Interior transmission eigenvalue problem for Maxwell's equations: the T-coercivity as an alternative approach

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Abstract. In this paper, we examine the interior transmission problem for Maxwell's equations in the case where both ε and μ, the physical parameters of the scattering medium, differ from ε_0 and μ_0 modeling the background medium. Using the T-coercivity method, we propose an alternative approach to the classical techniques to prove that this problem is of Fredholm type and that the so-called transmission eigenvalues form at most a discrete set. The T-coercivity approach allows us to deal with cases where $\varepsilon - \varepsilon_0$ and $\mu - \mu_0$ can change sign. We also provide results of localization and Faber-Krahn type inequalities for the transmission eigenvalues.

1. Introduction

The term "interior transmission eigenvalue problem" refers to a family of spectral problems which appear in scattering theory. In particular, they arise when one is interested in the reconstruction of an inclusion embedded in a background medium from multi-static measurements of scattered fields at a given frequency. These problems, unlike conventional transmission problems, involve two similar PDEs in the same domain, coupled via transmission conditions on the boundary. The values of the frequency for which the homogeneous interior transmission problem admits nontrivial solutions are called transmission eigenvalues. More physically, transmission eigenvalues can be seen as frequencies for which there exists an incident field, superposition of incident plane waves, such that the corresponding scattered field is arbitrary small. An important issue is to prove that transmission eigenvalues form at most a discrete set with infinity as the only accumulation point.

In this paper, we will concentrate on interior transmission eigenvalue problems in electromagnetism. The permittivity and permeability of the scattering medium are denoted ε and μ whereas the physical parameters of the background medium are ε_0 and μ_0. To simplify the notations, we introduce A and N such that $\varepsilon = \varepsilon_0\,N$ and $\mu = \mu_0\,A^{-1}$. Research has focused primarily on the case where the scattering medium is characterized by one contrast function: $A \neq Id, N = Id$ or $A = Id, N \neq Id$. Both the scalar [16, 27, 17, 18] and Maxwell [20, 11, 7] problems have been widely studied but questions remain open. We note that a nice step forward was made recently by Sylvester for the scalar problem in the case $A = Id, N = nId$, n being a scalar function. In [28], he indeed proved that the eigenvalues form at most a discrete set as soon as $n-1$ is positive (or negative) in a neighbourhood of the boundary. To our knowledge, it is still an open problem to prove an equivalent result when $A \neq Id, N = Id$.

In practice, it is quite restrictive to model the scattering medium by only one parameter. Therefore, some authors have introduced an interior transmission eigenvalue problem with $A \neq Id$ and $N \neq Id$ [6, 21, 13]. From a technical point of view, the sesquilinear form associated with the natural variational formulation of this interior transmission problem exhibits a sign-changing in its principal part. Consequently, the associated operator is not strongly elliptic and its study is not standard. One observes an equivalent difficulty in the study of the transmission problem between a dielectric and a negative metamaterial in the time-harmonic regime. To tackle it, we can use the T-coercivity technique [5, 2, 4, 8, 10]. The idea consists in testing, in variational formulations, not directly against the field, but against a simple geometrical transformation of the field. This allows one to restore some properties of positivity for the associated operators. In [4], thanks to this simple approach, we have been able to extend the results of [6, 13] for the scalar problem associated with (2): only the values of $A - Id$ in a neighbourhood of the boundary actually matter for determining whether or not the problem is of Fredholm type. In this paper, we complete the results obtained in [13] for the Maxwell problem. More precisely, we prove, using the T-coercivity method, that this problem is of Fredholm type and that transmission eigenvalues form at most a discrete set in situations for which $A - Id$ and $N - Id$ are either positive or negative in a neighbourhood of the boundary and can change sign inside the domain. Under more restrictive conditions on A and N, we also provide estimates for the first eigenvalue.

Another important question for these interior transmission eigenvalue problems is to prove the existence of transmission eigenvalues which can then help in determining the values of the physical parameters of the inclusion. This question will not be dealt with here. Indeed, up to now, the T-coercivity approach appears inefficient to show these kind of results because the formulation we work on, although it presents some useful property of positivity, is not symmetric. This prevents using the nice *min-max* arguments (see [18, 12, 9]). Hence, the question of existence of real transmission eigenvalues when $A-Id$ or $N - Id$ change sign, both for scalar and Maxwell problems, remains open.

This paper is organized as follows. In section 2, we formulate the interior transmission eigenvalue problem in some "$\boldsymbol{H}(\boldsymbol{curl}\,)$" type space $\boldsymbol{X}$. We then present the idea of the T-coercivity studying the scalar problem which appears in the Helmholtz decomposition of $\boldsymbol{X}$. Although we can restore some positivity property using the T-coercivity approach in $\boldsymbol{X}$, this is not sufficient to apply the analytic Fredholm theorem because $\boldsymbol{X}$ is not compactly imbedded in $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$ (D is the domain). Hence, we introduce in section 3 a formulation of the transmission eigenvalue problem in $\boldsymbol{X_0}$, which is the space "orthogonal" to gradients in the Helmholtz decomposition. The next section is dedicated to prove a result of compact imbedding of $\boldsymbol{X_0}$ into $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$. Then, we proceed to the study of the interior transmission eigenvalue problem using the analytical Fredholm theorem in the case where $A \leq A^\star Id$ in a neighbourhood of the boundary, where $A^\star$ is a constant such that $A^\star < 1$. In section 6, we summarize the equivalent results when $A_\star Id \leq A$ in a neighbourhood of the boundary, with $1 < A_\star$. Finally, we discuss the cases where $A-Id$ and/or $N-Id$ change sign in a neighbourhood of the boundary.

2. Setting of the problem

2.1. Basic definitions

Consider $D \subset \mathbb{R}^3$ a bounded simply connected domain with Lipschitz connected boundary ∂D. The unit outward normal vector to ∂D will be denoted ν. We study the problem of scattering of the electric field in the time-harmonic regime by an inclusion whose permeability and permittivity are given by $\varepsilon(\boldsymbol{x}) = \varepsilon_0\, N(\boldsymbol{x})$ and $\mu(\boldsymbol{x}) = \mu_0\, A(\boldsymbol{x})^{-1}$. To simplify the presentation, we assume that ε_0 and μ_0 are constant but considering a background medium which is not homogeneous would only induce minor corrections in the analysis we provide. Here, A, $N \in L^\infty(D, \mathbb{C}^{3\times3})$ are matrix valued functions such that $A(\boldsymbol{x})$ and $N(\boldsymbol{x})$ are hermitian for almost all $\boldsymbol{x} \in D$. Furthermore, we suppose that

A^{-1}, $N^{-1} \in L^\infty(D, \mathbb{C}^{3\times 3})$ and we denote

$$\begin{array}{llll} A_- := \inf_{\boldsymbol{x}\in D} \inf_{\xi\in\mathbb{C}^3,|\xi|=1} (\xi \cdot A(\boldsymbol{x})\overline{\xi}) > 0 & ; & A_+ := \sup_{\boldsymbol{x}\in D} \sup_{\xi\in\mathbb{C}^3,|\xi|=1} (\xi \cdot A(\boldsymbol{x})\overline{\xi}) < \infty \; ; \\ N_- := \inf_{\boldsymbol{x}\in D} \inf_{\xi\in\mathbb{C}^3,|\xi|=1} (\xi \cdot N(\boldsymbol{x})\overline{\xi}) > 0 & \text{and} & N_+ := \sup_{\boldsymbol{x}\in D} \sup_{\xi\in\mathbb{C}^3,|\xi|=1} (\xi \cdot N(\boldsymbol{x})\overline{\xi}) < \infty. \end{array} \tag{1}$$

In this paper, $A_\star > 1$, $A^\star < 1$, $N_\star > 1$ and $N^\star < 1$ will be constants which will allow to state some assumptions on the values of A and N in a neighborhood of the boundary. On the other hand, $\mathscr{V}$ will always refer to a neighbourhood of ∂D, i.e. an open set of $\mathbb{R}^3$ such that $\partial D \subset (\mathscr{V} \cap \overline{D})$.

If $\mathscr{O}$ is an open subset of $\mathbb{R}^3$, we denote indistinctly $(\cdot,\cdot)_{\mathscr{O}}$ the inner products of $L^2(\mathscr{O}) := L^2(\mathscr{O}, \mathbb{C})$ and $\boldsymbol{L}^2(\mathscr{O}) := L^2(\mathscr{O}, \mathbb{C}^3)$, and $\|\cdot\|_{\mathscr{O}}$ the associated norms. We also denote $H^1(D)$ instead of $H^1(D, \mathbb{C})$. The space $\boldsymbol{H}(\boldsymbol{curl}, D)$ is defined as the closure of $\mathscr{C}^\infty(\overline{D}, \mathbb{C}^3)$ for the norm

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\boldsymbol{curl},D)} := (\boldsymbol{u},\boldsymbol{u})^{1/2}_{\boldsymbol{H}(\boldsymbol{curl},D)} \;\text{ with }\; (\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{H}(\boldsymbol{curl},D)} := (\boldsymbol{u},\boldsymbol{v})_D + (\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{v})_D.$$

The subset of the elements of $\boldsymbol{H}(\boldsymbol{curl}, D)$ such that the tangential trace vanishes on ∂D is denoted $\boldsymbol{H_0}(\boldsymbol{curl}\,, D)$.

$$\boldsymbol{H_0}(\boldsymbol{curl}\,, D) := \{\boldsymbol{v} \in \boldsymbol{H}(\boldsymbol{curl}, D) \,|\, \boldsymbol{v} \times \nu = 0 \text{ on } \partial D\}.$$

The tangential trace is well-defined (see for example [19] or [26]).

