Construction of invisible defects for acoustic problems with a finite number of measurements

Lucas Chesnel[1]

Coll. with A. Bera[2], A.-S. Bonnet-Ben Dhia[2] and S.A. Nazarov[3].

[1]Defi team, CMAP, École Polytechnique, France
[2]Poems team, Ensta ParisTech, France
[3]FMM, St. Petersburg State University, Russia

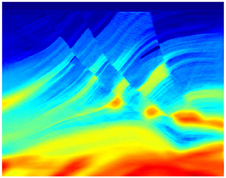
Institut Henri-Poincaré, Paris, 09/05/2016

General setting

▶ We are interested in methods based on the propagation of waves to determine the shape, the physical properties of objects, in an exact or qualitative manner, from given measurements.

▶ General principle of the methods:

i) send waves in the medium;
ii) measure the scattered field;
iii) deduce information on the structure.



- Many techniques: Xray, ultrasound imaging, seismic tomography, ...
- Many applications: biomedical imaging, non destructive testing of materials, geophysics, ...

Goal of the talk

- The goal of imaging techniques is to find features of the structure from the knowledge of measurements.

- In this talk, we are interested in questions of invisibility when one has a finite number of measurements.

Goal of the talk

- The goal of imaging techniques is to find features of the structure from the knowledge of measurements.

- In this talk, we are interested in questions of invisibility when one has a finite number of measurements.

  - Less ambitious than usual cloaking and therefore, more accessible.

  - Also relevant for applications.

Goal of the talk

- The goal of imaging techniques is to find features of the structure from the knowledge of measurements.

- In this talk, we are interested in questions of invisibility when one has a finite number of measurements.
  - Less ambitious than usual cloaking and therefore, more accessible.
  - Also relevant for applications.

- We will consider two types of problems:
  1. Scattering in free space
  2. Scattering in waveguides

Goal of the talk

- The goal of imaging techniques is to find features of the structure from the knowledge of measurements.

- In this talk, we are interested in questions of invisibility when one has a finite number of measurements.

  - Less ambitious than usual cloaking and therefore, more accessible.
  - Also relevant for applications.

- We will consider two types of problems:

  1. Scattering in free space
  2. Scattering in waveguides

- At least two reasons to study invisibility questions:

  - We can wish to hide objects.
  - It allows to understand limits of imaging techniques.

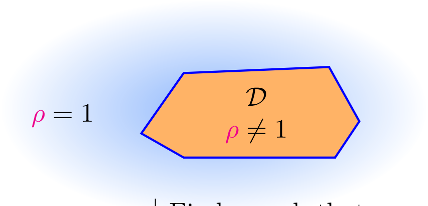
Outline of the talk

1. Invisibility in free space
   - The general scheme
   - The forbidden case
   - Numerical experiments

2. Invisibility for waveguide problems
   - Construction of invisible penetrable defects
   - Can one hide a small Dirichlet obstacle?
   - Can one hide a perturbation of the wall?

Model problem

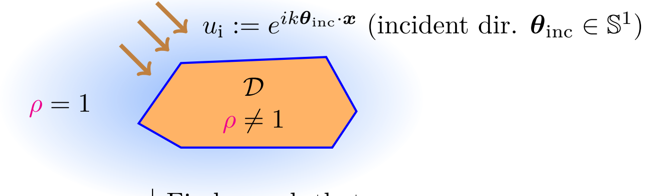
- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\mathbb{R}^2$.



$$\left|\begin{array}{l} \text{Find } u \text{ such that} \\ \begin{array}{rcll} -\Delta u & = & k^2 \rho\, u & \text{in } \mathbb{R}^2, \\ u & = & u_{\mathrm{i}} + u_{\mathrm{s}} & \text{in } \mathbb{R}^2, \end{array} \\ \displaystyle\lim_{r \to +\infty} \sqrt{r} \left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}} \right) = 0. \end{array}\right. \qquad (1)$$

Model problem

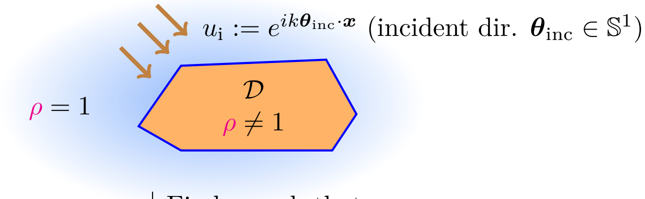
▶ Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\mathbb{R}^2$.



$$\left|\begin{array}{l} \text{Find } u \text{ such that} \\ \begin{aligned} -\Delta u &= k^2 \rho\, u && \text{in } \mathbb{R}^2, \\ u &= u_{\mathrm{i}} + u_{\mathrm{s}} && \text{in } \mathbb{R}^2, \end{aligned} \\ \lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) = 0. \end{array}\right. \tag{1}$$

Model problem

▶ Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\mathbb{R}^2$.



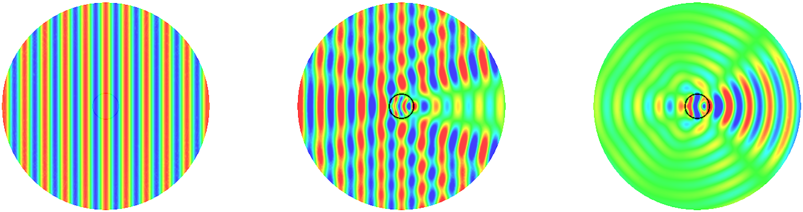
Find u such that

$$\begin{aligned} -\Delta u &= k^2 \rho\, u && \text{in } \mathbb{R}^2, \\ u &= u_\mathrm{i} + u_\mathrm{s} && \text{in } \mathbb{R}^2, \\ \lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_\mathrm{s}}{\partial r} - iku_\mathrm{s}\right) &= 0. \end{aligned} \tag{1}$$

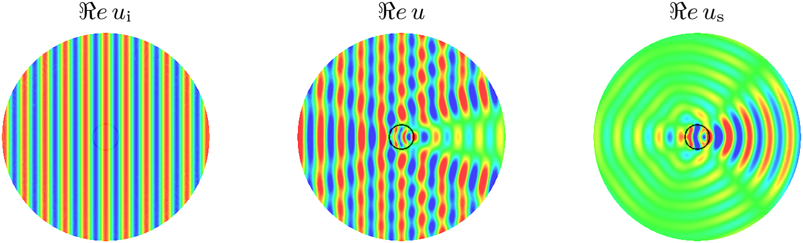
DEFINITION:
- u_i = incident field (data)
- u = total field (uniquely defined by (1))
- u_s = scattered field (uniquely defined by (1)).

Far field pattern

$\Re e\, u_{\rm i}$ $\Re e\, u$ $\Re e\, u_{\rm s}$



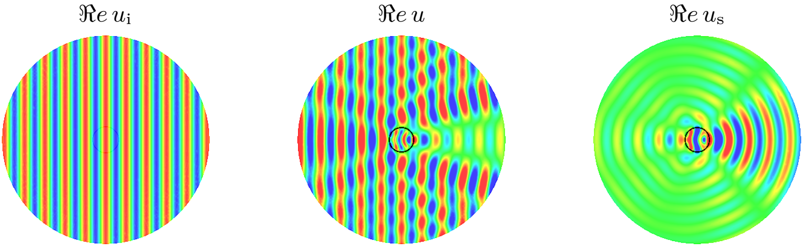
Far field pattern



- The scattered field of an incident plane wave of direction $\boldsymbol{\theta}_{\mathrm{inc}}$ behaves in each direction like a cylindrical wave at infinity:

$$u_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{\theta}_{\mathrm{inc}}) = \frac{e^{ikr}}{\sqrt{r}} \left(u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{sca}}, \boldsymbol{\theta}_{\mathrm{inc}}) + O(1/r) \right).$$

Far field pattern

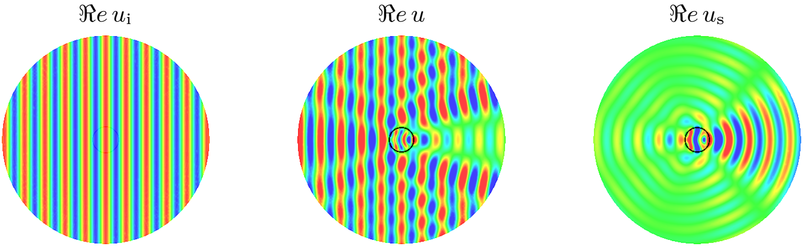


- The scattered field of an incident plane wave of direction $\boldsymbol{\theta}_{\mathrm{inc}}$ behaves in each direction like a cylindrical wave at infinity:

$$u_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{\theta}_{\mathrm{inc}}) = \frac{e^{ikr}}{\sqrt{r}} \left(\boxed{u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{sca}}, \boldsymbol{\theta}_{\mathrm{inc}})} + O(1/r) \right).$$

DEFINITION: The map $\boxed{u_{\mathrm{s}}^{\infty}(\cdot,\cdot)} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}$ is called the far field pattern.

Far field pattern



► The scattered field of an incident plane wave of direction $\boldsymbol{\theta}_{\rm inc}$ behaves in each direction like a cylindrical wave at infinity:

$$u_{\rm s}(\boldsymbol{x}, \boldsymbol{\theta}_{\rm inc}) = \frac{e^{ikr}}{\sqrt{r}} \left(\boxed{u_{\rm s}^{\infty}(\boldsymbol{\theta}_{\rm sca}, \boldsymbol{\theta}_{\rm inc})} + O(1/r) \right).$$

DEFINITION: The map $\boxed{u_{\rm s}^{\infty}(\cdot,\cdot)} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}$ is called the far field pattern.

At infinity, one measures the far field pattern (other terms are too small).

Setting

- The goal of imaging techniques is to find features of the inclusion from the knowledge of $u_s^\infty(\cdot,\cdot)$ on a subset of $\mathbb{S}^1 \times \mathbb{S}^1$.

  – In literature, most of the techniques require a continuum of data.

(Nachman, 1988, Sylvester & Uhlmann, 1987, Bukhgeim, 2008, Imanuvilov & Yamamoto, 2012)

  – In practice, one has a finite number of emitters and receivers.

Setting

- The goal of imaging techniques is to find features of the inclusion from the knowledge of $u_s^\infty(\cdot,\cdot)$ on a subset of $\mathbb{S}^1 \times \mathbb{S}^1$.

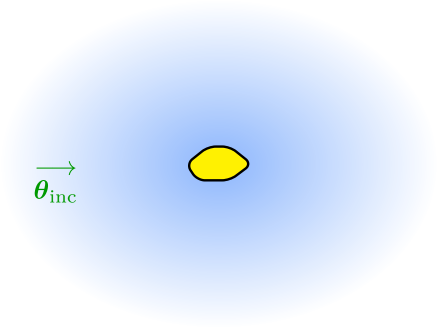
  – In literature, most of the techniques require a continuum of data.

(Nachman, 1988, Sylvester & Uhlmann, 1987, Bukhgeim, 2008, Imanuvilov & Yamamoto, 2012)

  – In practice, one has a finite number of emitters and receivers.

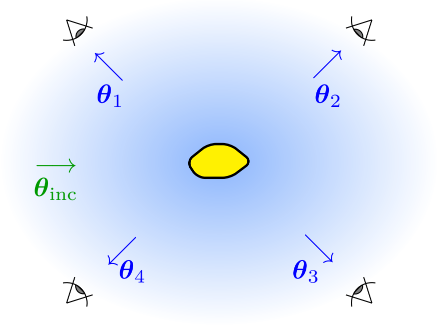
Setting

- To simplify the presentation, only one incident direction $\boldsymbol{\theta}_{\text{inc}}$ and N scattering directions $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ (given).



