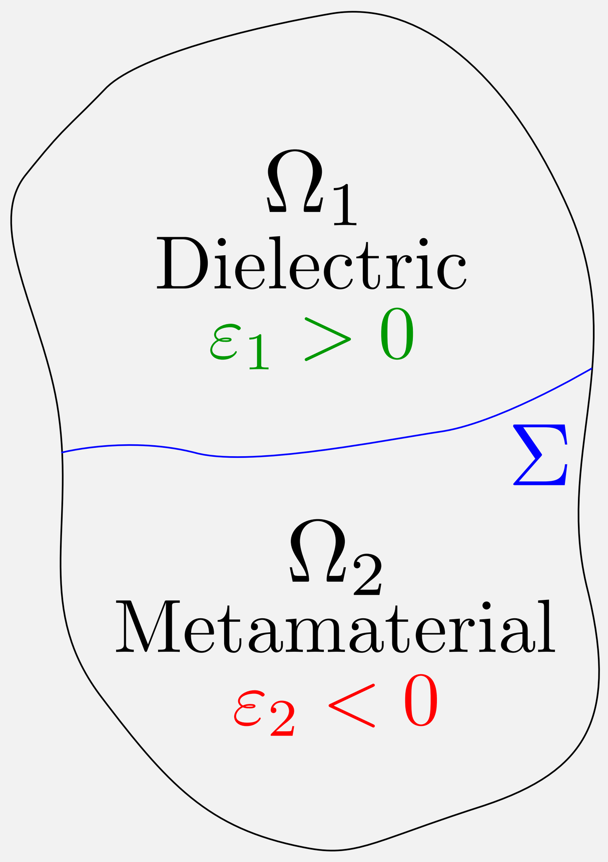
Radiation condition for a non-smooth interface between a dielectric and a metamaterial

L. Chesnel with A.S. Bonnet-Ben Dhia, P. Ciarlet Jr. and X. Claeys
POEMS, UMR 7231 CNRS-ENSTA-INRIA, École Polytechnique, Paris

Studied problem



- Time harmonic problem in a heterogeneous medium Ω. Difficulty concentrated in the electrostatic case.

- Define the space of finite energy fields:

$$H^1_0(\Omega) = \{v \in L^2(\Omega) \,|\, \int_\Omega |\nabla v|^2\, d\Omega < \infty;\ v|_{\partial\Omega} = 0\}.$$

$(\mathcal{P})$ Find $u \in H^1_0(\Omega)$ such that: $-\mathrm{div}(\varepsilon\,\nabla u) = f$ in Ω.

- $(\mathcal{P})$ is equivalent to the variational problem:

$(\mathcal{P}_V)$ Find $u \in H^1_0(\Omega)$ such that: $\displaystyle\int_\Omega \varepsilon\,\nabla u\cdot\nabla v\, d\Omega = \int_\Omega f v\, d\Omega\ ,\forall v \in H^1_0(\Omega).$

Difficulties :

- Loss of coercivity: there is no constant C such that

$$\int_\Omega \varepsilon\,|\nabla u|^2\, d\Omega > C\int_\Omega |\nabla u|^2\, d\Omega,\ \forall u \in H^1_0(\Omega).$$

- Add some dissipation (modeled by η) is not sufficient:

$$\left|\int_\Omega \varepsilon^\eta\,|\nabla u^\eta|^2\, d\Omega\right| > \frac{C}{\eta}\int_\Omega |\nabla u^\eta|^2\, d\Omega.$$

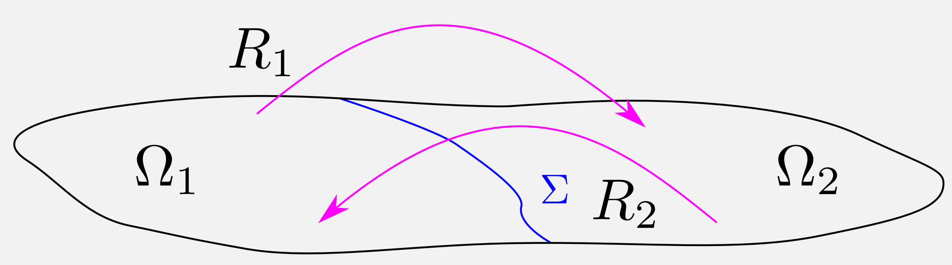
Questions :

- ▷ Is problem $(\mathcal{P})$ well-posed ?
- ▷ How to compute a numerical approximation of the solution ?
- ▷ New model when $(\mathcal{P})$ is ill-posed?