Definition 2.1 *The elements $k \in \mathbb{C}$ such that there exists a pair $(\boldsymbol{u},\boldsymbol{w}) \neq (0,0)$ solving the problem*

$$\left|\begin{array}{lll} \multicolumn{3}{l}{\textit{Find } (\boldsymbol{u},\boldsymbol{w}) \in \boldsymbol{H}(\boldsymbol{curl}, D) \times \boldsymbol{H}(\boldsymbol{curl}, D) \;\; \textit{such that:}} \\ \boldsymbol{curl}\,(A\boldsymbol{curl}\,\boldsymbol{u}) - k^2 N\boldsymbol{u} & = 0 & \textit{in } D \\ \boldsymbol{curl}\,\boldsymbol{curl}\,\boldsymbol{w} - k^2\boldsymbol{w} & = 0 & \textit{in } D \\ \nu \times (\boldsymbol{u} - \boldsymbol{w}) & = 0 & \textit{on } \partial D \\ \nu \times (A\boldsymbol{curl}\,\boldsymbol{u} - \boldsymbol{curl}\,\boldsymbol{w}) & = 0 & \textit{on } \partial D \end{array}\right. \tag{2}$$

are called transmission eigenvalues.

Here, $\boldsymbol{w}$ and $\boldsymbol{u}$ denote respectively the incident electric field which does not scatter and the total electric field inside the inclusion. One classically proves that $(\boldsymbol{u},\boldsymbol{w})$ satisfies (2) if and only if $(\boldsymbol{u},\boldsymbol{w})$ satisfies the problem

$$\left|\begin{array}{l} \text{Find } (\boldsymbol{u},\boldsymbol{w}) \in \boldsymbol{X} \text{ such that, for all } (\boldsymbol{u}',\boldsymbol{w}') \in \boldsymbol{X}, \\ \displaystyle\int_D A\boldsymbol{curl}\,\boldsymbol{u} \cdot \boldsymbol{curl}\,\boldsymbol{u}' - \boldsymbol{curl}\,\boldsymbol{w} \cdot \boldsymbol{curl}\,\boldsymbol{w}'\,d\boldsymbol{x} = k^2 \int_D N\boldsymbol{u}\cdot\boldsymbol{u}' - \boldsymbol{w}\cdot\boldsymbol{w}'\,d\boldsymbol{x}, \end{array}\right. \tag{3}$$

with $\boldsymbol{X} := \{(\boldsymbol{u},\boldsymbol{w}) \in \boldsymbol{H}(\boldsymbol{curl}, D) \times \boldsymbol{H}(\boldsymbol{curl}, D) \,|\, \boldsymbol{u} - \boldsymbol{w} \in \boldsymbol{H_0}(\boldsymbol{curl}\,, D)\}$. Let us introduce the sesquilinear form on $\boldsymbol{X} \times \boldsymbol{X}$

$$\begin{array}{rcl} a_k((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}')) & := & (A\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{u}')_D - (\boldsymbol{curl}\,\boldsymbol{w}, \boldsymbol{curl}\,\boldsymbol{w}')_D \\ & & -k^2\left((N\boldsymbol{u},\boldsymbol{u}')_D - (\boldsymbol{w},\boldsymbol{w}')_D\right). \end{array}$$

We remark that if $(\boldsymbol{u},\boldsymbol{w})$ satisfies problem (3), then for all $(\varphi,\psi) \in H^1(D)\times H^1(D)$ such that $\varphi - \psi \in H^1_0(D)$ (in this case, $(\nabla\varphi,\nabla\psi) \in \boldsymbol{X}$ because $(\nabla\varphi - \nabla\psi)\times\nu = 0$ on ∂D), we have

$$k^2\left((N\boldsymbol{u},\nabla\varphi)_D - (\boldsymbol{w},\nabla\psi)_D\right) = 0. \tag{4}$$

This leads us to introduce the spaces

$$\begin{array}{ll} S & := \{(\varphi,\psi)\in H^1(D)\times H^1(D)\,|\,\varphi-\psi\in H^1_0(D) \text{ and } (\varphi,1)_{\partial D}=(\psi,1)_{\partial D}=0\};\\ \boldsymbol{X_0} & := \{(\boldsymbol{u},\boldsymbol{w})\in \boldsymbol{X}\,|\,(N\boldsymbol{u},\nabla\varphi)_D-(\boldsymbol{w},\nabla\psi)_D=0,\quad \forall(\varphi,\psi)\in S\}. \end{array} \tag{5}$$

Here, $(\cdot,\cdot)_{\partial D}$ denotes the inner product of $L^2(\partial D)$. The condition $(\varphi,1)_{\partial D}=(\psi,1)_{\partial D}=0$ for elements (φ,ψ) of S is only used to set the constants: if $(c_1,c_2)\in S\cap\mathbb{C}^2$ then $c_1=c_2=0$. One can check that $((\varphi,\psi),(\varphi',\psi'))\mapsto((\varphi,\psi),(\varphi',\psi'))_S := (\nabla\varphi,\nabla\varphi')_D+(\nabla\psi,\nabla\psi')_D$ defines an inner product on S. Let us state a lemma characterizing the elements of $\boldsymbol{X_0}$.

Lemma 2.2 *Let $(\boldsymbol{u},\boldsymbol{w})$ be an element of $\boldsymbol{X}$. The pair $(\boldsymbol{u},\boldsymbol{w})$ belongs to $\boldsymbol{X_0}$ if and only if $div\,(N\boldsymbol{u})=div\,\boldsymbol{w}=0$ in D and $\nu\cdot(N\boldsymbol{u}-\boldsymbol{w})=0$ on ∂D.*

Proof Consider $(\boldsymbol{u},\boldsymbol{w})$ an element of $\boldsymbol{X_0}$. By definition, one has

$$(N\boldsymbol{u},\nabla\varphi)_D-(\boldsymbol{w},\nabla\psi)_D=0,\quad \forall(\varphi,\psi)\in S.$$

Taking, $(\varphi,\psi)=(\zeta,0)$ (resp. $(\varphi,\psi)=(0,\zeta)$) for $\zeta\in\mathscr{C}^\infty_0(D)$, one finds $div\,(N\boldsymbol{u})=0$ (resp. $div\,\boldsymbol{w}=0$). Now, if $\zeta\in H^1(D)$, one can write

$$\begin{array}{rl} & \langle\nu\cdot(N\boldsymbol{u}-\boldsymbol{w}),\zeta\rangle_{H^{-1/2}(\partial D),\,H^{1/2}(\partial D)}\\ = & (div\,(N\boldsymbol{u}-\boldsymbol{w}),\zeta)_D+(N\boldsymbol{u}-\boldsymbol{w},\nabla\zeta)_D\\ = & (N\boldsymbol{u},\nabla(\zeta-\lambda_\zeta))_D-(\boldsymbol{w},\nabla(\zeta-\lambda_\zeta))_D=0. \end{array}$$

Above, λ_ζ is the number $(\zeta,1)_{\partial D}/(1,1)_{\partial D}$.
Reciprocally, if $(\boldsymbol{u},\boldsymbol{w})\in\boldsymbol{X}$ satisfies $div\,(N\boldsymbol{u})=div\,\boldsymbol{w}=0$ in D and $\nu\cdot(N\boldsymbol{u}-\boldsymbol{w})=0$, then for $(\varphi,\psi)\in S$, one has

$$\begin{array}{rl} & (N\boldsymbol{u},\nabla\varphi)_D-(\boldsymbol{w},\nabla\psi)_D\\ = & \langle\nu\cdot N\boldsymbol{u},\varphi\rangle_{H^{-1/2}(\partial D),\,H^{1/2}(\partial D)}-\langle\nu\cdot\boldsymbol{w},\psi\rangle_{H^{-1/2}(\partial D),\,H^{1/2}(\partial D)}\\ = & \langle\nu\cdot(N\boldsymbol{u}-\boldsymbol{w}),\varphi\rangle_{H^{-1/2}(\partial D),\,H^{1/2}(\partial D)}=0. \end{array}$$

This ends the proof. ■
Now, let us consider the scalar problem

$$\left|\begin{array}{l} \text{Find } (\varphi,\psi)\in S \text{ such that, for all } (\varphi',\psi')\in S,\\ \displaystyle\int_D N\nabla\varphi\cdot\overline{\nabla\varphi'}-\nabla\psi\cdot\overline{\nabla\psi'}\,d\boldsymbol{x}=f((\varphi',\psi')), \end{array}\right. \tag{6}$$

where $f\in S'$ (the topological dual space to S). The study of this problem will be useful for two reasons. First, in the sequel, we will need informations about $\boldsymbol{X_0}\cap\nabla S$, which is exactly equal to the set of the gradients of the elements of the kernel of (6). Secondly, this will allow us to present the T-coercivity technique on a very simple case.