Setting

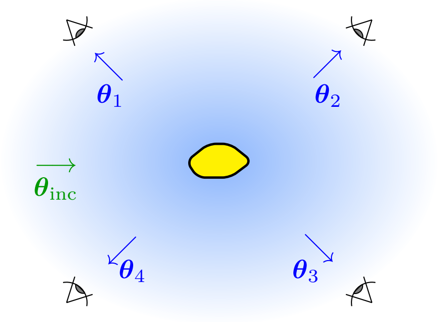
► To simplify the presentation, only one incident direction $\boldsymbol{\theta}_{\text{inc}}$ and N scattering directions $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ (given).



$\rightarrow$ We measure $u_s^\infty(\boldsymbol{\theta}_1), \ldots, u_s^\infty(\boldsymbol{\theta}_N)$.

Setting

- To simplify the presentation, only one incident direction $\boldsymbol{\theta}_{\text{inc}}$ and N scattering directions $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ (given).



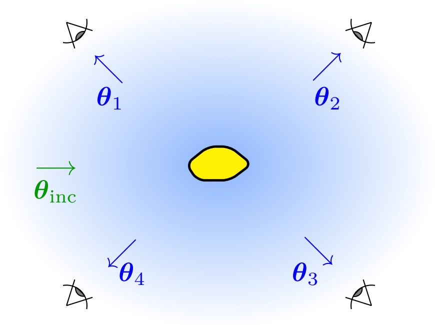
GOAL

We explain how to construct inclusions such that

$$u_{\text{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\text{s}}^{\infty}(\boldsymbol{\theta}_N) = 0.$$

Setting

- To simplify the presentation, only one incident direction $\boldsymbol{\theta}_{\text{inc}}$ and N scattering directions $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$ (given).



GOAL

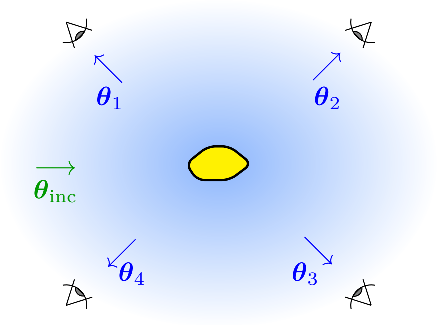
We explain how to construct inclusions such that

$$u_{\text{s}}^{\infty}(\boldsymbol{\theta}_1) = \dots = u_{\text{s}}^{\infty}(\boldsymbol{\theta}_N) = 0.$$

- These inclusions cannot be detected from far field measurements.

Setting

- To simplify the presentation, only one incident direction $\boldsymbol{\theta}_{\text{inc}}$ and N scattering directions $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ (given).



GOAL

We explain how to construct inclusions such that

$$u_s^\infty(\boldsymbol{\theta}_1) = \cdots = u_s^\infty(\boldsymbol{\theta}_N) = 0.$$

- These inclusions cannot be detected from far field measurements.
- We assume that k and the support of the inclusion $\overline{\mathcal{D}}$ are given.

Setting

- To simplify the presentation, only one incident direction θ_{inc} and N scattering directions $\theta_1, \dots, \theta_N$ (given).

Find a real valued function $\rho \not\equiv 1$, with $\rho - 1$ supported in $\overline{\mathcal{D}}$, such that the solution to the problem

$$\left|\begin{array}{l} \text{Find } u = u_{\text{s}} + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2 \rho\, u \quad \text{in } \mathbb{R}^2, \\ \qquad\qquad u_{\text{s}} \text{ is outgoing} \end{array}\right.$$

satisfies $u_{\text{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\text{s}}^{\infty}(\boldsymbol{\theta}_N) = 0$.

GOAL

We explain how to construct inclusions such that

$$u_{\text{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\text{s}}^{\infty}(\boldsymbol{\theta}_N) = 0.$$

- These inclusions cannot be detected from far field measurements.
- We assume that k and the support of the inclusion $\mathcal{D}$ are given.

Sketch of the method

- We will work as in the proof of the implicit functions theorem.

  The idea was used in Nazarov 11 to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

(N complex measurements ⇒ 2N real measurements)

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$
- No obstacle leads to null measurements $\Rightarrow$ $F(0) = 0$.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector

$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- We look for small perturbations of the reference medium: $\sigma = \varepsilon\mu$ where $\varepsilon > 0$ is a small parameter and where μ has be to determined.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector

$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = F(0) + \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector

$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector

$$F(\sigma) = (F_1(\sigma), \dots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \dots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \dots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t.} \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector

$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left|\begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array}\right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow$$

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:
$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad 0 = \varepsilon \sum_{n=1}^{2N} \tau_n dF(0)(\mu_n) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$$

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad 0 = \varepsilon\vec{\tau} + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$$

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \dots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \dots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \dots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad 0 = \varepsilon\vec{\tau} + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$$

where $\vec{\tau} = (\tau_1, \dots, \tau_{2N})^\top$

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t.} \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad \boxed{\vec{\tau} = G^\varepsilon(\vec{\tau})}$$

where $\vec{\tau} = (\tau_1, \ldots, \tau_{2N})^\top$

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \dots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \dots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t.} \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \dots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad \boxed{\vec{\tau} = G^\varepsilon(\vec{\tau})}$$

where $\vec{\tau} = (\tau_1, \dots, \tau_{2N})^\top$ and $G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu)$.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \dots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \dots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \dots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad \vec{\tau} = G^\varepsilon(\vec{\tau})$$

where $\vec{\tau} = (\tau_1, \dots, \tau_{2N})^\top$ and $G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu)$.

If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

Our goal: to find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

- Taylor: $F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad \boxed{\vec{\tau} = G^\varepsilon(\vec{\tau})}$$

where $\vec{\tau} = (\tau_1, \ldots, \tau_{2N})^\top$ and $G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu)$.

If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.
Set $\sigma^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $F(\sigma^{\mathrm{sol}}) = 0$ (invisible inclusion).

Calculus of $dF(0)$ 1/2

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \ldots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \ldots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

Calculus of $dF(0)$ 1/2

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \ldots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \ldots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

Calculus of $dF(0)$ 1/2

- For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^ε, u_s^ε the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_s^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\ -\Delta u^\varepsilon \quad = \quad k^2\rho^\varepsilon\, u^\varepsilon \qquad \text{in } \mathbb{R}^2. \end{array}\right.$$

Calculus of $dF(0)$

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1), \dots, \Re e\, u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N), \Im m\, u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1), \dots, \Im m\, u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N)).$$

To compute $dF(0)(\mu)$, we take $\boxed{\rho^{\varepsilon} = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^{ε}, $u_{\mathrm{s}}^{\varepsilon}$ the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^{\varepsilon} = u_{\mathrm{s}}^{\varepsilon} + e^{ik\boldsymbol{\theta}_{\mathrm{inc}}\cdot\boldsymbol{x}}, \text{with } u_{\mathrm{s}}^{\varepsilon} \text{ outgoing, such that} \\ -\Delta u^{\varepsilon} \;=\; k^2\rho^{\varepsilon}\, u^{\varepsilon} \quad \text{in } \mathbb{R}^2. \end{array}\right.$$

- $u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = c\, k^2 \displaystyle\int_{\mathcal{D}} (\rho^{\varepsilon} - 1)\,(u_{\mathrm{i}} + u_{\mathrm{s}}^{\varepsilon})\, e^{-ik\boldsymbol{\theta}_n\cdot\boldsymbol{x}}\, d\boldsymbol{x} \qquad \left(c = \dfrac{e^{i\pi/4}}{\sqrt{8\pi k}}\right).$

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^ε, u_s^ε the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_s^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\ -\Delta u^\varepsilon \;=\; k^2\rho^\varepsilon\, u^\varepsilon \quad \text{in } \mathbb{R}^2. \end{array}\right.$$

- $u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = c\, k^2 \displaystyle\int_{\mathcal{D}} (\rho^\varepsilon - 1)\,(u_i + u_s^\varepsilon)\, e^{-ik\boldsymbol{\theta}_n\cdot\boldsymbol{x}}\, d\boldsymbol{x}.$

Calculus of $dF(0)$ 1/2

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^ε, u_s^ε the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_s^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\ -\Delta u^\varepsilon \quad = \quad k^2 \rho^\varepsilon\, u^\varepsilon \qquad \text{in } \mathbb{R}^2. \end{array}\right.$$

- $u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = \varepsilon\, c\, k^2 \int_{\mathcal{D}} \mu\,(u_i + u_s^\varepsilon)\, e^{-ik\boldsymbol{\theta}_n\cdot\boldsymbol{x}}\, d\boldsymbol{x}.$

Calculus of $dF(0)$ 1/2

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^ε, u_s^ε the functions satisfying

$$\left| \begin{array}{l} \text{Find } u^\varepsilon = u_s^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\ -\Delta u^\varepsilon \;=\; k^2 \rho^\varepsilon\, u^\varepsilon \quad \text{in } \mathbb{R}^2. \end{array} \right.$$

- $u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = \varepsilon\, c\, k^2 \displaystyle\int_{\mathcal{D}} \mu\,(u_i + u_s^\varepsilon)\, e^{-ik\boldsymbol{\theta}_n\cdot\boldsymbol{x}}\, d\boldsymbol{x}.$

- We can prove that $u_s^\varepsilon = O(\varepsilon)$.

Calculus of $dF(0)$

- For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^ε, u_s^ε the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_s^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\ -\Delta u^\varepsilon \quad = \quad k^2 \rho^\varepsilon\, u^\varepsilon \qquad \text{in } \mathbb{R}^2. \end{array}\right.$$

- $u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = \varepsilon\, c\, k^2 \displaystyle\int_{\mathcal{D}} \mu\, u_i\, e^{-ik\boldsymbol{\theta}_n\cdot\boldsymbol{x}}\, d\boldsymbol{x} + O(\varepsilon^2).$

- We can prove that $u_s^\varepsilon = O(\varepsilon)$.