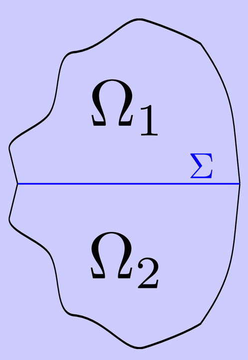
The T-coercivity approach

1. Consider $\mathtt{T}_1\, u = \begin{cases} u_1 & \text{in } \Omega_1 \\ -u_2 + 2R_1\, u_1 & \text{in } \Omega_2 \end{cases}$, where R_1 is such that $\mathtt{T}_1\, u \in H^1_0(\Omega)$.
2. $\displaystyle\int_\Omega \varepsilon\,\nabla u\cdot\nabla(\mathtt{T}_1\, u)\, d\Omega \geq C\int_\Omega |\nabla u|^2\, d\Omega$ for $\varepsilon_1 \geq \|R_1\|^2\,|\varepsilon_2|$.
3. Since $\mathtt{T}_1$ is an isomorphism of $H^1_0(\Omega)$ (notice that $\mathtt{T}_1^{-1} = \mathtt{T}_1$), $(\mathcal{P}_V)$, and so $(\mathcal{P})$, is well-posed when $\varepsilon_1 \geq \|R_1\|^2\,|\varepsilon_2|$.
4. One proceeds in the same way with $\mathtt{T}_2$ built from $R_2 : \Omega_2 \to \Omega_1$.

Theorem. If the contrast $\kappa_\varepsilon = \varepsilon_2/\varepsilon_1 \notin I_\Sigma = [-\|R_2\|^2; -1/\|R_1\|^2]$ (critical interval) then problem $(\mathcal{P})$ is well-posed.



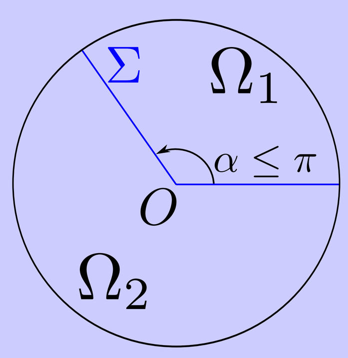
- This technique can be used to justify the classical finite element methods and to study the Maxwell's problem.



Symmetrical domain

$R_1 = S_\Sigma$ and $R_2 = S_\Sigma$ (symmetry).

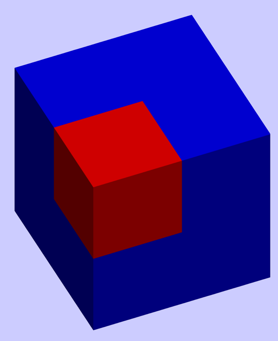
$I_\Sigma = \{-1\}$.



2D corner

R_1 and R_2 obtained from symmetry/dilatation w.r.t. θ.

$I_\Sigma = [-\frac{2\pi-\alpha}{\alpha}; -\frac{\alpha}{2\pi-\alpha}]$.