2.2. Outline of the T-coercivity technique: study of the scalar problem

Define the sesquilinear form $b((\varphi,\psi),(\varphi',\psi')) := \displaystyle\int_D N\nabla\varphi\cdot\overline{\nabla\varphi'}-\nabla\psi\cdot\overline{\nabla\psi'}\,d\boldsymbol{x}$ and, with the help of the Riesz representation theorem, the operator $\mathscr{B}$ from S to S such that,

$$\forall((\varphi,\psi),(\varphi',\psi'))\in S\times S,\ \ (\mathscr{B}(\varphi,\psi),(\varphi',\psi'))_S\ =\ b((\varphi,\psi),(\varphi',\psi')).$$

Notice that b is not coercive on S (nor "coercive+compact"). The idea of the T-coercivity (see [5, 2, 4]) consists in considering an equivalent formulation to (6) replacing b by b^T defined by

$$b^T((\varphi,\psi),(\varphi',\psi')) := b((\varphi,\psi),T(\varphi',\psi')), \quad \forall(\varphi',\psi') \in S,$$

where T is an *ad hoc* isomorphism of S. Indeed, (φ,ψ) satisfies $b((\varphi,\psi),(\varphi',\psi')) = f((\varphi',\psi'))$ for all $(\varphi',\psi') \in S$ if, and only if, it satisfies $b^T((\varphi,\psi),(\varphi',\psi')) = f(T(\varphi',\psi'))$ for all $(\varphi',\psi') \in S$. Let us consider for example $T(\varphi,\psi) = (\varphi - 2\psi, -\psi)$. Notice that $(\varphi - 2\psi) - (-\psi) = \varphi - \psi \in H_0^1(D)$ and $(\varphi - 2\psi, 1)_{\partial D} = (-\psi, 1)_{\partial D} = 0$, so $T(\varphi,\psi) \in S$. Moreover, since $T^2 = Id$, T is an isomorphism of S.
For all $(\varphi,\psi) \in S$ and all $\eta > 0$, one has, using Young's inequality,

$$\begin{array}{rcl} |b^T((\varphi,\psi),(\varphi,\psi))| & = & |(N\nabla\varphi,\nabla\varphi)_D + (\nabla\psi,\nabla\psi)_D - 2(N\nabla\varphi,\nabla\psi)_D| \\ & \geq & (N\nabla\varphi,\nabla\varphi)_D + (\nabla\psi,\nabla\psi)_D - 2|(N\nabla\varphi,\nabla\psi)_D| \\ & \geq & (N\nabla\varphi,\nabla\varphi)_D + (\nabla\psi,\nabla\psi)_D - \eta(N\nabla\varphi,\nabla\varphi)_D - \eta^{-1}(N\nabla\psi,\nabla\psi)_D \\ & \geq & (1-\eta)(N\nabla\varphi,\nabla\varphi)_D + (1-\eta^{-1}N_+)(\nabla\psi,\nabla\psi)_D. \end{array}$$

Suppose that $N_+ < 1$. Taking η such that $N_+ < \eta < 1$, one proves that b^T is coercive on S. Using Lax-Milgram theorem and since T is an isomorphism of S, one deduces that the operator $\mathscr{B}$ is an isomorphism of S when $N_+ < 1$. Working in the same way with $T(\varphi,\psi) = (\varphi, -\psi + 2\varphi)$ to deal with the case $1 < N_-$, one can state the following lemma.

Lemma 2.3 *Assume that $N_+ < 1$ or $1 < N_-$. Then the operator $\mathscr{B}$ associated with the scalar problem (6) is an isomorphism of S.*

Now, we wish to weaken the assumption on N.

Proposition 2.4 *Assume there exists a neighbourhood $\mathscr{V}$ of ∂D such that the function N satisfies $N \leq N^\star Id < Id$ or $Id < N_\star Id \leq N$ a.e. on $D \cap \mathscr{V}$. Then the operator $\mathscr{B}$ associated with the scalar problem (6) satisfies the equality $\mathscr{B} = \mathscr{I} + \mathscr{K}$ where $\mathscr{I}$ is an isomorphism of S and $\mathscr{K}$ a compact operator of S.*

Remark 2.5 *Under the assumptions of **Proposition 2.4**, one has classically (see [30, 25]) the alternative:*

- *either $\mathscr{B}$ is injective and then this operator is an isomorphism of S;*
- *or $\mathscr{B}$ has a non empty kernel of finite dimension $\ker \mathscr{B} = span((\varphi_1,\psi_1),\ldots,(\varphi_N,\psi_N))$ and then the problem (6) has a solution (defined up to a linear combination of the elements of $\ker \mathscr{B}$) if and only if the source term f satisfies the compatibility conditions $f((\varphi_k,\psi_k)) = 0$ for $k = 1\ldots N$.*

Proof Let $\chi \in \mathscr{C}^\infty(\overline{D},[0;1])$ be a cut-off function with support in $\mathscr{V} \cap \overline{D}$ equal to 1 in a neighbourhood of ∂D. Let us focus on the case $N \leq N^\star Id < Id$ a.e. on $D \cap \mathscr{V}$. Define $T(\varphi,\psi) = (\varphi - 2\chi\psi, -\psi)$. Given $(\varphi,\psi) \in S$, one checks that $T(\varphi,\psi) \in S$. Also, there holds $T^2 = Id$ so T is an isomorphism of S. For all $((\varphi,\psi),(\varphi',\psi')) \in S \times S$, one has

$$\begin{array}{rl} & b^T((\varphi,\psi),(\varphi',\psi')) \\ = & (N\nabla\varphi,\nabla\varphi')_D + (\nabla\psi,\nabla\psi')_D - 2(N\nabla\varphi,\nabla(\chi\psi'))_D \\ = & (N\nabla\varphi,\nabla\varphi')_D + (\nabla\psi,\nabla\psi')_D - 2(N\chi\nabla\varphi,\nabla\psi')_D - 2(N\nabla\varphi,\psi'\nabla\chi)_D. \end{array}$$

Define the continuous operator $\mathscr{I}$ from S to S such that for all $((\varphi,\psi),(\varphi',\psi')) \in S\times S$,

$$(\mathscr{I}(\varphi,\psi),T(\varphi',\psi'))_S \;=\; (N\nabla\varphi,\nabla\varphi')_D+(\nabla\psi,\nabla\psi')_D-2(N\chi\nabla\varphi,\nabla\psi')_D.$$

Let us prove that $\mathscr{I}$ is an isomorphism. For all $(\varphi,\psi)\in S$, using Young's inequality, one can write for all $\eta>0$

$$2|(N\chi\nabla\varphi,\nabla\psi)_D| = 2|(N\chi\nabla\varphi,\nabla\psi)_{\mathscr{V}}| \le \eta(N\nabla\varphi,\nabla\varphi)_{\mathscr{V}}+\eta^{-1}N^\star(\nabla\psi,\nabla\psi)_{\mathscr{V}}.$$

We deduce

$$\begin{array}{rcl}|(\mathscr{I}(\varphi,\psi),T(\varphi,\psi))_S| & \ge & (N\nabla\varphi,\nabla\varphi)_D+(\nabla\psi,\nabla\psi)_D-2|(N\chi\nabla\varphi,\nabla\psi)_D| \\ & \ge & (N\nabla\varphi,\nabla\varphi)_{D\setminus\overline{\mathscr{V}}}+(\nabla\psi,\nabla\psi)_{D\setminus\overline{\mathscr{V}}} \\ & & +(1-\eta)(N\nabla\varphi,\nabla\varphi)_{\mathscr{V}}+(1-\eta^{-1}N^\star)(\nabla\psi,\nabla\psi)_{\mathscr{V}}.\end{array}$$

Taking η such that $N^\star<\eta<1$, this proves that $((\varphi,\psi),(\varphi',\psi'))\mapsto(\mathscr{I}(\varphi,\psi),T(\varphi',\psi'))_S$ is coercive. Thus, $\mathscr{I}$ is an isomorphism from S. Now, define $\mathscr{K}:=\mathscr{B}-\mathscr{I}$. For all $((\varphi,\psi),(\varphi',\psi'))\in S\times S$, there holds

$$(\mathscr{K}(\varphi,\psi),T(\varphi',\psi'))_S \;=\; -2(N\nabla\varphi,\psi'\nabla\chi)_D.$$

Let us show that $\mathscr{K}$ is a compact operator. Consider a bounded sequence (φ_m,ψ_m) of elements of S. Define $(\varphi'_m,\psi'_m):=T^{-1}\mathscr{K}(\varphi_m,\psi_m)$. The sequence (φ'_m,ψ'_m) is bounded in S because T^{-1} and $\mathscr{K}$ are continuous operators. Since the imbedding of S in $L^2(D)\times L^2(D)$ is compact, we can extract a subsequence from (φ_m,ψ_m) (still denoted (φ_m,ψ_m)) such that (φ'_m,ψ'_m) converges in $L^2(D)\times L^2(D)$. Define $\varphi_{lm}:=\varphi_l-\varphi_m$, $\psi_{lm}:=\psi_l-\psi_m$ and $\psi'_{lm}:=\psi'_l-\psi'_m$. One has

$$(\mathscr{K}(\varphi_{lm},\psi_{lm}),\mathscr{K}(\varphi_{lm},\psi_{lm}))_S=-2(N\nabla\varphi_{lm},\psi'_{lm}\nabla\chi)_D\le C\,\|\psi'_{lm}\|_D.$$

Thus, the sequence $\mathscr{K}(\varphi_{lm},\psi_{lm})$ is a Cauchy sequence of S, so it converges. This proves that $\mathscr{K}$ is a compact operator of S. One proceeds in the same way to deal with the case $Id<N_\star Id\le N$ a.e. on $D\cap\mathscr{V}$ working this time with $T(\varphi,\psi)=(\varphi,-\psi+2\chi\varphi)$. ■

3. A sufficient condition for the discreteness of transmission eigenvalues

Let us go back to the study of problem (3). If $\boldsymbol{T}$ is an isomorphism of $\boldsymbol{X}$, then $(\boldsymbol{u},\boldsymbol{w})$ is a solution of (3) if and only if $(\boldsymbol{u},\boldsymbol{w})$ satisfies

$$a_k^{\boldsymbol{T}}((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}')):=a_k((\boldsymbol{u},\boldsymbol{w}),\boldsymbol{T}(\boldsymbol{u}',\boldsymbol{w}'))=0,\quad\forall(\boldsymbol{u}',\boldsymbol{w}')\in\boldsymbol{X}.\qquad(7)$$