Calculus of $dF(0)$ 1/2

- For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \ldots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \ldots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^ε, u_s^ε the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_s^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\ -\Delta u^\varepsilon \quad = \quad k^2 \rho^\varepsilon\, u^\varepsilon \qquad \text{in } \mathbb{R}^2. \end{array}\right.$$

- We obtain the expansion (Born approx.), for small ε

$$u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = 0 + \varepsilon\, c\, k^2 \int_{\mathcal{D}} \mu\, e^{ik(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n)\cdot\boldsymbol{x}}\, d\boldsymbol{x} \; + O(\varepsilon^2).$$

Calculus of $dF(0)$

- For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = \Big(\Re e\, \frac{u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1)}{ck^2}, \dots, \Re e\, \frac{u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N)}{ck^2}, \Im m\, \frac{u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1)}{ck^2}, \dots, \Im m\, \frac{u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N)}{ck^2}\Big).$$

To compute $dF(0)(\mu)$, we take $\rho^{\varepsilon} = 1 + \varepsilon\mu$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^{ε}, $u_{\mathrm{s}}^{\varepsilon}$ the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^{\varepsilon} = u_{\mathrm{s}}^{\varepsilon} + e^{ik\boldsymbol{\theta}_{\mathrm{inc}}\cdot\boldsymbol{x}}, \text{with } u_{\mathrm{s}}^{\varepsilon} \text{ outgoing, such that} \\ -\Delta u^{\varepsilon} \;=\; k^2\rho^{\varepsilon}\, u^{\varepsilon} \quad \text{in } \mathbb{R}^2. \end{array}\right.$$

- We obtain the expansion (Born approx.), for small ε

$$u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = 0 + \varepsilon\, c\, k^2 \int_{\mathcal{D}} \mu\, e^{ik(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_n)\cdot\boldsymbol{x}}\, d\boldsymbol{x} + O(\varepsilon^2).$$

Calculus of $dF(0)$

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = \Big(\Re e\, \frac{u_s^\infty(\boldsymbol{\theta}_1)}{ck^2}, \dots, \Re e\, \frac{u_s^\infty(\boldsymbol{\theta}_N)}{ck^2}, \Im m\, \frac{u_s^\infty(\boldsymbol{\theta}_1)}{ck^2}, \dots, \Im m\, \frac{u_s^\infty(\boldsymbol{\theta}_N)}{ck^2}\Big).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon\mu$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^ε, u_s^ε the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_s^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\ -\Delta u^\varepsilon \; = \; k^2\rho^\varepsilon\, u^\varepsilon \quad \text{in } \mathbb{R}^2. \end{array}\right.$$

- We obtain the expansion (Born approx.), for small ε

$$u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = 0 + \varepsilon\, c\, k^2 \boxed{\int_{\mathcal{D}} \mu\, e^{ik(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n)\cdot\boldsymbol{x}}\, d\boldsymbol{x}} + O(\varepsilon^2).$$

$$\begin{aligned} dF(0)(\mu) = \Big(&\int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \\ &\int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}\Big) \end{aligned}$$

Calculus of $dF(0)$ 2/2

$$
\begin{aligned}
dF(0)(\mu) = \Big(&\int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \ldots, \int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \\
&\int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \ldots, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x} \Big)
\end{aligned}
$$

Calculus of $dF(0)$ 2/2

$$dF(0)(\mu) = \Big(\int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1) \cdot \boldsymbol{x}) \, d\boldsymbol{x}, \ldots, \int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N) \cdot \boldsymbol{x}) \, d\boldsymbol{x},$$
$$\int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1) \cdot \boldsymbol{x}) \, d\boldsymbol{x}, \ldots, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N) \cdot \boldsymbol{x}) \, d\boldsymbol{x} \Big)$$

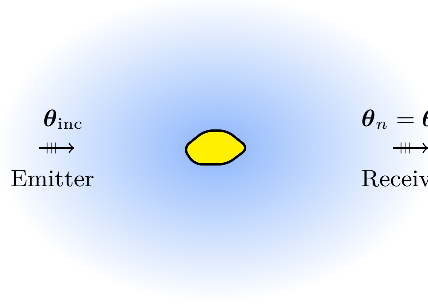
Is $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ onto ?

Calculus of $dF(0)$ 2/2

$$dF(0)(\mu) = \Big(\int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot \boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot \boldsymbol{x})\, d\boldsymbol{x},$$
$$\int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot \boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot \boldsymbol{x})\, d\boldsymbol{x}\Big)$$

Is $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ onto ?

- Clearly, we need to avoid the configuration $\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n = 0$.

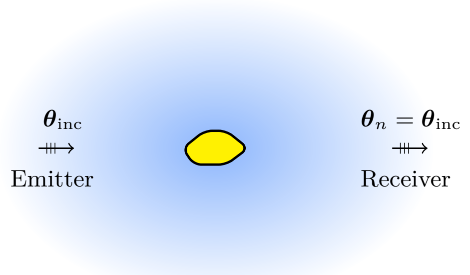


Calculus of $dF(0)$

$$dF(0)(\mu) = \Big(\int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot \boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot \boldsymbol{x})\, d\boldsymbol{x}, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot \boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot \boldsymbol{x})\, d\boldsymbol{x} \Big)$$

Is $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ onto ?

- Clearly, we need to avoid the configuration $\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n = 0$.



- And one can prove that if $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \dots, N$, the answer is yes.

Main result

Proposition: Assume that $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$ for $n = 1, \dots, N$. For $\boldsymbol{\varepsilon}$ **small enough**, define $\rho^{\text{sol}} = 1 + \varepsilon\mu^{\text{sol}}$ with

$$\mu^{\text{sol}} = \mu_0 + \sum_{n=1}^{2N} \tau_n^{\text{sol}} \mu_n.$$

Then the solution of the scattering problem

$$\left| \begin{array}{l} \text{Find } u^\varepsilon = u_{\text{s}}^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2 \rho^{\text{sol}} u \quad \text{in } \mathbb{R}^2, \\ \qquad\qquad u_{\text{s}} \text{ is outgoing} \end{array} \right.$$

satisfies $u_{\text{s}}^\infty(\boldsymbol{\theta}_1) = \cdots = u_{\text{s}}^\infty(\boldsymbol{\theta}_N) = 0$.

Comments:

→ We need $\boldsymbol{\varepsilon}$ to be **small enough** to prove that G^ε is a contraction.

→ We have $\mu^{\text{sol}} \not\equiv 0$ (non trivial inclusion). To see it, compute $dF(0)(\mu^{\text{sol}})$.

Main result

Proposition: Assume that $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$ for $n = 1, \dots, N$. For $\boldsymbol{\varepsilon}$ **small enough**, define $\rho^{\text{sol}} = 1 + \varepsilon\mu^{\text{sol}}$ with

$$\mu^{\text{sol}} = \mu_0 + \sum_{n=1}^{2N} \tau_n^{\text{sol}} \mu_n.$$

Then the solution of the scattering problem

$$\left| \begin{array}{l} \text{Find } u^\varepsilon = u_{\text{s}}^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2 \rho^{\text{sol}} u \quad \text{in } \mathbb{R}^2, \\ \qquad\qquad u_{\text{s}} \text{ is outgoing} \end{array} \right.$$

satisfies $u_{\text{s}}^\infty(\boldsymbol{\theta}_1) = \cdots = u_{\text{s}}^\infty(\boldsymbol{\theta}_N) = 0$.

Comments:

→ We need $\boldsymbol{\varepsilon}$ to be **small enough** to prove that G^ε is a contraction.

→ We have $\mu^{\text{sol}} \not\equiv 0$ (non trivial inclusion). To see it, compute $dF(0)(\mu^{\text{sol}})$.

→ We have $\tau^{\text{sol}} = O(\varepsilon) \Rightarrow \mu^{\text{sol}} \approx \mu_0$. We control the main form of the defect.

Main result

Proposition: Assume that $\theta_n \neq \theta_{\text{inc}}$ for $n = 1, \dots, N$. For ε **small enough**, define $\rho^{\text{sol}} = 1 + \varepsilon\mu^{\text{sol}}$ with

$$\mu^{\text{sol}} = \mu_0 + \sum_{n=1}^{2N} \tau_n^{\text{sol}} \mu_n.$$

Then the solution of the scattering problem

$$-\Delta u = k^2 \rho^{\text{sol}} u \quad \text{in } \mathbb{R}^2,$$

u_s is outgoing

satisfies $u_s^\infty(\theta_1) = \dots = u_s^\infty(\theta_N) = 0$.

Existence of invisible inclusions may appear not so surprising since there are $2N$ measurements and $\rho \in \mathrm{L}^\infty(\mathcal{D})$.

Comments:

→ We need ε to be **small enough** to prove that G^ε is a contraction.

→ We have $\mu^{\text{sol}} \not\equiv 0$ (non trivial inclusion). To see it, compute $dF(0)(\mu^{\text{sol}})$.

→ We have $\tau^{\text{sol}} = O(\varepsilon) \Rightarrow \mu^{\text{sol}} \approx \mu_0$. We control the main form of the defect.

Main result

Proposition: Assume that $\theta_n \neq \theta_{\text{inc}}$ for $n = 1, \dots, N$. For ε **small enough**, define $\rho^{\text{sol}} = 1 + \varepsilon \mu^{\text{sol}}$ with

$$\mu^{\text{sol}} = \mu_0 + \sum_{n=1}^{2N} \tau_n^{\text{sol}} \mu_n.$$

Then the solution of the scattering problem

Existence of invisible inclusions may appear not so surprising since there are $2N$ measurements and $\rho \in \mathrm{L}^\infty(\mathcal{D})$. *Let us see the case* $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$...

$$-\Delta u = k^2 \rho^{\text{sol}} u \quad \text{in } \mathbb{R}^2,$$
$$u_s \text{ is outgoing}$$

satisfies $u_s^\infty(\theta_1) = \cdots = u_s^\infty(\theta_N) = 0$.

Comments:

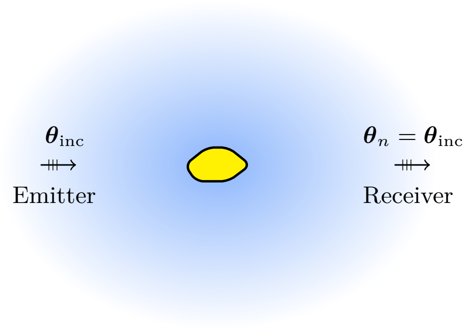
→ We need ε to be **small enough** to prove that G^ε is a contraction.

→ We have $\mu^{\text{sol}} \not\equiv 0$ (non trivial inclusion). To see it, compute $dF(0)(\mu^{\text{sol}})$.

→ We have $\tau^{\text{sol}} = O(\varepsilon) \Rightarrow \mu^{\text{sol}} \approx \mu_0$. We control the main form of the defect.

The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

- In the previous approach, we needed to assume $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \ldots, N$.
  What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$?



The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

- In the previous approach, we needed to assume $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \ldots, N$.
  What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$?

The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

- In the previous approach, we needed to assume $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \ldots, N$.
  What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$?

- There holds

$$u_{\text{s}}^{\infty}(\boldsymbol{\theta}_n) = c\, k^2 \int_{\mathcal{D}} (\rho - 1)\,(u_{\text{i}} + u_{\text{s}})\, e^{-ik\boldsymbol{\theta}_n \cdot \boldsymbol{x}}\, d\boldsymbol{x}.$$

The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

- In the previous approach, we needed to assume $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \ldots, N$.
  What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$?

- There holds

$$u_{\text{s}}^{\infty}(\boldsymbol{\theta}_{\text{inc}}) = c\, k^2 \int_{\mathcal{D}} (\rho - 1)\,(u_{\text{i}} + u_{\text{s}})\,\overline{u_{\text{i}}}\, d\boldsymbol{x}.$$

The case $\boldsymbol{\theta}_{\mathrm{inc}} = \boldsymbol{\theta}_n$

- In the previous approach, we needed to assume $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\mathrm{inc}}$, $n = 1, \ldots, N$.
  What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\mathrm{inc}}$?

- There holds

$$u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{inc}}) = c\, k^2 \int_{\mathcal{D}} (\rho - 1)\, (u_{\mathrm{i}} + u_{\mathrm{s}})\, \overline{u_{\mathrm{i}}}\, d\boldsymbol{x}.$$

- This allows to prove the optical theorem (already obtained by Rayleigh)

$$\Im m\, (c^{-1}\, u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{inc}})) = k \int_{\mathbb{S}^1} |u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta})|^2\, d\boldsymbol{\theta}.$$

The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

- In the previous approach, we needed to assume $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \dots, N$.
  What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$?

- There holds

$$u_{\text{s}}^{\infty}(\boldsymbol{\theta}_{\text{inc}}) = c\, k^2 \int_{\mathcal{D}} (\rho - 1)\,(u_{\text{i}} + u_{\text{s}})\,\overline{u_{\text{i}}}\, d\boldsymbol{x}.$$

- This allows to prove the optical theorem (already obtained by Rayleigh)

$$\Im m\,(c^{-1}\, u_{\text{s}}^{\infty}(\boldsymbol{\theta}_{\text{inc}})) = k \int_{\mathbb{S}^1} |u_{\text{s}}^{\infty}(\boldsymbol{\theta})|^2\, d\boldsymbol{\theta}.$$

Imposing invisibility in the direction $\boldsymbol{\theta}_{\text{inc}}$ requires to impose invisibility in all directions $\boldsymbol{\theta} \in \mathbb{S}^1$! *(not only for small but for all defects)*

The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

- In the previous approach, we needed to assume $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \dots, N$. What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$?

