# Design of a mode converter using thin resonant ligaments

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**Abstract**

The goal of this work is to design an acoustic mode converter. The wave number is fixed so that two modes can propagate. We explain how to construct geometries such that the energy of the modes is completely transmitted and additionally the mode 1 is converted into the mode 2 and conversely.

*Keywords:* acoustic waveguide, mode converter, asymptotic analysis, thin resonators, scattering coefficients, complex resonance

## 1 Setting of the problem

To create our mode converter, we work in a specific geometry that we start by describing.

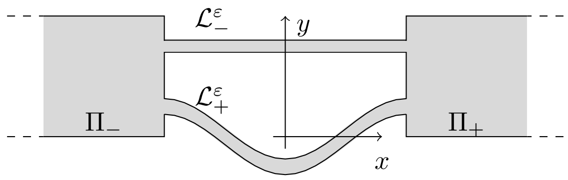


Figure 1: Geometry of the waveguide $\Omega^\varepsilon$.

Define the domains $\Pi_\pm := \{(x,y) \in \mathbb{R}^2 \,|\, (\pm x, y) \in (1/2;+\infty) \times (0;1)\}$, connect them by two thin non intersecting ligaments $\mathcal{L}_-^\varepsilon$, $\mathcal{L}_+^\varepsilon$ of constant width $\varepsilon > 0$ and finally set (see Figure 1)

$$\Omega^\varepsilon := \Pi_- \cup \mathcal{L}_-^\varepsilon \cup \mathcal{L}_+^\varepsilon \cup \Pi_+.$$

Considering the propagation of acoustic waves in $\Omega^\varepsilon$ leads us to study the problem

$$\left|\begin{array}{rcll} \Delta u^\varepsilon + k^2 u^\varepsilon &=& 0 & \text{in } \Omega^\varepsilon \\ \partial_\nu u^\varepsilon &=& 0 & \text{on } \partial\Omega^\varepsilon. \end{array}\right. \qquad (1)$$

We fix $k \in (\pi; 2\pi)$ so that only the modes

$$w_1^\pm(x,y) = e^{\pm i\beta_1 x}\varphi_1(y),\; w_2^\pm(x,y) = e^{\pm i\beta_2 x}\varphi_2(y)$$

with $\varphi_1(y) = \beta_1^{-1/2}$, $\varphi_2(y) = \beta_2^{-1/2}\sqrt{2}\cos(\pi y)$ and $\beta_1 = k$, $\beta_2 = \sqrt{k^2 - \pi^2}$, can propagate. We are interested in the solutions to the diffraction problem (1) generated by the incoming waves $w_1^+$, $w_2^+$ in the channel $\Pi_-$. They admit the decompositions

$$u_1^\varepsilon = \left|\begin{array}{ll} w_1^+ \circ s^+ + \sum_{j=1}^2 r_{1j}^\varepsilon w_j^- \circ s^+ + \dots & \text{in } \Pi_- \\ \sum_{j=1}^2 t_{1j}^\varepsilon w_j^+ \circ s^- + \dots & \text{in } \Pi_+ \end{array}\right.$$

$$u_2^\varepsilon = \left|\begin{array}{ll} w_2^+ \circ s^+ + \sum_{j=1}^2 r_{2j}^\varepsilon w_j^- \circ s^+ + \dots & \text{in } \Pi_- \\ \sum_{j=1}^2 t_{2j}^\varepsilon w_j^+ \circ s^- + \dots & \text{in } \Pi_+. \end{array}\right.$$

Here the shifts $s^\pm(x,y) = (x \pm 1/2, y)$ are introduced only to simplify notation below and $r_{ij}^\varepsilon$, $t_{ij}^\varepsilon \in \mathbb{C}$ are reflection and transmission coefficients. Moreover ellipsis stand for remainders which decay exponentially at infinity. We define the reflection and transmission matrices

$$R^\varepsilon := \begin{pmatrix} r_{11}^\varepsilon & r_{12}^\varepsilon \\ r_{21}^\varepsilon & r_{22}^\varepsilon \end{pmatrix}, \quad T^\varepsilon := \begin{pmatrix} t_{11}^\varepsilon & t_{12}^\varepsilon \\ t_{21}^\varepsilon & t_{22}^\varepsilon \end{pmatrix}.$$

In general, due to the geometrical features of $\Omega^\varepsilon$, almost no energy of the incident waves passes through the thin ligaments $\mathcal{L}_\pm^\varepsilon$ and one observes almost complete reflection (see Figure 2). More precisely, as $\varepsilon$ tends to zero, one gets

$$R^\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + o(1),\; T^\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + o(1).$$