Again, as for the scalar problem, the idea is to use an *ad hoc* isomorphism $\boldsymbol{T}$ of $\boldsymbol{X}$ to restore some property of positivity for the principal part of $a_k^{\boldsymbol{T}}$. However, this is not enough to apply the analytic Fredholm theorem because the imbedding of $\boldsymbol{X}$ in $\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)$ is not compact. Classically for Maxwell's equations, the compactness will be obtained by taking into account the free divergence condition working in the space $\boldsymbol{X_0}$. If k is a non-trivial transmission eigenvalue, we know, according to (4), that the associated pair of eigenvectors belongs to $\boldsymbol{X_0}$. This leads us to introduce the problem

$$\left|\begin{array}{l}\text{Find } (\boldsymbol{u},\boldsymbol{w}) \in \boldsymbol{X_0} \text{ such that, for all } (\boldsymbol{u}',\boldsymbol{w}') \in \boldsymbol{X_0},\\ a_k^{\boldsymbol{T}}((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}')) = l((\boldsymbol{u}',\boldsymbol{w}')),\end{array}\right. \tag{8}$$

where $l \in \boldsymbol{X_0}'$ (the topological dual space to $\boldsymbol{X_0}$). Define the operator $\mathscr{A}_k^{\boldsymbol{T}}$ from $\boldsymbol{X_0}$ to $\boldsymbol{X_0}$ such that, for all $((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}')) \in \boldsymbol{X_0} \times \boldsymbol{X_0}$,

$$(\mathscr{A}_k^{\boldsymbol{T}}(\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}'))_{\boldsymbol{H}(\boldsymbol{curl},D)^2} = a_k^{\boldsymbol{T}}((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}')). \tag{9}$$

If $(\boldsymbol{u},\boldsymbol{w})$ is a pair of eigenvectors associated with the transmission eigenvalue $k \neq 0$, then we have $\mathscr{A}_k^{\boldsymbol{T}}(\boldsymbol{u},\boldsymbol{w}) = 0$. Consequently, to show that the interior transmission eigenvalues form at most a discrete set, it is sufficient to prove that $\mathscr{A}_k^{\boldsymbol{T}}$ is injective for all $k \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of the complex plane. In the next section, we prove a result of compact imbedding of $\boldsymbol{X_0}$ into $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$ so that, as previously announced, we can use the analytic Fredholm theorem.

Remark 3.1 *Assume that the scalar operator $\mathscr{B}$ is an isomorphism of S (a sufficient condition for this assumption to be satisfied is $N_+ < 1$ or $1 < N_-$). Then one can easily show that if $\mathscr{A}_k^{\boldsymbol{T}}$ is not injective, then k is a transmission eigenvalue. In this case, $k \neq 0$ is a transmission eigenvalue if and only if $\mathscr{A}_k^{\boldsymbol{T}}$ is not injective.*

4. Study of the space $\boldsymbol{X_0}$

4.1. Compactness property of $\boldsymbol{X_0}$

Theorem 4.1 *Assume that there exists a neighbourhood $\mathscr{V}$ of ∂D such that the function N satisfies $N \leq N^\star Id < Id$ or $Id < N_\star Id \leq N$ a.e. on $D \cap \mathscr{V}$. Then $\boldsymbol{X_0}$ is compactly imbedded in $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$.*

Proof Define the classical spaces, for $\xi \in L^\infty(D,\mathbb{C})$,

$$\begin{array}{lcl}\boldsymbol{V}_N(\xi;\, D) & := & \{\boldsymbol{u} \in \boldsymbol{H}(\boldsymbol{curl}\,,\, D) \,|\, div\,(\xi\boldsymbol{u}) = 0 \text{ in } D,\, \boldsymbol{u}\times\boldsymbol{n} = 0 \text{ on } \partial D\}\,,\\ \boldsymbol{V}_T(\xi;\, D) & := & \{\boldsymbol{u} \in \boldsymbol{H}(\boldsymbol{curl}\,,\, D) \,|\, div\,(\xi\boldsymbol{u}) = 0 \text{ in } D,\, \xi\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial D\}\,.\end{array}$$

Consider a bounded sequence $(\boldsymbol{u}_m,\boldsymbol{w}_m)$ of elements of $\boldsymbol{X_0}$. According to **Lemma 2.2**, one has $div\,(N\boldsymbol{u}_m + \boldsymbol{w}_m) = 0$ in D which has a connected boundary. Hence, there exists (see theorem 3.12 in [1]) an element $\boldsymbol{s}_m \in \boldsymbol{V}_T(1;\,D)$ such that $N\boldsymbol{u}_m + \boldsymbol{w}_m = \boldsymbol{curl}\,\boldsymbol{s}_m$. On the other hand, since $div\,(N\boldsymbol{u}_m - \boldsymbol{w}_m) = 0$ in the simply connected domain D and $\nu\cdot(N\boldsymbol{u}_m - \boldsymbol{w}_m) = 0$ on ∂D (again **Lemma 2.2**), there exists according to theorem 3.17 in [1] an element $\boldsymbol{d}_m \in \boldsymbol{V}_N(1;\,D)$ such that $N\boldsymbol{u}_m - \boldsymbol{w}_m = \boldsymbol{curl}\,\boldsymbol{d}_m$. Then define $\boldsymbol{\varphi}_m := (\boldsymbol{s}_m + \boldsymbol{d}_m)/2$ and $\boldsymbol{\psi}_m := (\boldsymbol{s}_m - \boldsymbol{d}_m)/2$. One has $\boldsymbol{u}_m = N^{-1}\,\boldsymbol{curl}\,\boldsymbol{\varphi}_m$ and $\boldsymbol{w}_m = \boldsymbol{curl}\,\boldsymbol{\psi}_m$.

Let us show that we can extract subsequences from $(\boldsymbol{curl}\,\boldsymbol{\varphi}_m)$ and $(\boldsymbol{curl}\,\boldsymbol{\psi}_m)$ which converge in $\boldsymbol{L}^2(D)$. Define the space

$$\tilde{\boldsymbol{X}}_0 := \{(\boldsymbol{\varphi},\boldsymbol{\psi}) \in \boldsymbol{X} \,|\, div\,\boldsymbol{\varphi} = div\,\boldsymbol{\psi} = 0 \text{ in } D,\, \nu\cdot(\boldsymbol{\varphi}+\boldsymbol{\psi}) = 0 \text{ on } \partial D\}. \tag{10}$$

In **Lemma 4.2** below, we prove that $\tilde{\boldsymbol{X}}_0$ is compactly imbedded in $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$ and that the map $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \mapsto (\|\boldsymbol{curl}\, \boldsymbol{\varphi}\|_D^2 + \|\boldsymbol{curl}\, \boldsymbol{\psi}\|_D^2)^{1/2}$ defines on this space a norm equivalent to the canonical norm. Consequently, the sequence $(\boldsymbol{\varphi}_m, \boldsymbol{\psi}_m)$ is bounded in $\tilde{\boldsymbol{X}}_0$, and there is a subsequence (still denoted $(\boldsymbol{\varphi}_m, \boldsymbol{\psi}_m)$) which converges in $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$. Define $\boldsymbol{\varphi}_{lm} := \boldsymbol{\varphi}_l - \boldsymbol{\varphi}_m$, $\boldsymbol{\psi}_{lm} := \boldsymbol{\psi}_l - \boldsymbol{\psi}_m$, $\boldsymbol{u}_{lm} := \boldsymbol{u}_l - \boldsymbol{u}_m$ and $\boldsymbol{w}_{lm} := \boldsymbol{w}_l - \boldsymbol{w}_m$. Then, we write

$$\begin{array}{rcl} \boldsymbol{curl}\, N^{-1} \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm} & = & \boldsymbol{curl}\, \boldsymbol{u}_{lm} \\ \boldsymbol{curl}\, \boldsymbol{curl}\, \boldsymbol{\psi}_{lm} & = & \boldsymbol{curl}\, \boldsymbol{w}_{lm}. \end{array} \tag{11}$$

Consider, as for the study of the scalar problem, a cut-off function $\chi \in \mathscr{C}^\infty(\overline{D}, [0;1])$ with support in $\mathscr{V} \cap \overline{D}$ and equal to 1 in a neighbourhood of ∂D. Let us study the case $Id < N_\star Id \le N$ a.e. on $D \cap \mathscr{V}$. Multiply the equations (11) respectively by $\boldsymbol{\varphi}_{lm} - 2\chi \boldsymbol{\psi}_{lm}$ and $\boldsymbol{\psi}_{lm}$ (to deal with the case $N \le N^\star Id < Id$ a.e. on $D \cap \mathscr{V}$, just multiply by $\boldsymbol{\varphi}_{lm}$ and $\boldsymbol{\psi}_{lm} - 2\chi \boldsymbol{\psi}_{lm}$). Integrating by parts, we find

$$\begin{array}{l} (N^{-1} \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\,(\boldsymbol{\varphi}_{lm} - 2\chi \boldsymbol{\psi}_{lm}))_D \\ + \langle \nu \times (N^{-1} \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}), (\nu \times (\boldsymbol{\varphi}_{lm} - 2\chi \boldsymbol{\psi}_{lm})) \times \nu \rangle_{\partial D} = (\boldsymbol{curl}\, \boldsymbol{u}_{lm}, \boldsymbol{\varphi}_{lm} - 2\chi \boldsymbol{\psi}_{lm})_D \end{array} \tag{12}$$

and

$$\begin{array}{rl} & (\boldsymbol{curl}\, \boldsymbol{\psi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\psi}_{lm})_D + \langle \nu \times \boldsymbol{curl}\, \boldsymbol{\psi}_{lm}, (\nu \times \boldsymbol{\psi}_{lm}) \times \nu \rangle_{\partial D} \\ = & (\boldsymbol{curl}\, \boldsymbol{w}_{lm}, \boldsymbol{\psi}_{lm})_D. \end{array} \tag{13}$$