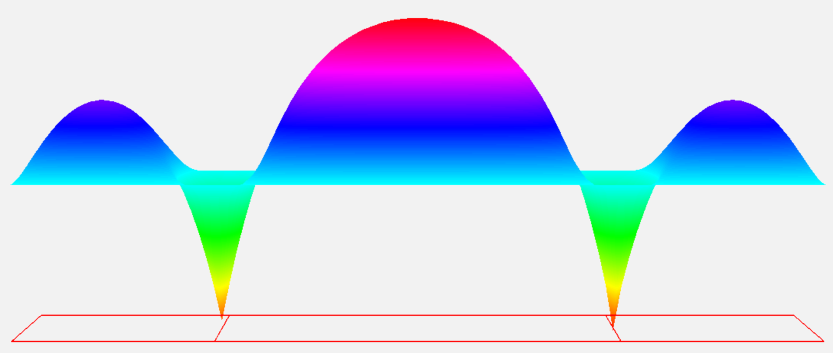


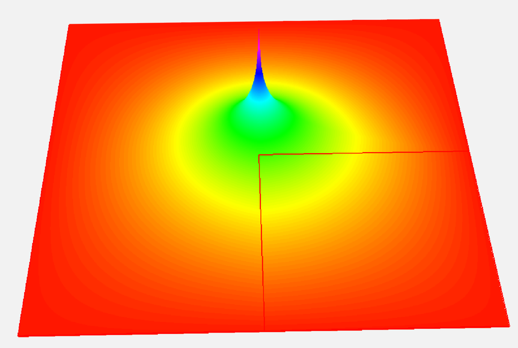
Fichera's corner

R_1 and R_2 obtained from the symmetries S_{Ox}, S_{Oy}, S_{Oz}.

$I_\Sigma = [-7; -1/7]$.

- If Σ is smooth, $(\mathcal{P})$ is well-posed for $\kappa_\varepsilon \neq -1$.
- If Σ has a corner, $(\mathcal{P})$ is well-posed for $\kappa_\varepsilon \notin I_\Sigma$ (open interval). But one observes a field of strong intensity in a neighbourhood of the corner. $\Rightarrow$ What happens in the critical interval I_Σ?



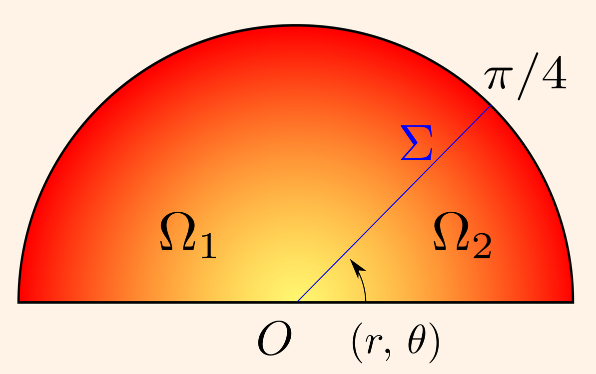


A black hole phenomena

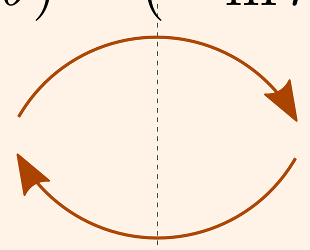
Singularity problem

Helmholtz equation in the bounded sector Ω

$$-\mathrm{div}(\varepsilon\,\nabla u) = -r^{-2}(\varepsilon(r\partial_r)^2 + \partial_\theta\varepsilon\partial_\theta)u = f.$$



$(z,\theta) = (-\ln r, \theta)$

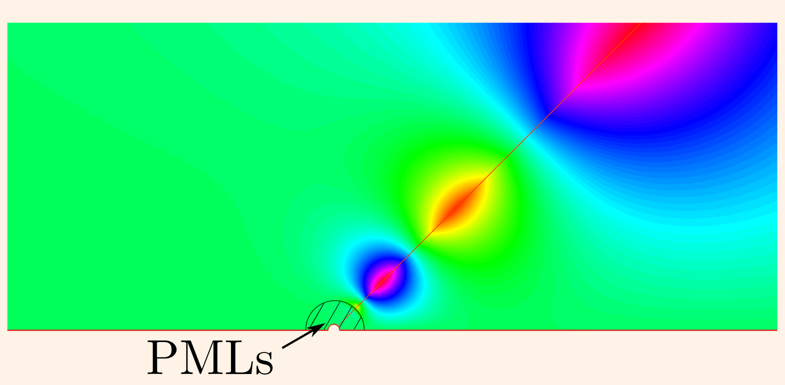


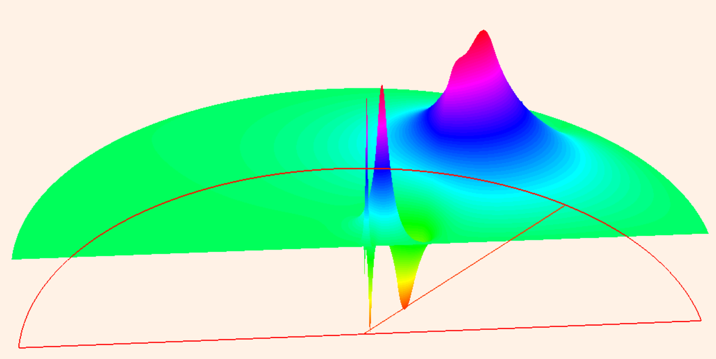
$(r,\theta) = (e^{-z}, \theta)$

- For $\kappa_\varepsilon \in (-1; -1/3)$, propagative singularities appear:

$s_1^\mp(r,\theta) = \varphi_1(\theta)e^{\mp i\eta\ln r}$ where η is a real number which depends on κ_ε.