- There holds

$$u_{\text{s}}^{\infty}(\boldsymbol{\theta}_{\text{inc}}) = c\,k^2 \int_{\mathcal{D}} (\rho - 1)\,(u_{\text{i}} + u_{\text{s}})\,\overline{u_{\text{i}}}\,d\boldsymbol{x}.$$

- This allows to prove the optical theorem (already obtained by Rayleigh)

$$\Im m\,(c^{-1}\,u_{\text{s}}^{\infty}(\boldsymbol{\theta}_{\text{inc}})) = k \int_{\mathbb{S}^1} |u_{\text{s}}^{\infty}(\boldsymbol{\theta})|^2\,d\boldsymbol{\theta}.$$

Imposing invisibility in the direction $\boldsymbol{\theta}_{\text{inc}}$ requires to impose invisibility in all directions $\boldsymbol{\theta} \in \mathbb{S}^1$! *(not only for small but for all defects)*

Rellich's lemma $\Rightarrow$ $u_{\text{s}} \equiv 0$ in $\mathbb{R}^2 \setminus \overline{\mathcal{D}}$. **An incident plane wave which produces a scattered field null outside the defect...** Is it possible?

The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

- In the previous approach, we needed to assume $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \ldots, N$.
  What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$?

- There holds

$$u_{\text{s}}^{\infty}(\boldsymbol{\theta}_{\text{inc}}) = c\,k^2 \int_{\mathcal{D}} (\rho - 1)\,(u_{\text{i}} + u_{\text{s}})\,\overline{u_{\text{i}}}\,d\boldsymbol{x}.$$

- No solution if $\mathcal{D}$ has corners and under certain assumptions on ρ.

- Corners always scatter, E. Blåsten, L. Päivärinta, J. Sylvester, 2014
- Corners and edges always scatter, J. Elschner, G. Hu, 2015

- And if $\mathcal{D}$ is smooth? $\Rightarrow$ The problem seems open.

Imposing invisibility in the direction $\boldsymbol{\theta}_{\text{inc}}$ requires to impose invisibility in all directions $\boldsymbol{\theta} \in \mathbb{S}^1$! *(not only for small but for all defects)*

Rellich's lemma $\Rightarrow$ $u_{\text{s}} \equiv 0$ in $\mathbb{R}^2 \setminus \overline{\mathcal{D}}$. **An incident plane wave which produces a scattered field null outside the defect...** Is it possible?

Data and algorithm

- We can solve the fixed point problem using an iterative procedure: we set $\vec{\tau}^{\,0} = (0, \dots, 0)^\top$ then define

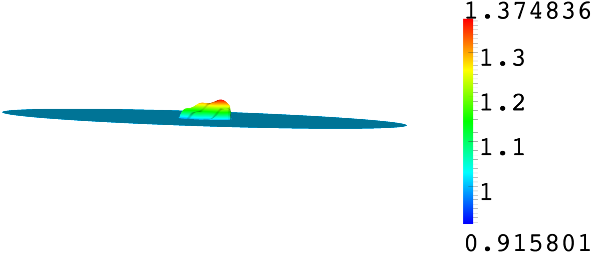
$$\vec{\tau}^{\,n+1} = G^{\varepsilon}(\vec{\tau}^{\,n}).$$

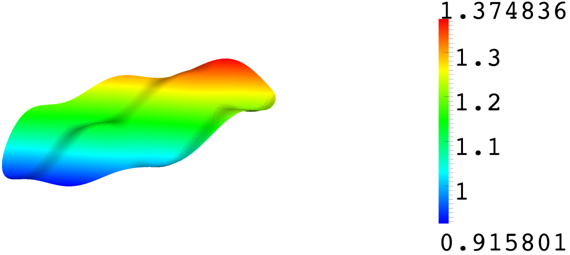
- At each step, we solve a scattering problem. We use a P2 finite element method set on the ball B_8. On $\partial\mathrm{B}_8$, a truncated Dirichlet-to-Neumann map with 13 harmonics serves as a transparent boundary condition.

- For the numerical experiments, we take $\mathcal{D} = \mathrm{B}_1$, $M = 3$ (3 directions of observation) and

$$\left|\begin{array}{ll} \boldsymbol{\theta}_{\text{inc}} = (\cos(\psi_{\text{inc}}), \sin(\psi_{\text{inc}})), & \psi_{\text{inc}} = 0^\circ \\ \boldsymbol{\theta}_1 = (\cos(\psi_1), \sin(\psi_1)), & \psi_1 = 90^\circ \\ \boldsymbol{\theta}_2 = (\cos(\psi_2), \sin(\psi_2)), & \psi_2 = 180^\circ \\ \boldsymbol{\theta}_3 = (\cos(\psi_3), \sin(\psi_3)), & \psi_3 = 225^\circ \end{array}\right.$$

Results: coefficient ρ at the end of the process





Results: scattered field

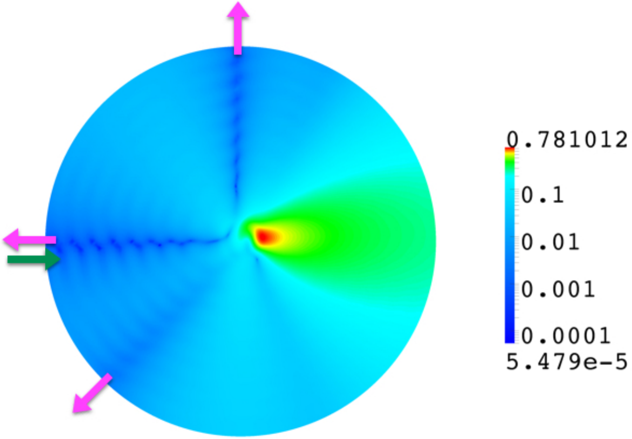


Figure: $|u_s|$ at the end of the fixed point procedure in logarithmic scale. As desired, we see it is very small far from $\mathcal{D}$ in the directions corresponding to the angles 90°, 180° and 225°. The domain is equal to B_8.

Results: far field pattern

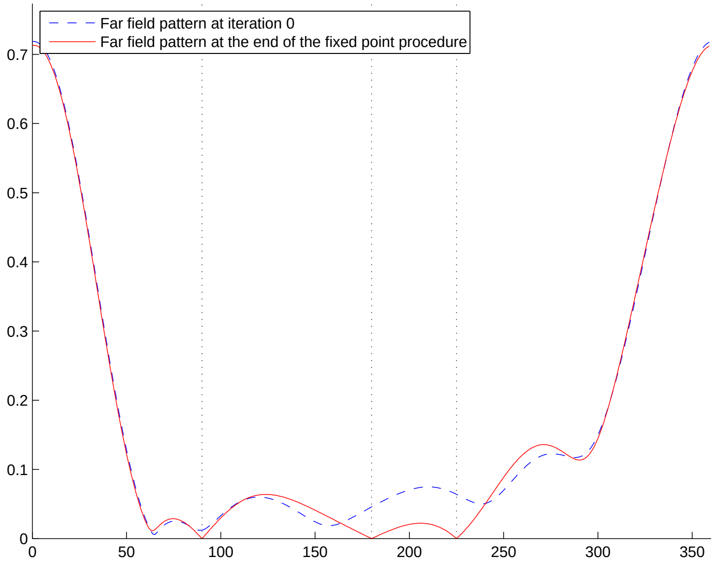
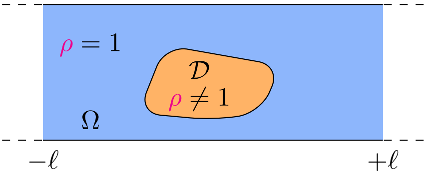


Figure: The dotted lines show the directions where we want u_s^∞ to vanish.

Waveguide problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\Omega := \{(x, y) \in \mathbb{R} \times (0; 1)\}$.



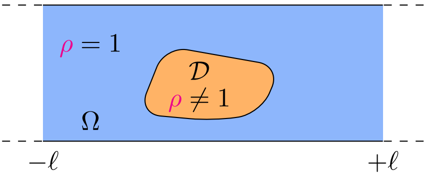
Find $u = u_{\mathrm{i}} + u_{\mathrm{s}}$ s. t.

$$\begin{aligned} -\Delta u &= k^2 \rho\, u && \text{in } \Omega, \\ \partial_n u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

u_{s} is outgoing.

Waveguide problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\Omega := \{(x, y) \in \mathbb{R} \times (0; 1)\}$.

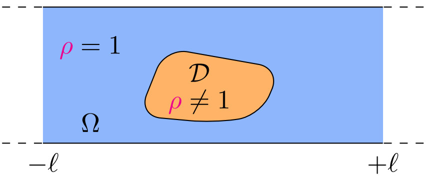


$$\left|\begin{array}{lcll} \multicolumn{4}{l}{\text{Find } u = u_{\mathrm{i}} + u_{\mathrm{s}} \text{ s. t.}} \\ -\Delta u & = & k^2 \rho\, u & \text{in } \Omega, \\ \partial_n u & = & 0 & \text{on } \partial\Omega, \\ \multicolumn{4}{c}{u_{\mathrm{s}} \text{ is outgoing.}} \end{array}\right.$$

- Set $k \in (0; \pi)$, $\quad w^{\pm} = e^{\pm ikx}/\sqrt{2k}$ $\quad$ and $\quad u_{\mathrm{i}} = w^{-}$.

Waveguide problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\Omega := \{(x, y) \in \mathbb{R} \times (0; 1)\}$.



Find $u = u_\mathrm{i} + u_\mathrm{s}$ s. t.

$$\begin{array}{rcll} -\Delta u & = & k^2 \rho\, u & \text{in } \Omega, \\ \partial_n u & = & 0 & \text{on } \partial\Omega, \end{array}$$

u_s is outgoing.

- Set $k \in (0; \pi)$, $\quad w^\pm = e^{\pm ikx}/\sqrt{2k}\quad$ and $\quad u_\mathrm{i} = w^-$.

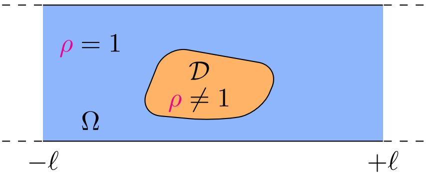
- u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$$\boxed{u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s},}$$

where $\tilde{u}_\mathrm{s}$ is exponentially decaying at $\pm\infty$.
(*$\chi^\pm$ are smooth cut-off functions s.t. $\chi^\pm = 1$ for $\pm x \geq 2\ell$, $\chi^\pm = 0$ for $\pm x \leq \ell$*)

Waveguide problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\Omega := \{(x,y) \in \mathbb{R} \times (0;1)\}$.



Find $u = u_{\mathrm{i}} + u_{\mathrm{s}}$ s. t.

$$\begin{array}{rcll} -\Delta u & = & k^2 \rho\, u & \text{in } \Omega, \\ \partial_n u & = & 0 & \text{on } \partial\Omega, \\ & & u_{\mathrm{s}} \text{ is outgoing.} & \end{array}$$

- Set $k \in (0;\pi)$, $\quad w^{\pm} = e^{\pm ikx}/\sqrt{2k} \quad$ and $\quad u_{\mathrm{i}} = w^{-}$.