Since $N^{-1} \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm} - \boldsymbol{curl}\, \boldsymbol{\psi}_{lm} = \boldsymbol{u}_{lm} - \boldsymbol{w}_{lm}$, the function $N^{-1} \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm} - \boldsymbol{curl}\, \boldsymbol{\psi}_{lm}$ belongs to $\boldsymbol{H}_{\boldsymbol{0}}(\boldsymbol{curl}\,, D)$. Remembering that $\boldsymbol{\varphi}_{lm} - \boldsymbol{\psi}_{lm} \in \boldsymbol{H}_{\boldsymbol{0}}(\boldsymbol{curl}\,, D)$, we obtain

$$\begin{array}{l} \langle \nu \times (N^{-1} \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}), (\nu \times (\boldsymbol{\varphi}_{lm} - 2\chi \boldsymbol{\psi}_{lm})) \times \nu \rangle_{\partial D} + \langle \nu \times \boldsymbol{curl}\, \boldsymbol{\psi}_{lm}, (\nu \times \boldsymbol{\psi}_{lm}) \times \nu \rangle_{\partial D} \\ = \langle \nu \times \boldsymbol{curl}\, \boldsymbol{\psi}_{lm}, (\nu \times (-\boldsymbol{\varphi}_{lm})) \times \nu \rangle_{\partial D} + \langle \nu \times \boldsymbol{curl}\, \boldsymbol{\psi}_{lm}, (\nu \times \boldsymbol{\varphi}_{lm}) \times \nu \rangle_{\partial D} = 0. \end{array}$$

Therefore, adding up (12) and (13) leads to

$$\begin{array}{rl} & (N^{-1} \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm})_D + (\boldsymbol{curl}\, \boldsymbol{\psi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\psi}_{lm})_D \\ & -2|(N^{-1} \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\,(\chi \boldsymbol{\psi}_{lm}))_D| \\ \le & C\,(\|\boldsymbol{curl}\, \boldsymbol{u}_{lm}\|_D \|\boldsymbol{\varphi}_{lm}\|_D + \|\boldsymbol{curl}\, \boldsymbol{u}_{lm}\|_D \|\boldsymbol{\psi}_{lm}\|_D + \|\boldsymbol{curl}\, \boldsymbol{w}_{lm}\|_D \|\boldsymbol{\psi}_{lm}\|_D). \end{array} \tag{14}$$

But, for all $\alpha > 0$, $\beta > 0$, we have, according to Young's inequality,

$$\begin{array}{rl} & 2\,|(N^{-1}\, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\,(\chi \boldsymbol{\psi}_{lm}))_D| \\ \le & 2\,|(\chi\, N^{-1}\, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\psi}_{lm})_D| + 2\,|(N^{-1}\, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \nabla\chi \times \boldsymbol{\psi}_{lm})_D| \\ \le & \alpha\,(N^{-1}\, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm})_{\mathscr{V}} + \alpha^{-1}\,(N^{-1}\, \boldsymbol{curl}\, \boldsymbol{\psi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\psi}_{lm})_{\mathscr{V}} \\ & +\beta\,(N^{-1}\, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm})_{\mathscr{V}} + \beta^{-1}\,(N^{-1}(\nabla\chi \times \boldsymbol{\psi}_{lm}), \nabla\chi \times \boldsymbol{\psi}_{lm})_{\mathscr{V}} \\ \le & \alpha\,(N^{-1}\, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm})_{\mathscr{V}} + \alpha^{-1} N_\star^{-1}\,(\boldsymbol{curl}\, \boldsymbol{\psi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\psi}_{lm})_{\mathscr{V}} \\ & +\beta\,(N^{-1}\, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm}, \boldsymbol{curl}\, \boldsymbol{\varphi}_{lm})_{\mathscr{V}} + C\,\beta^{-1}\,(\boldsymbol{\psi}_{lm}, \boldsymbol{\psi}_{lm})_{\mathscr{V}} \end{array} \tag{15}$$

with $C>0$ which only depends on χ and N. Plugging (15) in (14), we obtain

$$
\begin{array}{rl}
& (N^{-1}\boldsymbol{curl}\,\boldsymbol{\varphi}_{lm},\boldsymbol{curl}\,\boldsymbol{\varphi}_{lm})_{D\setminus\overline{\mathscr{V}}}+(\boldsymbol{curl}\,\boldsymbol{\psi}_{lm},\boldsymbol{curl}\,\boldsymbol{\psi}_{lm})_{D\setminus\overline{\mathscr{V}}} \\
& +(1-\alpha-\beta)(N^{-1}\boldsymbol{curl}\,\boldsymbol{\varphi}_{lm},\boldsymbol{curl}\,\boldsymbol{\varphi}_{lm})_{\mathscr{V}} \\
& +(1-\alpha^{-1}N_\star^{-1})(\boldsymbol{curl}\,\boldsymbol{\psi}_{lm},\boldsymbol{curl}\,\boldsymbol{\psi}_{lm})_{\mathscr{V}} \\
\leq & C\,(\|\boldsymbol{curl}\,\boldsymbol{u}_{lm}\|_D\|\boldsymbol{\varphi}_{lm}\|_D+\|\boldsymbol{curl}\,\boldsymbol{w}_{lm}\|_D\|\boldsymbol{\psi}_{lm}\|_D+\|\boldsymbol{curl}\,\boldsymbol{u}_{lm}\|_D\|\boldsymbol{\psi}_{lm}\|_D \\
& \quad +\beta^{-1}\|\boldsymbol{\psi}_{lm}\|_D^2).
\end{array}
\tag{16}
$$

Since $1<N_\star$, we can choose $\alpha<1$ such that $(1-\alpha^{-1}N_\star^{-1})>0$. Taking $0<\beta<1-\alpha$, we obtain the estimate

$$
\begin{array}{rcl}
\|\boldsymbol{curl}\,\boldsymbol{\varphi}_{lm}\|_D^2+\|\boldsymbol{curl}\,\boldsymbol{\psi}_{lm}\|_D^2 & \leq & C\,(\|\boldsymbol{curl}\,\boldsymbol{u}_{lm}\|_D\|\boldsymbol{\varphi}_{lm}\|_D+\|\boldsymbol{curl}\,\boldsymbol{u}_{lm}\|_D\|\boldsymbol{\psi}_{lm}\|_D \\
& & +\|\boldsymbol{curl}\,\boldsymbol{w}_{lm}\|_D\|\boldsymbol{\psi}_{lm}\|_D+\|\boldsymbol{\psi}_{lm}\|_D^2).
\end{array}
$$

Thus, the sequences $(\boldsymbol{curl}\,\boldsymbol{\varphi}_m)$ and $(\boldsymbol{curl}\,\boldsymbol{\psi}_m)$ are Cauchy sequences for the $\boldsymbol{L}^2(D)$ norm. This proves that $(\boldsymbol{u}_m,\boldsymbol{w}_m)=(N^{-1}\boldsymbol{curl}\,\boldsymbol{\varphi}_m,\boldsymbol{curl}\,\boldsymbol{\psi}_m)$ converges in $\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)$. ■

Lemma 4.2 *The space $\tilde{\boldsymbol{X}}_0$ defined in (10) is compactly imbedded in $\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)$. Moreover, the map $(\boldsymbol{\varphi},\boldsymbol{\psi})\mapsto(\|\boldsymbol{curl}\,\boldsymbol{\varphi}\|_D^2+\|\boldsymbol{curl}\,\boldsymbol{\psi}\|_D^2)^{1/2}$ defines on $\tilde{\boldsymbol{X}}_0$ a norm equivalent to the canonical norm.*

Proof Let $(\boldsymbol{\varphi}_m,\boldsymbol{\psi}_m)$ be a bounded sequence of elements of $\tilde{\boldsymbol{X}}_0$. The sequences $(\boldsymbol{\varphi}_m-\boldsymbol{\psi}_m)$ and $(\boldsymbol{\varphi}_m+\boldsymbol{\psi}_m)$ are respectively bounded in $\boldsymbol{V}_N(1;\,D)$ and $\boldsymbol{V}_T(1;\,D)$. According to the Weber theorem [29], we can extract from $(\boldsymbol{\varphi}_m-\boldsymbol{\psi}_m)$ and $(\boldsymbol{\varphi}_m+\boldsymbol{\psi}_m)$ subsequences which converge in $\boldsymbol{L}^2(D)$. Writing, $\boldsymbol{\varphi}_m=(\boldsymbol{\varphi}_m+\boldsymbol{\psi}_m)/2+(\boldsymbol{\varphi}_m-\boldsymbol{\psi}_m)/2$ and $\boldsymbol{\psi}_m=(\boldsymbol{\varphi}_m+\boldsymbol{\psi}_m)/2-(\boldsymbol{\varphi}_m-\boldsymbol{\psi}_m)/2$, this proves that we can extract a subsequence from $(\boldsymbol{\varphi}_m,\boldsymbol{\psi}_m)$ which converges in $\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)$. Moreover, since the map $\boldsymbol{v}\mapsto\|\boldsymbol{curl}\,\boldsymbol{v}\|_D$ defines a norm on $\boldsymbol{V}_N(1;\,D)$ and $\boldsymbol{V}_T(1;\,D)$, we obtain the second part of the lemma. ■

4.2. Equivalent norms on $\boldsymbol{X_0}$

In this paragraph, we want to determine under which criterion the map $(\boldsymbol{u},\boldsymbol{w})\mapsto(\|\boldsymbol{curl}\,\boldsymbol{u}\|_D^2+\|\boldsymbol{curl}\,\boldsymbol{w}\|_D^2)^{1/2}$ defines on $\boldsymbol{X_0}$ a norm which is equivalent to the canonical norm.