- Radiation condition in O to select the good singularity.
- We use PMLs in a neighbourhood of O to approach the solution which is not of finite energy.

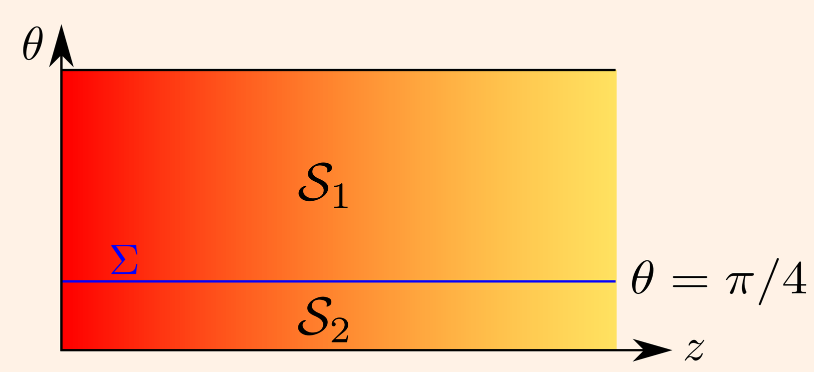




Waveguide problem

Helmholtz equation in the strip $\mathcal{S}$

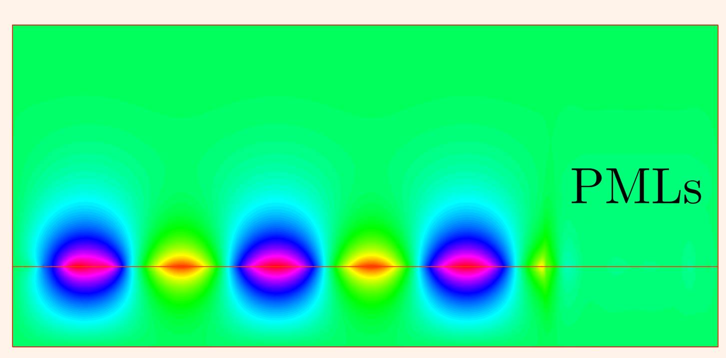
$$-\mathrm{div}(\varepsilon\,\nabla u) = -(\varepsilon\partial_z^2 + \partial_\theta\varepsilon\partial_\theta)u = e^{-2z}f.$$

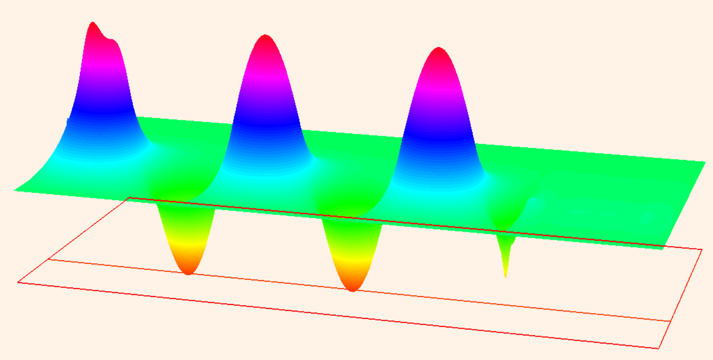


- For $\kappa_\varepsilon \in (-1; -1/3)$, propagative modes appear:

$m_1^\pm(z,\theta) = \varphi_1(\theta)e^{\pm i\eta z}$ where η is a real number which depends on κ_ε.

- Radiation condition in $+\infty$ to select the outgoing mode.
- We use PMLs to bound the domain and to implement classical finite element methods in the truncated strip.





chesnel@ensta-paristech.fr