- u_{s} is outgoing means that there are some $s^{\pm} \in \mathbb{C}$ such that

$$\boxed{u_{\mathrm{s}} = \chi^{+} s^{+} w^{+} + \chi^{-} s^{-} w^{-} + \tilde{u}_{\mathrm{s}},}$$

where $\tilde{u}_{\mathrm{s}}$ is exponentially decaying at $\pm\infty$.
($\chi^{\pm}$ *are smooth cut-off functions s.t.* $\chi^{\pm} = 1$ *for* $\pm x \geq 2\ell$, $\chi^{\pm} = 0$ *for* $\pm x \leq \ell$)

- Conservation of energy implies $|s^{-}|^2 + |1 + s^{+}|^2 = 1$.

Invisibility for waveguides

- u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$$u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s},$$

where $\tilde{u}_\mathrm{s}$ is exponentially decaying at $\pm\infty$.

At infinity, one measures the reflection coefficient s^- and/or the transmission coefficient $1 + s^+$ (other terms are too small).

Invisibility for waveguides

- u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$$u_s = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_s,$$

where $\tilde{u}_s$ is exponentially decaying at $\pm\infty$.

At infinity, one measures the reflection coefficient s^- and/or the transmission coefficient $1 + s^+$ (other terms are too small).

Definition: Inclusion is said
- non reflective if $s^- = 0$
- completely invisible if $s^+ = 0$.

Invisibility for waveguides

- u_s is outgoing means that there are some $s^{\pm} \in \mathbb{C}$ such that

$$u_s = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_s,$$

where $\tilde{u}_s$ is exponentially decaying at $\pm\infty$.

At infinity, one measures the reflection coefficient s^- and/or the transmission coefficient $1 + s^+$ (other terms are too small).

Definition: Inclusion is said	non reflective if $s^- = 0$ completely invisible if $s^+ = 0$.

- Due to conservation of energy $|s^-|^2 + |1 + s^+|^2 = 1$,

$$s^+ = 0 \quad \Rightarrow \quad s^- = 0 \qquad \text{(and } u_s \text{ is expo. decay. at } \pm\infty\text{)}.$$

The converse is wrong ($s^- = 0 \not\Rightarrow s^+ = 0$).

Penetrable inclusion

- For $\sigma = \rho - 1$ (contrast), gather the measurements in

$$F(\sigma) = (\Re e\,(s^-/(ik)), \Im m\,(s^-/(ik)), \Re e\,(s^+/(ik)), \Im m\,(s^+/(ik))).$$

Again, we wish to find $\sigma \not\equiv 0$ such that $F(\sigma) = 0$.

Penetrable inclusion

- For $\sigma = \rho - 1$ (contrast), gather the measurements in

$$F(\sigma) = (\Re e\,(s^-/(ik)), \Im m\,(s^-/(ik)), \Re e\,(s^+/(ik)), \Im m\,(s^+/(ik))).$$

- As in free space, we obtain

$$dF(0)(\mu) = \left(\int_{\mathcal{D}} \mu \cos(2kx)\, d\boldsymbol{x}, \int_{\mathcal{D}} \mu \sin(2kx)\, d\boldsymbol{x}, \int_{\mathcal{D}} \mu \, d\boldsymbol{x},\ 0 \right).$$

Penetrable inclusion

- For $\sigma = \rho - 1$ (contrast), gather the measurements in

$$F(\sigma) = (\Re e\,(s^-/(ik)), \Im m\,(s^-/(ik)), \Re e\,(s^+/(ik)), \Im m\,(s^+/(ik))).$$

- As in free space, we obtain

$$dF(0)(\mu) = \left(\int_{\mathcal{D}} \mu \cos(2kx)\, d\boldsymbol{x}, \int_{\mathcal{D}} \mu \sin(2kx)\, d\boldsymbol{x}, \int_{\mathcal{D}} \mu\, d\boldsymbol{x}, 0\right).$$

- We can get $s^- = 0$ but $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^4$ is not onto.

Penetrable inclusion

- For $\sigma = \rho - 1$ (contrast), gather the measurements in

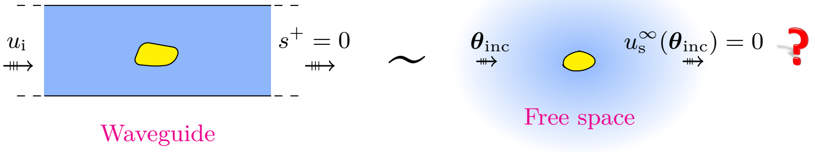
$$F(\sigma) = (\Re e\,(s^-/(ik)), \Im m\,(s^-/(ik)), \Re e\,(s^+/(ik)), \Im m\,(s^+/(ik))).$$

- As in free space, we obtain

$$dF(0)(\mu) = \left(\int_{\mathcal{D}} \mu \cos(2kx)\,d\boldsymbol{x}, \int_{\mathcal{D}} \mu \sin(2kx)\,d\boldsymbol{x}, \int_{\mathcal{D}} \mu\,d\boldsymbol{x}, 0\right).$$

- We can get $s^- = 0$ but $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^4$ is not onto.

Can we get $s^+ = 0$ or



Penetrable inclusion

- For $\sigma = \rho - 1$ (contrast), gather the measurements in

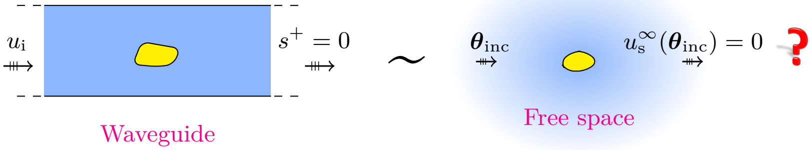
$$F(\sigma) = (\Re e\,(s^-/(ik)), \Im m\,(s^-/(ik)), \Re e\,(s^+/(ik)), \Im m\,(s^+/(ik))).$$

- As in free space, we obtain

$$dF(0)(\mu) = \left(\int_{\mathcal{D}} \mu \cos(2kx)\,d\boldsymbol{x}, \int_{\mathcal{D}} \mu \sin(2kx)\,d\boldsymbol{x}, \int_{\mathcal{D}} \mu\,d\boldsymbol{x}, 0\right).$$

- We can get $s^- = 0$ but $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^4$ is not onto.

Can we get $s^+ = 0$ or



Impose $\left|\begin{array}{l} s^- = 0 \\ \Im m\, s^+ = 0 \end{array}\right.$. *Then,* $\left|\begin{array}{l} |s^-|^2 + |1 + s^+|^2 = 1 \\ s^+ = O(\varepsilon) \end{array}\right.$ $\Rightarrow$ $s^+ = 0$.

Penetrable inclusion

- For $\sigma = \rho - 1$ (contrast), gather the measurements in

$$F(\sigma) = \left(\Re e\,(s^-/(ik)), \Im m\,(s^-/(ik)), \Re e\,(s^+/(ik)), \Im m\,(s^+/(ik))\right).$$

- As in free space, we obtain

$$dF(0)(\mu) = \left(\int_{\mathcal{D}} \mu \cos(2kx)\,dx, \int_{\mathcal{D}} \mu \sin(2kx)\,dx, \int_{\mathcal{D}} \mu\,dx, 0\right).$$

With this approach, we produce small contrast invisible perturbations of the reference medium.

- [illegible] 0 but $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^4$ is not onto.

Can we get $s^+ = 0$ or

Impose $\left| \begin{array}{l} s^- = 0 \\ \Im m\, s^+ = 0 \end{array} \right.$. Then, $\left| \begin{array}{l} |s^-|^2 + |1 + s^+|^2 = 1 \\ s^+ = O(\varepsilon) \end{array} \right.$ $\Rightarrow$ $s^+ = 0$.

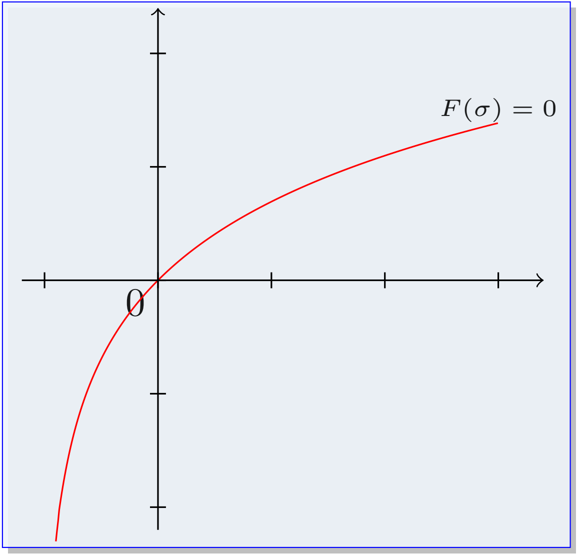
Penetrable inclusion

With this approach, we produce small contrast invisible perturbations of the reference medium.

Can we increase the perturbation to obtain larger defects ?

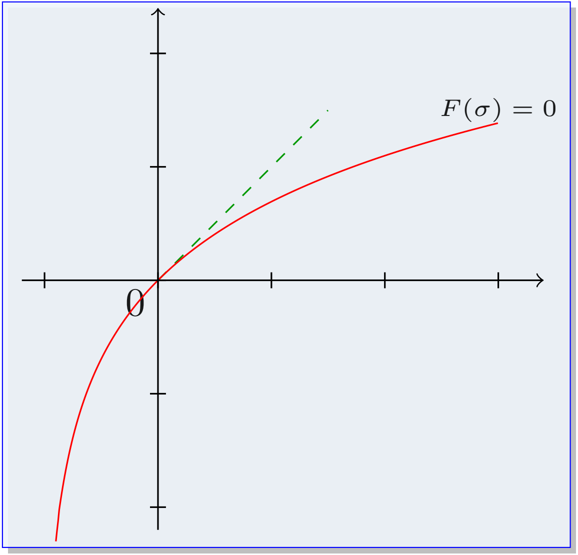
Can we increase the perturbation?

- Schematic view of what we did ($F : \mathbb{R}^2 \to \mathbb{R}$ is the measurements map):



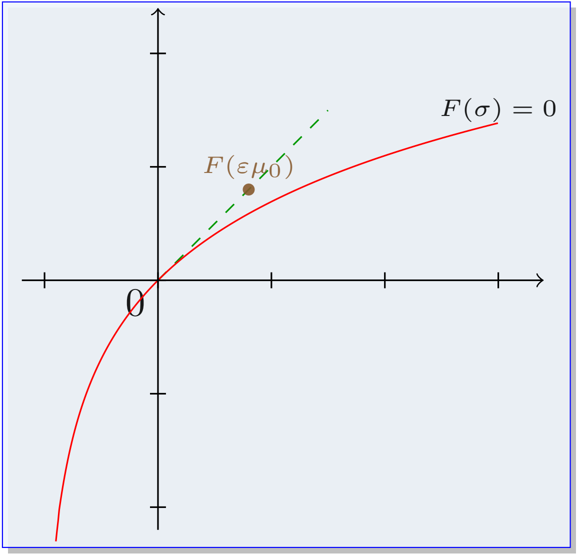
Can we increase the perturbation?