Proposition 4.3 *Suppose there exists a neighbourhood $\mathscr{V}$ of ∂D such that the function N satisfies $N\leq N^\star Id<Id$ or $Id<N_\star Id\leq N$ a.e. on $D\cap\mathscr{V}$. Suppose also that the operator $\mathscr{B}$ associated with the scalar problem (6) is injective ($\nabla\ker\mathscr{B}=\boldsymbol{X_0}\cap\nabla S=\{0\}$). Then the map $(\boldsymbol{u},\boldsymbol{w})\mapsto(\|\boldsymbol{curl}\,\boldsymbol{u}\|_D^2+\|\boldsymbol{curl}\,\boldsymbol{w}\|_D^2)^{1/2}$ defines on $\boldsymbol{X_0}$ a norm equivalent to the canonical norm.*

Definition 4.4 *Under the assumptions of **Proposition 4.3**, we will denote $C_P>0$ the smallest constant such that*

$$\|\boldsymbol{u}\|_D^2+\|\boldsymbol{w}\|_D^2\leq C_P(\|\boldsymbol{curl}\,\boldsymbol{u}\|_D^2+\|\boldsymbol{curl}\,\boldsymbol{w}\|_D^2),\quad\forall(\boldsymbol{u},\boldsymbol{w})\in\boldsymbol{X_0}.\tag{17}$$

Proof of Proposition 4.3 It is sufficient to prove that (17) holds for some $C_P > 0$. Suppose there exists a sequence $(\boldsymbol{u}_m, \boldsymbol{w}_m)$ of elements of $\boldsymbol{X_0}$ such that

$$\forall m \in \mathbb{N},\ \|\boldsymbol{u}_m\|_D^2 + \|\boldsymbol{w}_m\|_D^2 = 1 \text{ and } \lim_{m\to\infty} \|\boldsymbol{curl}\,\boldsymbol{u}_m\|_D^2 + \|\boldsymbol{curl}\,\boldsymbol{w}_m\|_D^2 = 0.$$

By **Theorem 4.1**, we can extract from $(\boldsymbol{u}_m, \boldsymbol{w}_m)$ a sequence (still denoted $(\boldsymbol{u}_m, \boldsymbol{w}_m)$) which converges to some $(\boldsymbol{u}, \boldsymbol{w}) \in \boldsymbol{X_0}$ in $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$. By construction, one has $\|\boldsymbol{u}\|_D^2 + \|\boldsymbol{w}\|_D^2 = 1$ and $\boldsymbol{curl}\,\boldsymbol{u} = \boldsymbol{curl}\,\boldsymbol{w} = 0$. Since ∂D is simply connected, one deduces (see [14], theorem 8) that there exists $(\varphi, \psi) \in S$ such that $(\boldsymbol{u}, \boldsymbol{w}) = (\nabla\varphi, \nabla\psi)$. We then notice that $\mathscr{B}(\nabla\varphi, \nabla\psi) = (0, 0)$. Since we have supposed that $\mathscr{B}$ was injective, one deduces $(\boldsymbol{u}, \boldsymbol{w}) = (0, 0)$. This leads to a contradiction because we must have $\|\boldsymbol{u}\|_D^2 + \|\boldsymbol{w}\|_D^2 = 1$. ■

5. Case $A \le A^\star Id$, with $A^\star < 1$, in a neighbourhood of the boundary

Let us go back to the study of the operator $\mathscr{A}_k^{\boldsymbol{T}}$ defined in (9), where, for the moment, $\boldsymbol{T}$ is an abstract isomorphism of $\boldsymbol{X}$. In this paragraph, we suppose there exists a neighbourhood $\mathscr{V}$ of ∂D such that $A \le A^\star Id$ a.e. in $\mathscr{V}$, with $A^\star < 1$. Again, $\chi \in \mathscr{C}^\infty(\overline{D}, [0; 1])$ designates a cut-off function with support in $\mathscr{V} \cap \overline{D}$ and equal to 1 in a neighbourhood ∂D. Define the operator $\boldsymbol{T} : \boldsymbol{X} \to \boldsymbol{X}$ such that

$$\boldsymbol{T}(\boldsymbol{u}, \boldsymbol{w}) = (\boldsymbol{u} - 2\chi\boldsymbol{w}, -\boldsymbol{w}). \tag{18}$$

It is an isomorphism because $\boldsymbol{T}^2 = Id$.

5.1. Fredholm property for the operator $\mathscr{A}_k^{\boldsymbol{T}}$

Lemma 5.1 *Assume that $A \le A^\star Id < Id$ and $N \le N^\star Id < Id$ a.e. on $D \cap \mathscr{V}$. Then there exists $k = i\kappa$, with $\kappa \in \mathbb{R}$, such that the operator $\mathscr{A}_k^{\boldsymbol{T}}$ is an isomorphism of $\boldsymbol{X_0}$.*

Proof Let us show that the sesquilinear form $a_{i\kappa}^{\boldsymbol{T}}$ is coercive for some $\kappa \in \mathbb{R}$. For all $(\boldsymbol{u}, \boldsymbol{w}) \in \boldsymbol{X_0}$, one can write

$$\begin{aligned}
&\left|a_{i\kappa}^{\boldsymbol{T}}((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w}))\right| \\
=\ & |(A\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{u})_D + (\boldsymbol{curl}\,\boldsymbol{w}, \boldsymbol{curl}\,\boldsymbol{w})_D - 2(A\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,(\chi\boldsymbol{w}))_D \\
& + \kappa^2 ((N\boldsymbol{u}, \boldsymbol{u})_D + (\boldsymbol{w}, \boldsymbol{w})_D - 2(N\boldsymbol{u}, \chi\boldsymbol{w})_D)| \\
\ge\ & (A\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{u})_D + (\boldsymbol{curl}\,\boldsymbol{w}, \boldsymbol{curl}\,\boldsymbol{w})_D + \kappa^2 ((N\boldsymbol{u}, \boldsymbol{u})_D + (\boldsymbol{w}, \boldsymbol{w})_D) \\
& -2\,|(A\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,(\chi\boldsymbol{w}))_D| - 2\kappa^2\,|(N\boldsymbol{u}, \chi\boldsymbol{w})_D|\,.
\end{aligned} \tag{19}$$

But, for all $\eta > 0$, $\alpha > 0$, one has, according to Young's inequality,

$$
\begin{array}{rl}
 & 2\,|(A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,(\chi\boldsymbol{w}))_D| \\
\leq & 2\,|(\chi A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,\boldsymbol{w})_D| + 2\,|(A\boldsymbol{curl}\,\boldsymbol{u},\nabla\chi\times\boldsymbol{w})_D| \\
\leq & \eta\,(A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,\boldsymbol{u})_{\mathscr{V}} + \eta^{-1}\,(A\boldsymbol{curl}\,\boldsymbol{w},\boldsymbol{curl}\,\boldsymbol{w})_{\mathscr{V}} \\
 & +\alpha\,(A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,\boldsymbol{u})_{\mathscr{V}} + \alpha^{-1}\,(A(\nabla\chi\times\boldsymbol{w}),\nabla\chi\times\boldsymbol{w})_{\mathscr{V}} \\
\leq & \eta\,(A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,\boldsymbol{u})_{\mathscr{V}} + \eta^{-1}A^*\,(\boldsymbol{curl}\,\boldsymbol{w},\boldsymbol{curl}\,\boldsymbol{w})_{\mathscr{V}} \\
 & +\alpha\,(A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,\boldsymbol{u})_{\mathscr{V}} + C\,\alpha^{-1}\,(\boldsymbol{w},\boldsymbol{w})_{\mathscr{V}}
\end{array}
\tag{20}
$$

with $C > 0$ which only depends on χ and on A. On the other hand, for all $\beta > 0$,

$$
\begin{array}{rcl}
2\,|(N\boldsymbol{u},\chi\boldsymbol{w})_D| & \leq & \beta(N\boldsymbol{u},\boldsymbol{u})_{\mathscr{V}} + \beta^{-1}(N\boldsymbol{w},\boldsymbol{w})_{\mathscr{V}} \\
 & \leq & \beta(N\boldsymbol{u},\boldsymbol{u})_{\mathscr{V}} + \beta^{-1}N^*(\boldsymbol{w},\boldsymbol{w})_{\mathscr{V}}.
\end{array}
\tag{21}
$$

Thus, plugging (20) and (21) in (19), one obtains, for all $\eta > 0$, $\alpha > 0$, $\beta > 0$,

$$
\begin{array}{rl}
 & \left|a^{\boldsymbol{T}}_{i\kappa}((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u},\boldsymbol{w}))\right| \\
\geq & (A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,\boldsymbol{u})_{D\setminus\overline{\mathscr{V}}} + (\boldsymbol{curl}\,\boldsymbol{w},\boldsymbol{curl}\,\boldsymbol{w})_{D\setminus\overline{\mathscr{V}}} \\
 & +\kappa^2\left((N\boldsymbol{u},\boldsymbol{u})_{D\setminus\overline{\mathscr{V}}} + (\boldsymbol{w},\boldsymbol{w})_{D\setminus\overline{\mathscr{V}}}\right) \\
 & +(1-\eta-\alpha)(A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,\boldsymbol{u})_{\mathscr{V}} + (1-\eta^{-1}A^*)(\boldsymbol{curl}\,\boldsymbol{w},\boldsymbol{curl}\,\boldsymbol{w})_{\mathscr{V}} \\
 & +\kappa^2(1-\beta)(N\boldsymbol{u},\boldsymbol{u})_{\mathscr{V}} + (\kappa^2(1-\beta^{-1}\,N^*) - C\,\alpha^{-1})(\boldsymbol{w},\boldsymbol{w})_{\mathscr{V}}.
\end{array}
\tag{22}
$$