- Schematic view of what we did ($F : \mathbb{R}^2 \to \mathbb{R}$ is the measurements map):



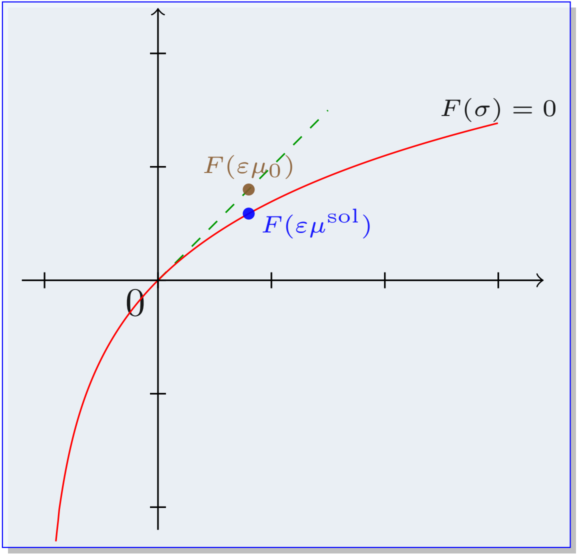
Can we increase the perturbation?

- Schematic view of what we did ($F : \mathbb{R}^2 \to \mathbb{R}$ is the measurements map):



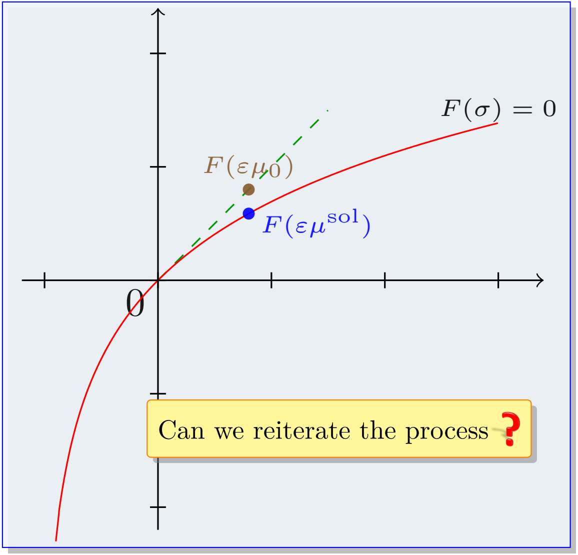
Can we increase the perturbation?

- Schematic view of what we did ($F : \mathbb{R}^2 \to \mathbb{R}$ is the measurements map):



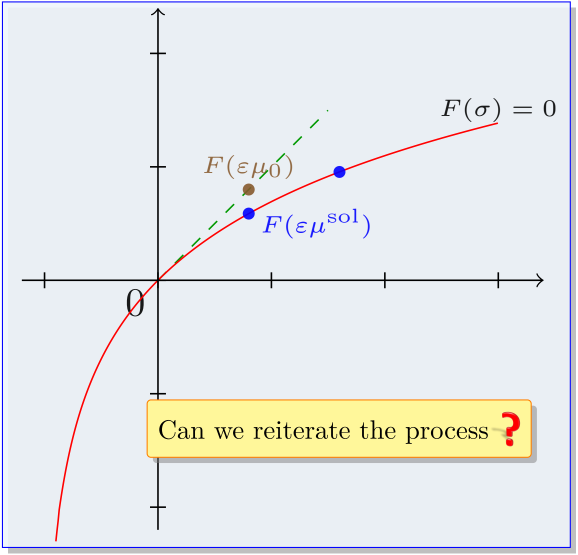
Can we increase the perturbation?

- Schematic view of the process to construct larger invisible defects:



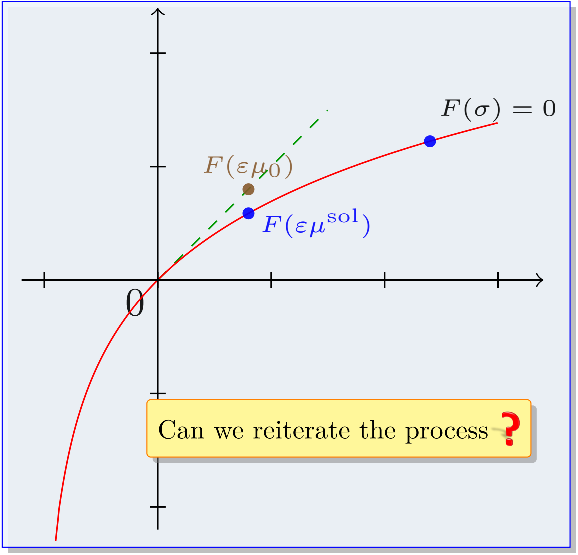
Can we increase the perturbation?

- Schematic view of the process to construct larger invisible defects:



Can we increase the perturbation?

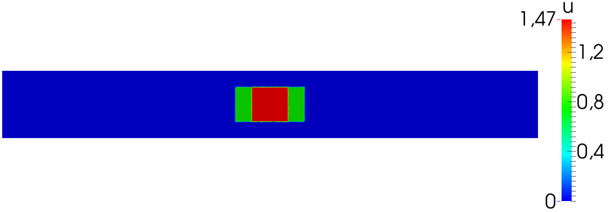
- Schematic view of the process to construct larger invisible defects:



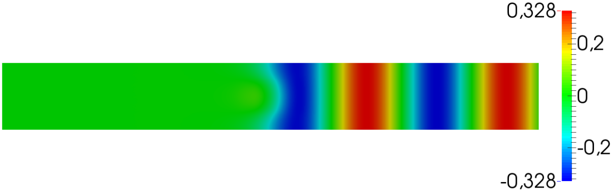
Numerical results to impose $s^- = 0$

- We set $k = 3$, $\mathcal{D} = (-\pi/k; \pi/k) \times (1/4; 3/4)$, 3 steps of iterations.

$\sigma = \rho - 1$



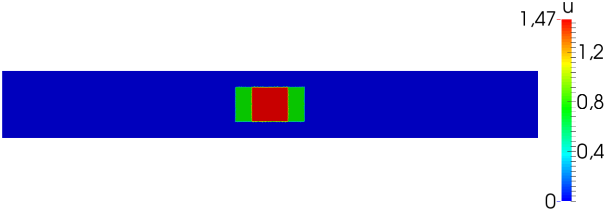
$\Re e\, u_{\mathrm{s}}$



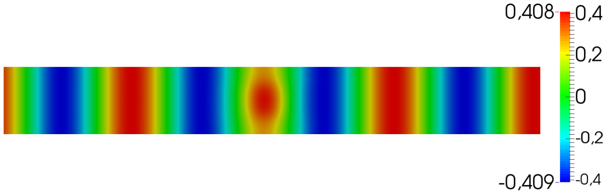
Numerical results to impose $s^- = 0$

- We set $k = 3$, $\mathcal{D} = (-\pi/k; \pi/k) \times (1/4; 3/4)$, 3 steps of iterations.

$\sigma = \rho - 1$



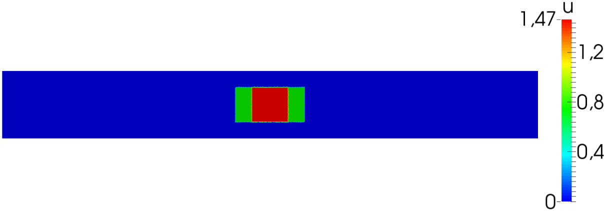
$\Re e\, u$



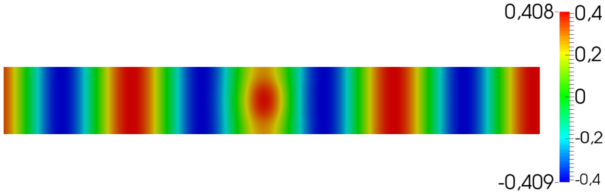
Numerical results to impose $s^- = 0$

- We set $k = 3$, $\mathcal{D} = (-\pi/k; \pi/k) \times (1/4; 3/4)$, 3 steps of iterations.

$\sigma = \rho - 1$



$\Re e\, u$

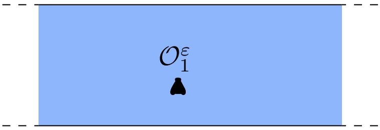


$\rightarrow$ First results are encouraging. Still some questions: at each step, how to choose the new directions?

$\rightarrow$ We are not able to prove that $ds^-(\sigma) : \mathrm{L}^\infty(\mathcal{D}) \rightarrow \mathbb{C}$ is onto for $\sigma \not\equiv 0$.

Small Dirichlet obstacle

- Can one hide a small Dirichlet obstacle centered at M_1?



Find $u = u_\mathrm{i} + u_\mathrm{s}$ s. t.

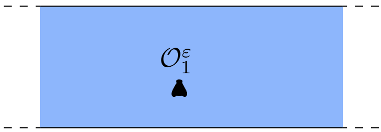
$$\begin{aligned} -\Delta u &= k^2 u \quad \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ u &= 0 \quad \text{on } \partial\Omega^\varepsilon, \\ & u_\mathrm{s} \text{ is outgoing.} \end{aligned}$$

- Again, u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$$\boxed{u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s},}$$ with $\tilde{u}_\mathrm{s}$ is expo. decaying at $\pm\infty$.

Small Dirichlet obstacle

- Can one hide a small Dirichlet obstacle centered at M_1?



Find $u = u_\mathrm{i} + u_\mathrm{s}$ s. t.

$$\begin{array}{rcll} -\Delta u &=& k^2 u & \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ u &=& 0 & \text{on } \partial\Omega^\varepsilon, \\ & & u_\mathrm{s} \text{ is outgoing.} & \end{array}$$

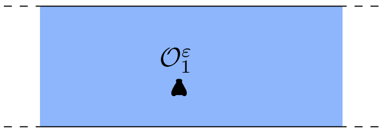
- Again, u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$$\boxed{u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s},} \text{ with } \tilde{u}_\mathrm{s} \text{ is expo. decaying at } \pm\infty.$$

Due to Dirichlet B.C., $w^\pm$ are not the same as previously (but this not important).

Small Dirichlet obstacle

- Can one hide a small Dirichlet obstacle centered at M_1?



Find $u = u_\mathrm{i} + u_\mathrm{s}$ s. t.

$$\begin{aligned} -\Delta u &= k^2 u \quad \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ u &= 0 \quad \text{on } \partial\Omega^\varepsilon, \\ &u_\mathrm{s} \text{ is outgoing.} \end{aligned}$$

- Again, u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

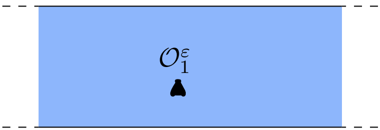
$$\boxed{u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s},}$$ with $\tilde{u}_\mathrm{s}$ is expo. decaying at $\pm\infty$.

- In 3D, we obtain

$$\begin{aligned} s^- &= 0 + \varepsilon\ (4i\pi\, \mathrm{cap}(\mathcal{O}) w^+(M_1)^2)\ + O(\varepsilon^2) \\ s^+ &= 0 + \varepsilon\ (4i\pi\, \mathrm{cap}(\mathcal{O}) |w^+(M_1)|^2)\ + O(\varepsilon^2). \end{aligned}$$

Small Dirichlet obstacle

- Can one hide a small Dirichlet obstacle centered at M_1?



Find $u = u_{\mathrm{i}} + u_{\mathrm{s}}$ s. t.

$$\begin{aligned} -\Delta u &= k^2 u && \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ u &= 0 && \text{on } \partial\Omega^\varepsilon, \\ & u_{\mathrm{s}} \text{ is outgoing.} \end{aligned}$$

- Again, u_{s} is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$$u_{\mathrm{s}} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_{\mathrm{s}},$$ with $\tilde{u}_{\mathrm{s}}$ is expo. decaying at $\pm\infty$.