Let us choose first $\eta > 0$ to have both $(1-\eta) > 0$ and $(1-\eta^{-1}\,A^\star) > 0$ (recall that $A^\star < 1$). Then, let us take $\alpha > 0$ such that $(1-\eta-\alpha) > 0$. Finally, let us choose $\beta > 0$ so that $(1-\beta) > 0$ and $(1-\beta^{-1}\,N^\star) > 0$ (recall that $N^\star < 1$). It just remains to take a value of κ sufficiently large (in absolute value) to obtain

$$
\left|a^{\boldsymbol{T}}_{i\kappa}((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u},\boldsymbol{w}))\right| \geq c\,(\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\boldsymbol{curl},D)} + \|\boldsymbol{w}\|^2_{\boldsymbol{H}(\boldsymbol{curl},D)})
$$

where c is a constant independent of $(\boldsymbol{u},\boldsymbol{w}) \in \boldsymbol{X_0}$. Thus, for a value of κ large enough, $a^{\boldsymbol{T}}_{i\kappa}$ is coercive. With the Lax-Milgram theorem, one can then conclude that $\mathscr{A}^{\boldsymbol{T}}_{i\kappa}$ is an isomorphism of $\boldsymbol{X_0}$ for such a κ. ■

We deduce the

Theorem 5.2 *Assume that $A \leq A^\star Id < Id$ a.e. on $D \cap \mathscr{V}$. Assume also that $N \leq N^\star Id < Id$ or $Id < N_\star Id \leq N$ a.e. on $D \cap \mathscr{V}$. Then for all $k \in \mathbb{C}$, the operator $\mathscr{A}^{\boldsymbol{T}}_k$ satisfies the equality $\mathscr{A}^{\boldsymbol{T}}_k = \mathscr{I} + \mathscr{K}_k$ where $\mathscr{I}$ is an isomorphism of $\boldsymbol{X_0}$ that is independent of k, and $\mathscr{K}_k$ is a compact operator of $\boldsymbol{X_0}$.*

Proof Introduce $\mathscr{I}$ the operator such that, for all $((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}')) \in \boldsymbol{X_0} \times \boldsymbol{X_0}$,

$$
(\mathscr{I}(\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}'))_{\boldsymbol{H}(\boldsymbol{curl},D)^2} = a_{i\kappa,1/2}((\boldsymbol{u},\boldsymbol{w}),\boldsymbol{T}(\boldsymbol{u}',\boldsymbol{w}')),
\tag{23}
$$

with $\boldsymbol{T}$ defined in (18) and

$$
\begin{array}{rcl}
a_{i\kappa,1/2}((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{u}',\boldsymbol{w}')) & = & (A\boldsymbol{curl}\,\boldsymbol{u},\boldsymbol{curl}\,\boldsymbol{u}')_D - (\boldsymbol{curl}\,\boldsymbol{w},\boldsymbol{curl}\,\boldsymbol{w}')_D \\
 & & +\kappa^2\,((2^{-1}\boldsymbol{u},\boldsymbol{u}')_D - (\boldsymbol{w},\boldsymbol{w}')_D)\,.
\end{array}
$$

According to **Lemma 5.1** (notice that $2^{-1}Id < Id$), we can choose $\kappa \in \mathbb{R}$ such that $\mathscr{I}$ is an isomorphism of $\boldsymbol{X_0}$. Since, by assumption, $N \le N^\star Id < Id$ or $Id < N_\star Id \le N$ a.e. on $D \cap \mathscr{V}$, **Theorem 4.1** indicates that the imbedding of $\boldsymbol{X_0}$ in $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$ is compact. This proves that $\mathscr{A}_k^{\boldsymbol{T}} - \mathscr{I}$ is a compact operator of $\boldsymbol{X_0}$. ■

5.2. Discreteness of the transmission eigenvalues

Suppose the assumptions of **Theorem 5.2** to be true. Reintroduce $\mathscr{I} : \boldsymbol{X_0} \to \boldsymbol{X_0}$ the isomorphism defined in (23). For all $((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}', \boldsymbol{w}')) \in \boldsymbol{X_0} \times \boldsymbol{X_0}$, one has

$$\begin{aligned}((\mathscr{A}_k^{\boldsymbol{T}} - \mathscr{I})(\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}', \boldsymbol{w}'))_{\boldsymbol{H}(\boldsymbol{curl},D)^2} = &-k^2\left((N\boldsymbol{u}, \boldsymbol{u}')_D + (\boldsymbol{w}, \boldsymbol{w}')_D - 2(N\boldsymbol{u}, \chi\boldsymbol{w}')_D\right)\\ &-\kappa^2\left((2^{-1}\boldsymbol{u}, \boldsymbol{u}')_D + (\boldsymbol{w}, \boldsymbol{w}')_D - 2(2^{-1}\boldsymbol{u}, \chi\boldsymbol{w}')_D\right).\end{aligned}$$

This leads us to define the operators $\mathscr{F}$ and $\mathscr{G}$ from $\boldsymbol{X_0}$ to $\boldsymbol{X_0}$ such that, for all $((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}', \boldsymbol{w}')) \in \boldsymbol{X_0} \times \boldsymbol{X_0}$,

$$(\mathscr{F}(\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}', \boldsymbol{w}'))_{\boldsymbol{H}(\boldsymbol{curl},D)^2} = -\left((N\boldsymbol{u}, \boldsymbol{u}')_D + (\boldsymbol{w}, \boldsymbol{w}')_D - 2(N\boldsymbol{u}, \chi\boldsymbol{w}')_D\right);$$

$$(\mathscr{G}(\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}', \boldsymbol{w}'))_{\boldsymbol{H}(\boldsymbol{curl},D)^2} = -\kappa^2\left((2^{-1}\boldsymbol{u}, \boldsymbol{u}')_D + (\boldsymbol{w}, \boldsymbol{w}')_D - 2(2^{-1}\boldsymbol{u}, \chi\boldsymbol{w}')_D\right).$$

One has $\mathscr{A}_k^{\boldsymbol{T}} = \mathscr{I} + k^2\mathscr{F} + \mathscr{G} \Leftrightarrow \mathscr{A}_k^{\boldsymbol{T}}\mathscr{I}^{-1} = Id + k^2\mathscr{F}\mathscr{I}^{-1} + \mathscr{G}\mathscr{I}^{-1}$. Here, Id denotes the identity operator of $\boldsymbol{X_0}$. According to **Theorem 4.1**, under the assumptions of **Theorem 5.2**, the imbedding of $\boldsymbol{X_0}$ in $\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$ is compact. Consequently $\mathscr{F}$, $\mathscr{G}$, and thus $\mathscr{F}\mathscr{I}^{-1}$, $\mathscr{G}\mathscr{I}^{-1}$, are compact operators from $\boldsymbol{X_0}$ to $\boldsymbol{X_0}$. In addition, the map $k \mapsto k^2\mathscr{F}\mathscr{I}^{-1} + \mathscr{G}\mathscr{I}^{-1}$ from $\mathbb{C}$ to the Banach space of bounded operators from $\boldsymbol{X_0}$ to $\boldsymbol{X_0}$ is polynomial and so analytic. Thanks to the analytical Fredholm theorem (see [15, theorem 8.26] or [23, corollary 1.1.1]), one distinguishes two cases for the family of operators $\{\mathscr{A}_k^{\boldsymbol{T}}\mathscr{I}^{-1}\}_{k\in\mathbb{C}}$ or equivalently for the family of operators $\{\mathscr{A}_k^{\boldsymbol{T}}\}_{k\in\mathbb{C}}$. Either, for all $k \in \mathbb{C}$, $\mathscr{A}_k^{\boldsymbol{T}}$ is not injective. Or there exists $k \in \mathbb{C}$ such that $\mathscr{A}_k^{\boldsymbol{T}}$ is injective and then $\mathscr{A}_k^{\boldsymbol{T}}$ is injective for all $k \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of the complex plane.

Theorem 5.3 *Assume that $A \le A^\star Id < Id$ and $N \le N^\star Id < Id$ a.e. on $D \cap \mathscr{V}$. Then the set of transmission eigenvalues is at most discrete in $\mathbb{C}$.*

Proof Lemma 5.1 ensures that there exists $\kappa \in \mathbb{R}$ such that $\mathscr{A}_{i\kappa}^{\boldsymbol{T}}$ is an isomorphism of $\boldsymbol{X_0}$. Thus, $\mathscr{A}_k^{\boldsymbol{T}}$ is injective for all $k \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of the complex plane. For $k \in \mathbb{C}\backslash\mathscr{S}$, this implies that the only solution of the problem (7) (and consequently of the problems (2) and (3)) is the zero solution. ■

Theorem 5.4 *Assume that $A_+ < 1$. Suppose also that the operator $\mathscr{B}$ associated with the scalar problem (6) is injective (a sufficient condition for this assumption to be satisfied is $N_+ < 1$ or $1 < N_-$). Then the set of transmission eigenvalues is at most discrete in $\mathbb{C}$.*

Proof Using the proof of **Lemma 5.1** with $\chi = 1$, one finds there exist two constants C_1, $C_2 > 0$ independent of k such that, for all $(\boldsymbol{u}, \boldsymbol{w}) \in \boldsymbol{X_0}$,

$$\left|a_k^{\boldsymbol{T}}((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w}))\right| \ge C_1(\|\boldsymbol{curl}\,\boldsymbol{u}\|_D^2 + \|\boldsymbol{curl}\,\boldsymbol{w}\|_D^2) - C_2|k|^2(\|\boldsymbol{u}\|_D^2 + \|\boldsymbol{w}\|_D^2).$$

Using the **Proposition 4.3** on equivalent norms on $\boldsymbol{X_0}$, one deduces that $a_k^{\boldsymbol{T}}$ is coercive on $\boldsymbol{X_0} \times \boldsymbol{X_0}$ for $|k|^2 < C_1/(C_2\, C_P)$, where C_P is defined in (17). Thus, the operator $\mathscr{A}_k^{\boldsymbol{T}}$ is an isomorphism of $\boldsymbol{X_0}$ for small values (in modulus) of k. One can conclude using the analytical Fredholm theorem. ■

5.3. Localization of the transmission eigenvalues

Now, using a trick from [22] (see the proof of theorem 3.6.1 p.102), we show a result of localization of the transmission eigenvalues.