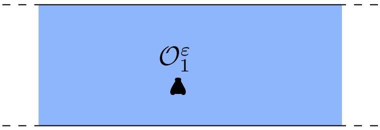
- In 3D, we obtain

$$\begin{aligned} s^- &= 0 + \varepsilon\, \left(4i\pi \operatorname{cap}(\mathcal{O}) w^+(M_1)^2\right) + O(\varepsilon^2) \\ s^+ &= 0 + \varepsilon\, \left(4i\pi \operatorname{cap}(\mathcal{O}) |w^+(M_1)|^2\right) + O(\varepsilon^2). \end{aligned}$$

Non zero terms!
($\operatorname{cap}(\mathcal{O}) > 0$)

Small Dirichlet obstacle

- Can one hide a small Dirichlet obstacle centered at M_1?



Find $u = u_\mathrm{i} + u_\mathrm{s}$ s. t.

$$\begin{aligned} -\Delta u &= k^2 u \quad \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ u &= 0 \quad \text{on } \partial\Omega^\varepsilon, \\ & u_\mathrm{s} \text{ is outgoing.} \end{aligned}$$

- Again, u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$$u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s},$$ with $\tilde{u}_\mathrm{s}$ is expo. decaying at $\pm\infty$.

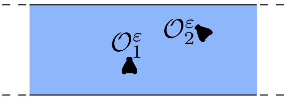
- In 3D, we obtain

$$\begin{aligned} s^- &= 0 + \varepsilon \, \left(4i\pi \, \mathrm{cap}(\mathcal{O}) w^+(M_1)^2\right) + O(\varepsilon^2) \\ s^+ &= 0 + \varepsilon \, \left(4i\pi \, \mathrm{cap}(\mathcal{O}) |w^+(M_1)|^2\right) + O(\varepsilon^2). \end{aligned}$$

Non zero terms!
($\mathrm{cap}(\mathcal{O}) > 0$)

$\Rightarrow$ One single small obstacle cannot even be non reflective.

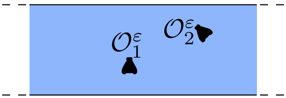
Small Dirichlet obstacles



- Let us try with two small Dirichlet obstacles centered at M_1, M_2.

- We obtain
$$\begin{aligned} s^- &= 0 + \varepsilon\,(4i\pi\,\mathrm{cap}(\mathcal{O})\sum_{n=1}^{2} w^+(M_n)^2) + O(\varepsilon^2) \\ s^+ &= 0 + \varepsilon\,(4i\pi\,\mathrm{cap}(\mathcal{O})\sum_{n=1}^{2} |w^+(M_n)|^2) + O(\varepsilon^2). \end{aligned}$$

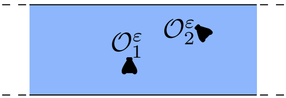
Small Dirichlet obstacles



- Let us try with two small Dirichlet obstacles centered at M_1, M_2.

- We obtain
$$s^- = 0 + \varepsilon \boxed{\left(4i\pi\,\mathrm{cap}(\mathcal{O}) \sum_{n=1}^{2} w^+(M_n)^2\right)} + O(\varepsilon^2)$$
$$s^+ = 0 + \varepsilon \left(4i\pi\,\mathrm{cap}(\mathcal{O}) \sum_{n=1}^{2} |w^+(M_n)|^2\right) + O(\varepsilon^2).$$

Small Dirichlet obstacles



- Let us try with two small Dirichlet obstacles centered at M_1, M_2.

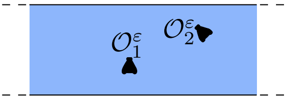
- We obtain

$$s^- = 0 + \varepsilon \left(4i\pi \,\mathrm{cap}(\mathcal{O}) \sum_{n=1}^{2} w^+(M_n)^2\right) + O(\varepsilon^2)$$

$$s^+ = 0 + \varepsilon \left(4i\pi \,\mathrm{cap}(\mathcal{O}) \sum_{n=1}^{2} |w^+(M_n)|^2\right) + O(\varepsilon^2).$$

We can find M_1, M_2 such that $s^- = O(\varepsilon^2)$.

Small Dirichlet obstacles



- Let us try with two small Dirichlet obstacles centered at M_1, M_2.

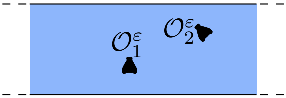
- We obtain

$$s^- = 0 + \varepsilon \left(4i\pi \,\mathrm{cap}(\mathcal{O}) \sum_{n=1}^{2} w^+(M_n)^2\right) + O(\varepsilon^2)$$

$$s^+ = 0 + \varepsilon \left(4i\pi \,\mathrm{cap}(\mathcal{O}) \sum_{n=1}^{2} |w^+(M_n)|^2\right) + O(\varepsilon^2).$$

We can find M_1, M_2 such that $s^- = O(\varepsilon^2)$. Then moving $\mathcal{O}_1^\varepsilon$ from M_1 to $M_1 + \varepsilon\tau$, and choosing a good $\tau \in \mathbb{R}^3$ (fixed point), we can get $s^- = 0$.

Small Dirichlet obstacles



- Let us try with two small Dirichlet obstacles centered at M_1, M_2.

- We obtain

$$\begin{aligned} s^- &= 0 + \varepsilon \boxed{\left(4i\pi \,\text{cap}(\mathcal{O}) \sum_{n=1}^{2} w^+(M_n)^2\right)} + O(\varepsilon^2) \\ s^+ &= 0 + \varepsilon \left(4i\pi \,\text{cap}(\mathcal{O}) \sum_{n=1}^{2} |w^+(M_n)|^2\right) + O(\varepsilon^2). \end{aligned}$$

We can find M_1, M_2 such that $s^- = O(\varepsilon^2)$. Then moving $\mathcal{O}_1^\varepsilon$ from M_1 to $M_1 + \varepsilon\tau$, and choosing a good $\tau \in \mathbb{R}^3$ (fixed point), we can get $s^- = 0$.

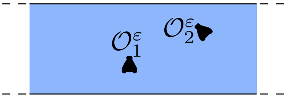
COMMENTS:

→ Hard part is to justify the asymptotics for the fixed point problem.

→ We cannot impose $s^+ = 0$ with this strategy.

→ When there are more propagative waves, we need more obstacles.

Small Dirichlet obstacles



- Let us try with two small Dirichlet obstacles centered at M_1, M_2.

- We obtain
$$s^- = 0 + \varepsilon \left(4i\pi\,\mathrm{cap}(\mathcal{O}) \sum_{n=1}^{2} w^+(M_n)^2\right) + O(\varepsilon^2)$$
$$s^+ = 0 + \varepsilon \left(4i\pi\,\mathrm{cap}(\mathcal{O}) \sum_{n=1}^{2} |w^+(M_n)|^2\right) + O(\varepsilon^2).$$

We can find M_1, M_2 such that $s^- = O(\varepsilon^2)$. Then moving $\mathcal{O}_1^\varepsilon$ from M_1 to $M_1 + \varepsilon\tau$, and choosing a good $\tau \in \mathbb{R}^3$ (fixed point), we can get $s^- = 0$.

Comments:

→ Hard part is to justify the asymptotics for the fixed point problem.

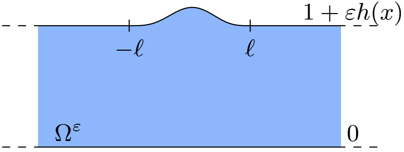
→ We cannot impose $s^+ = 0$ with this strategy.

→ When there are more propagative waves, we need more obstacles.

Acting as a team, flies can become invisible!

Can one hide a perturbation of the wall?

- Pick $h \in \mathscr{C}_0^\infty(-\ell;\ell)$, $\ell > 0$. Set $k \in (0;\pi)$, $w^\pm = e^{\pm ikx}/\sqrt{2k}$, $u_\mathrm{i} = w^-$.



Find $u = u_\mathrm{i} + u_\mathrm{s}$ s. t.

$$\begin{array}{rcll} -\Delta u &=& k^2 u & \text{in } \Omega^\varepsilon, \\ \partial_n u &=& 0 & \text{on } \partial\Omega^\varepsilon, \end{array}$$

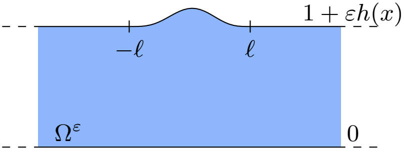
u_s is outgoing.

- Again, u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$$\boxed{u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s},}$$ with $\tilde{u}_\mathrm{s}$ is expo. decaying at $\pm\infty$.

Can one hide a perturbation of the wall?

- Pick $h \in \mathscr{C}_0^\infty(-\ell;\ell)$, $\ell > 0$. Set $k \in (0;\pi)$, $w^\pm = e^{\pm ikx}/\sqrt{2k}$, $u_\mathrm{i} = w^-$.



Find $u = u_\mathrm{i} + u_\mathrm{s}$ s. t.

$$\begin{array}{rcll} -\Delta u &=& k^2 u & \text{in } \Omega^\varepsilon, \\ \partial_n u &=& 0 & \text{on } \partial\Omega^\varepsilon, \end{array}$$

u_s is outgoing.

- Again, u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

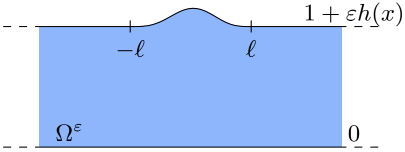
$u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s}$, with $\tilde{u}_\mathrm{s}$ is expo. decaying at $\pm\infty$.

- We obtain

$$\begin{array}{rcl} s^- &=& 0 + \varepsilon \left(-\frac{1}{2} \int_{-\ell}^{\ell} \partial_x h(x) (w^+(x))^2 \, dx \right) + O(\varepsilon^2) \\ s^+ &=& 0 + \varepsilon \; 0 \; + O(\varepsilon^2). \end{array}$$

Can one hide a perturbation of the wall?

- Pick $h \in \mathscr{C}_0^\infty(-\ell;\ell)$, $\ell > 0$. Set $k \in (0;\pi)$, $w^\pm = e^{\pm ikx}/\sqrt{2k}$, $u_\mathrm{i} = w^-$.



Find $u = u_\mathrm{i} + u_\mathrm{s}$ s. t.

$$\begin{array}{rcll} -\Delta u &=& k^2 u & \text{in } \Omega^\varepsilon, \\ \partial_n u &=& 0 & \text{on } \partial\Omega^\varepsilon, \end{array}$$

u_s is outgoing.

- Again, u_s is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

$u_\mathrm{s} = \chi^+ s^+ w^+ + \chi^- s^- w^- + \tilde{u}_\mathrm{s}$, with $\tilde{u}_\mathrm{s}$ is expo. decaying at $\pm\infty$.

- We obtain

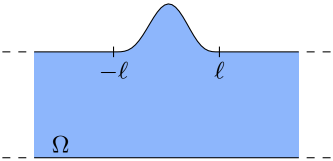
$$\begin{array}{rcl} s^- &=& 0 + \varepsilon \left(-\dfrac{1}{2} \displaystyle\int_{-\ell}^{\ell} \partial_x h(x) (w^+(x))^2 \, dx \right) + O(\varepsilon^2) \\ s^+ &=& 0 + \varepsilon \, 0 + O(\varepsilon^2). \end{array}$$

$\Rightarrow$ With this approach, we can impose $s^- = 0$ but not $s^+ = 0$.

Can one hide a perturbation of the wall?

- More generally, for any Neumann waveguide, one can show that $s^+ = 0$ implies

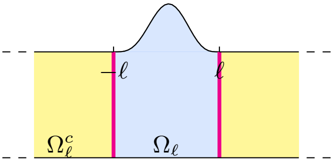
$$\int_\Omega |\nabla u_\mathrm{s}|^2 - k^2 |u_\mathrm{s}|^2 \, d\boldsymbol{x} = 0.$$



Can one hide a perturbation of the wall?