Theorem 5.5 *Assume that $A \leq A^\star Id < Id$ and $N \leq N^\star Id < Id$ a.e. on $D \cap \mathscr{V}$. Then there exist positive constants ρ and δ such that if $k \in \mathbb{C}$ satisfies $|k| > \rho$ and $|\Re e\, k| < \delta\, |\Im m\, k|$ then k is not a transmission eigenvalue.*

Proof Let $k = i\kappa$, $\kappa \in \mathbb{R}$. **Lemma 5.1** shows that, for $|\kappa|$ sufficiently large, one has the estimate, for all $(\boldsymbol{u}, \boldsymbol{w}) \in \boldsymbol{X_0}$,

$$\left|a_k^{\boldsymbol{T}}((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w}))\right| \geq C_1(\|\boldsymbol{curl}\, \boldsymbol{u}\|_D^2 + \|\boldsymbol{curl}\, \boldsymbol{w}\|_D^2) + C_2\, \kappa^2(\|\boldsymbol{u}\|_D^2 + \|\boldsymbol{w}\|_D^2), \tag{24}$$

where the constants $C_1 > 0$, $C_2 > 0$ are independent of κ.
Let us consider now $k = i\kappa\, e^{i\theta}$ with $\theta \in [-\pi/2; \pi/2]$. One checks that

$$\left|a_k^{\boldsymbol{T}}((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w})) - a_{i\kappa}^{\boldsymbol{T}}((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w}))\right| \leq C_3\, \kappa^2 \left|1 - e^{2i\theta}\right| (\|\boldsymbol{u}\|_D^2 + \|\boldsymbol{w}\|_D^2), \tag{25}$$

with $C_3 > 0$ independent of κ. Combining (24) and (25), one finds

$$\begin{array}{rl} & \left|a_k^{\boldsymbol{T}}((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w}))\right| \\ \geq & \left|a_{i\kappa}^{\boldsymbol{T}}((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w}))\right| - C_3\, \kappa^2 \left|1 - e^{2i\theta}\right| (\|\boldsymbol{u}\|_D^2 + \|\boldsymbol{w}\|_D^2) \\ \geq & C_1(\|\boldsymbol{curl}\, \boldsymbol{u}\|_D^2 + \|\boldsymbol{curl}\, \boldsymbol{w}\|_D^2) + (C_2 - C_3 \left|1 - e^{2i\theta}\right|)\kappa^2(\|\boldsymbol{u}\|_D^2 + \|\boldsymbol{w}\|_D^2). \end{array}$$

Taking θ sufficiently small to have, for example, $C_3 \left|1 - e^{2i\theta}\right| \leq C_2/2$, one then deduces the result. ■

5.4. An estimate for the first transmission eigenvalue

Theorem 5.6 *Assume that $A_+ < 1$. Assume also that the operator $\mathscr{B}$ associated with the scalar problem (6) is injective (a sufficient condition for this assumption to be satisfied is $N_+ < 1$ or $1 < N_-$). If $k \in \mathbb{C}$ satisfies the estimate $|k|^2 < (A_-(1 - \sqrt{A_+}))/(C_P \max(N_+, 1)(1 + \sqrt{N_+}))$, with C_P defined in (17), then k is not a transmission eigenvalue.*

Proof Following the lines of the proof of **Lemma 5.1** with $\chi = 1$, one can write, for all $(\boldsymbol{u}, \boldsymbol{w}) \in \boldsymbol{X_0}$,

$$\begin{aligned} & \left|a_k^{\boldsymbol{T}}((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w}))\right| \\ = \; & |(A\,\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{u})_D + (\boldsymbol{curl}\,\boldsymbol{w}, \boldsymbol{curl}\,\boldsymbol{w})_D - 2(A\,\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{w})_D \\ & -k^2\left((N\boldsymbol{u}, \boldsymbol{u})_D + (\boldsymbol{w}, \boldsymbol{w})_D - 2(N\boldsymbol{u}, \boldsymbol{w})_D\right)| \\ \geq \; & (A\,\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{u})_D + (\boldsymbol{curl}\,\boldsymbol{w}, \boldsymbol{curl}\,\boldsymbol{w})_D - 2|(A\,\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{w})_D| \\ & -|k|^2\left((N\boldsymbol{u}, \boldsymbol{u})_D + (\boldsymbol{w}, \boldsymbol{w})_D + 2|(N\boldsymbol{u}, \boldsymbol{w})_D|\right) \\ \geq \; & (1 - \sqrt{A_+})((A\,\boldsymbol{curl}\,\boldsymbol{u}, \boldsymbol{curl}\,\boldsymbol{u})_D + (\boldsymbol{curl}\,\boldsymbol{w}, \boldsymbol{curl}\,\boldsymbol{w})_D) \\ & -|k|^2(1 + \sqrt{N_+})((N\boldsymbol{u}, \boldsymbol{u})_D + (\boldsymbol{w}, \boldsymbol{w})_D). \end{aligned}$$

Therefore, for $k \in \mathbb{C}$ such that $|k|^2 < (A_-(1 - \sqrt{A_+}))/(C_P \max(N_+, 1)(1 + \sqrt{N_+}))$, $a_k^{\boldsymbol{T}}$ is coercive. ■

6. Case $A_\star Id \leq A$, with $1 < A_\star$, in a neighbourhood of the boundary

In this paragraph, we suppose that there exists a neighbourhood $\mathscr{V}$ of ∂D such that $A_\star Id \leq A$ a.e. in $\mathscr{V}$, with $1 < A_\star$. Again, $\chi \in \mathscr{C}^\infty(\overline{D}, [0; 1])$ designates a cut-off function with support in $\mathscr{V} \cap \overline{D}$ equal to 1 in a neighbourhood of ∂D. Define the operator $\boldsymbol{T} : \boldsymbol{X} \to \boldsymbol{X}$ such that $\boldsymbol{T}(\boldsymbol{u}, \boldsymbol{w}) = (\boldsymbol{u}, -\boldsymbol{w} + 2\chi\boldsymbol{u})$. It is an isomorphism because $\boldsymbol{T}^2 = Id$. As in the previous section, we prove the following results.

Theorem 6.1 *Assume that $Id < A_\star Id \leq A$ a.e. on $D \cap \mathscr{V}$. Assume also that $N \leq N^\star Id < Id$ or $Id < N_\star Id \leq N$ a.e. on $D \cap \mathscr{V}$. Then for all $k \in \mathbb{C}$, the operator $\mathscr{A}_k^{\boldsymbol{T}}$ satisfies the equality $\mathscr{A}_k^{\boldsymbol{T}} = \mathscr{I} + \mathscr{K}_k$ where $\mathscr{I}$ is an isomorphism of $\boldsymbol{X_0}$ that is independent of k, and $\mathscr{K}_k$ is a compact operator of $\boldsymbol{X_0}$.*

Theorem 6.2 *Assume that $Id < A_\star Id \leq A$ and $Id < N_\star Id \leq N$ a.e. on $D \cap \mathscr{V}$. Then the set of transmission eigenvalues is at most discrete in $\mathbb{C}$. Moreover, there exists positive constants ρ and δ such that if $k \in \mathbb{C}$ satisfies $|k| > \rho$ and $|\Re e\, k| < \delta\, |\Im m\, k|$ then k is not a transmission eigenvalue.*

Theorem 6.3 *Assume that $1 < A_-$. Assume also that the operator $\mathscr{B}$ associated with the scalar problem (6) is injective (a sufficient condition for this assumption to be satisfied is $N_+ < 1$ or $1 < N_-$). Then the set of transmission eigenvalues is at most discrete in $\mathbb{C}$. Moreover, if $k \in \mathbb{C}$ satisfies the estimate $|k|^2 < (1 - 1/\sqrt{A_-})/(C_P \max(N_+, 1)(1 + 1/\sqrt{N_-}))$, with C_P defined in (17), then k is not a transmission eigenvalue.*

7. Discussion

All over this paper, we have been obliged to suppose that $A - Id$ and $N - Id$ were positive or negative in a neighbourhood of the boundary to use the T-coercivity technique. A

natural question then is: what happens if $A - Id$ and/or $N - Id$ change sign in a neighbourhood of the boundary?
In [24], using the Shapiro-Lopatinskii condition, the authors provide necessary and sufficient conditions for ellipticity of the scalar interior transmission problem associated with (2) in the case where A and N are smooth. Actually, when $A - Id$ changes sign, or worse, vanishes in a neighbourhood of the boundary, as written in [4], we think there are geometries and values of A for which the scalar interior transmission problem is not Fredholm in H^1 because of the appearance of "strong" singularities‡. This result is proved in [2] for the transmission problem between a dielectric and a negative metamaterial, and in [3], we derive a functional framework in which Fredholmness is recovered. To be precise, let us mention that the situation we studied in these two articles corresponds to a situation where $A-Id$ changes sign in one point of the boundary for the scalar interior transmission problem. The case where $A - Id$ vanishes on a non-empty open subset of the boundary is much more intricate and the definition of a functional framework in which Fredholmness could be recovered, as it has been done naturally in the very particular case $A = Id$, is an open problem.

For the Maxwell problem, the situation is even more obscure. Indeed, the coefficient N also matters in establishing the Fredholm property because it determines whether or not the perturbation of the principal part is compact. The determination of an appropriate functional framework to study the interior transmission problem for Maxwell's equations when $A - Id$ and/or $N - Id$ change sign/vanish on the boundary is far from being clear.

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‡ Remark that for the scalar problem, since the imbedding of $H^1(D)$ in $L^2(D)$ is compact, only the function A, which appears in the principal part of the operator, matters in the Fredholmness property.

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