- More generally, for any Neumann waveguide, one can show that $s^+ = 0$ implies

$$\int_\Omega |\nabla u_s|^2 - k^2|u_s|^2 \, d\boldsymbol{x} = 0.$$



- Decomposing in Fourier series, one finds

$$\int_{\Omega_\ell^c} |\nabla u_s|^2 - k^2|u_s|^2 \, d\boldsymbol{x} \geq 0.$$

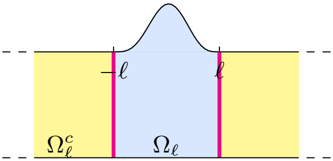
- Note that $s^+ = 0 \Rightarrow u_s \in \mathrm{Y} := \{v \in \mathrm{H}^1(\Omega_\ell) \,|\, \int_{x=\pm\ell} v \, d\sigma = 0\}$. Define

$$\lambda_\dagger := \inf_{v \in \mathrm{Y}\setminus\{0\}} \left(\int_{\Omega_\ell} |\nabla v|^2 \, d\boldsymbol{x}\right) \Big/ \left(\int_{\Omega_\ell} |v|^2 \, d\boldsymbol{x}\right) > 0.$$

Can one hide a perturbation of the wall?

- More generally, for any Neumann waveguide, one can show that $s^+ = 0$ implies

$$\int_\Omega |\nabla u_s|^2 - k^2 |u_s|^2 \, d\boldsymbol{x} = 0.$$



- Decomposing in Fourier series, one finds

$$\int_{\Omega_\ell^c} |\nabla u_s|^2 - k^2 |u_s|^2 \, d\boldsymbol{x} \geq 0.$$

- Note that $s^+ = 0 \Rightarrow u_s \in \mathrm{Y} := \{v \in \mathrm{H}^1(\Omega_\ell) \,|\, \int_{x=\pm\ell} v \, d\sigma = 0\}$. Define

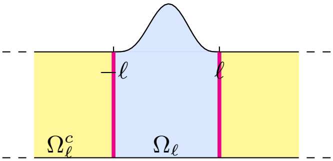
$$\lambda_\dagger := \inf_{v \in \mathrm{Y}\setminus\{0\}} \left(\int_{\Omega_\ell} |\nabla v|^2 \, d\boldsymbol{x}\right) \Big/ \left(\int_{\Omega_\ell} |v|^2 \, d\boldsymbol{x}\right) > 0.$$

PROPOSITION: For a given shape, $s^+ = 0$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

Can one hide a perturbation of the wall?

- More generally, for any Neumann waveguide, one can show that $s^+ = 0$ implies

$$\int_\Omega |\nabla u_s|^2 - k^2 |u_s|^2 \, d\boldsymbol{x} = 0.$$



- Decomposing in Fourier series, one finds

$$\int_{\Omega_\ell^c} |\nabla u_s|^2 - k^2 |u_s|^2 \, d\boldsymbol{x} \geq 0.$$

- Note that $s^+ = 0 \Rightarrow u_s \in \mathrm{Y} := \{v \in \mathrm{H}^1(\Omega_\ell) \,|\, \int_{x=\pm\ell} v \, d\sigma = 0\}$. Define

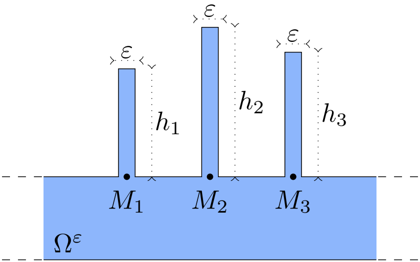
$$\lambda_\dagger := \inf_{v \in \mathrm{Y} \setminus \{0\}} \left(\int_{\Omega_\ell} |\nabla v|^2 \, d\boldsymbol{x} \right) \Big/ \left(\int_{\Omega_\ell} |v|^2 \, d\boldsymbol{x} \right) > 0.$$

PROPOSITION: For a given shape, $s^+ = 0$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

→ To impose invisibility at low frequency, we need to work with special shapes.

Non smooth perturbation of the wall

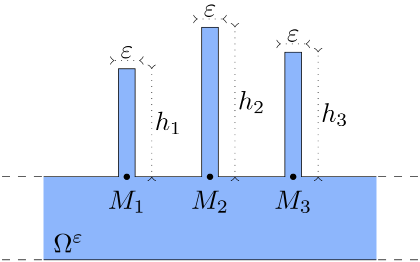
- We study the same problem in the geometry Ω^ε



- We obtain
$$\begin{aligned} s^- &= 0 + \varepsilon \left(ik \textstyle\sum_{n=1}^3 (w^+(M_n))^2 \tan(kh_n) \right) + O(\varepsilon^2) \\ s^+ &= 0 + \varepsilon \left(i/2 \textstyle\sum_{n=1}^3 \tan(kh_n) \right) + O(\varepsilon^2) \end{aligned}$$

Non smooth perturbation of the wall

- We study the same problem in the geometry Ω^ε



- We obtain

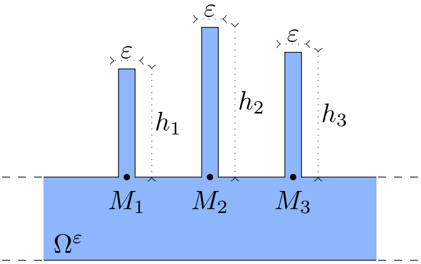
$$s^- = 0 + \varepsilon \left(ik \sum_{n=1}^{3} (w^+(M_n))^2 \tan(kh_n) \right) + O(\varepsilon^2)$$

$$s^+ = 0 + \varepsilon \left(i/2 \sum_{n=1}^{3} \tan(kh_n) \right) + O(\varepsilon^2)$$

1) We can find M_n, h_n such that $s^- = O(\varepsilon^2)$ *and* $s^+ = O(\varepsilon^2)$.

Non smooth perturbation of the wall

- We study the same problem in the geometry Ω^ε



- We obtain

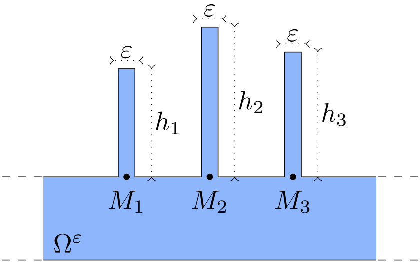
$$\begin{aligned} s^- &= 0 + \varepsilon \left(ik \sum_{n=1}^{3} (w^+(M_n))^2 \tan(kh_n) \right) + O(\varepsilon^2) \\ s^+ &= 0 + \varepsilon \left(i/2 \sum_{n=1}^{3} \tan(kh_n) \right) + O(\varepsilon^2) \end{aligned}$$

1) We can find M_n, h_n such that $s^- = O(\varepsilon^2)$ *and* $s^+ = O(\varepsilon^2)$.

2) Then changing h_n into $h_n + \tau_n$, and choosing a good $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ (fixed point), we can get $s^- = 0$ *and* $\Im m\, s^+ = 0$.

Non smooth perturbation of the wall

- We study the same problem in the geometry Ω^ε



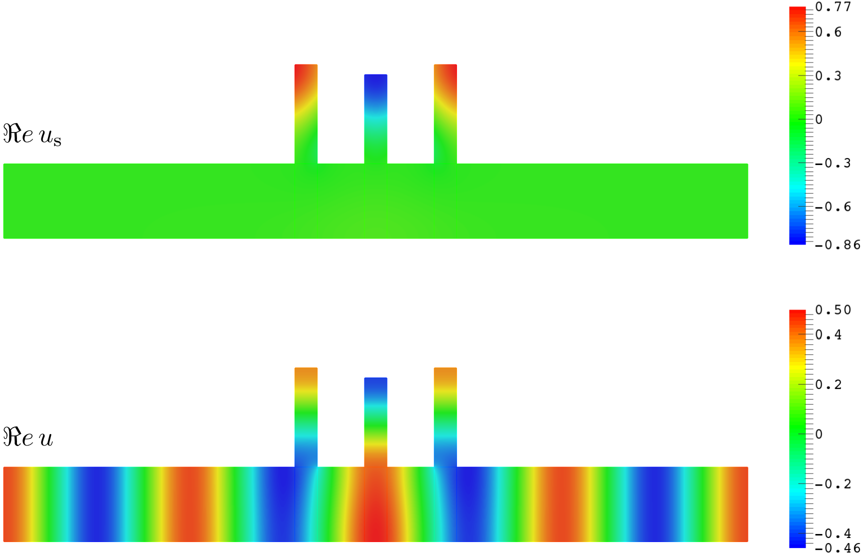
- We obtain
$$s^- = 0 + \varepsilon \left(ik \sum_{n=1}^{3} (w^+(M_n))^2 \tan(kh_n) \right) + O(\varepsilon^2)$$
$$s^+ = 0 + \varepsilon \left(i/2 \sum_{n=1}^{3} \tan(kh_n) \right) + O(\varepsilon^2)$$

1) We can find M_n, h_n such that $s^- = O(\varepsilon^2)$ and $s^+ = O(\varepsilon^2)$.

2) Then changing h_n into $h_n + \tau_n$, and choosing a good $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ (fixed point), we can get $s^- = 0$ and $\Im m\, s^+ = 0$.

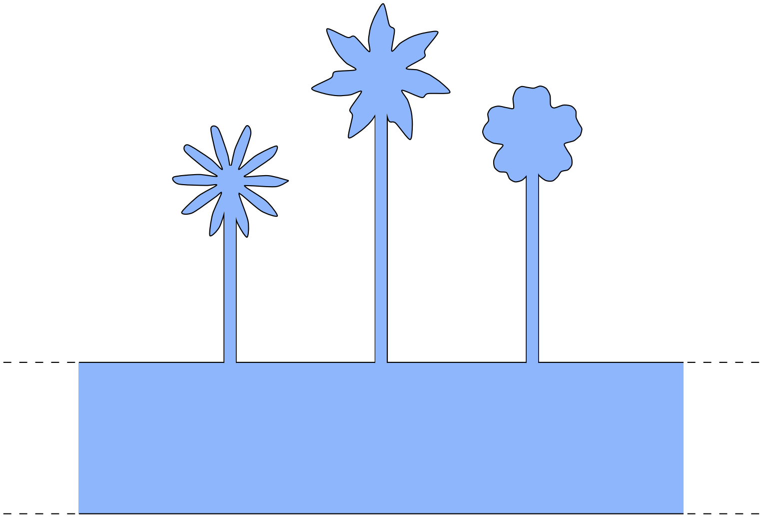
3) Energy conservation + $[s^+ = O(\varepsilon)]$ $\Rightarrow$ $s^+ = 0$.

Numerical results



Remark

- We could also have worked with gardens of flowers!



1. Invisibility in free space
   - The general scheme
   - The forbidden case
   - Numerical experiments
2. Invisibility for waveguide problems
   - Construction of invisible penetrable defects
   - Can one hide a small Dirichlet obstacle?
   - Can one hide a perturbation of the wall?

Conclusion

What we did

- ♠ We explained how to construct invisible perturbations of a reference situation in a setting with a finite number of measurements.

Future work

1) We want to continue the analysis of the reiteration process to construct large invisible defects of the reference medium.

2) It would be interesting to consider other models (Maxwell, elasticity, ...) and to investigate cases where the differential is not onto.

3) For a given perturbation, can we study the frequencies (invisible modes) such that invisibility holds?

4) We wish to better understand the link between the invisible modes and the so-called trapped modes in waveguides.

Thank you for your attention!!!