Our goal is to show that by choosing carefully the parameters defining the thin ligaments, we can get almost complete transmission of energy of the incident waves $w_1^+$, $w_2^+$ and additionally, we can ensure that all the energy carried by the incident mode 1 is transferred on the mode 2 only and vice versa. More precisely, we establish that as $\varepsilon$ tends to zero, we can have

$$R^\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + o(1), \quad T^\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + o(1).$$

## 2 Exploiting symmetry

To diminish the number of parameters to play with, we assume that $\Omega^\varepsilon$ is symmetric with respect to $(Oy)$. Then we set $\omega^\varepsilon := \{(x,y) \in \Omega^\varepsilon \,|\, x < 0\}$. Introduce the two problems

$$\left|\begin{array}{rcll} \Delta u_N^\varepsilon + k^2 u_N^\varepsilon &=& 0 & \text{in } \omega^\varepsilon \\ \partial_\nu u_N^\varepsilon &=& 0 & \text{on } \partial\omega^\varepsilon \end{array}\right. \qquad (2)$$

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$$\left| \begin{array}{rcll} \Delta u^\varepsilon_D + k^2 u^\varepsilon_D & = & 0 & \text{in } \omega^\varepsilon \\ \partial_\nu u^\varepsilon_D & = & 0 & \text{on } \partial\omega^\varepsilon \cap \partial\Omega^\varepsilon \\ u^\varepsilon_D & = & 0 & \text{on } \partial\omega^\varepsilon \setminus \partial\Omega^\varepsilon. \end{array} \right. \qquad (3)$$

For $i = 1, 2$, they admit the solutions

$$\begin{array}{l} u^\varepsilon_{Ni} = w^+_i \circ s^+ + \sum_{j=1}^2 r^{\varepsilon N}_{ij} w^-_j \circ s^+ + \dots \ \text{in } \omega^\varepsilon \\ u^\varepsilon_{Di} = w^+_i \circ s^+ + \sum_{j=1}^2 r^{\varepsilon D}_{ij} w^-_j \circ s^+ + \dots \ \text{in } \omega^\varepsilon. \end{array}$$

We define the reflection matrices

$$R^\varepsilon_N := \begin{pmatrix} r^{\varepsilon N}_{11} & r^{\varepsilon N}_{12} \\ r^{\varepsilon N}_{21} & r^{\varepsilon N}_{22} \end{pmatrix}, \quad R^\varepsilon_D := \begin{pmatrix} r^{\varepsilon D}_{11} & r^{\varepsilon D}_{12} \\ r^{\varepsilon D}_{21} & r^{\varepsilon D}_{22} \end{pmatrix}.$$

Playing with symmetries, one shows that

$$R^\varepsilon = \frac{R^\varepsilon_N + R^\varepsilon_D}{2}, \quad T^\varepsilon = \frac{R^\varepsilon_N - R^\varepsilon_D}{2}.$$

Therefore our goal is to find geometries $\omega^\varepsilon$ where, as $\varepsilon$ tends to zero,

$$R^\varepsilon_N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + o(1), \ R^\varepsilon_D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + o(1). \quad (4)$$

## 3 Asymptotic analysis

Denote by $L^\varepsilon_\pm$ the half ligaments $\mathcal{L}^\varepsilon_\pm|_{\omega^\varepsilon}$. Pick two different $y_\pm \in (0; 1)$ and set $A_\pm := (-1/2, y_\pm)$. We assume that $L^\varepsilon_\pm$ are connected to $\Pi_-$ at the points $A_\pm$. Moreover we assume that $L^\varepsilon_\pm$ is of length $\ell^\varepsilon_\pm := \ell_\pm + \varepsilon \ell'_\pm$ where the values $\ell_\pm > 0$, $\ell'_\pm > 0$ will be fixed below.

Next we derive asymptotic expansions of the $u^\varepsilon_{Ni}$, $u^\varepsilon_{Di}$ as $\varepsilon \to 0$. We employ the technique of matched asymptotic expansions. In the process, the properties of the 1D problems

$$(\mathscr{P}^\pm_N) \left| \begin{array}{ll} \partial^2_s v + k^2 v = 0 & \text{in } (0; \ell_\pm) \\ v(0) = \partial_s v(\ell_\pm) = 0 & \end{array} \right.$$

$$(\mathscr{P}^\pm_D) \left| \begin{array}{ll} \partial^2_s v + k^2 v = 0 & \text{in } (0; \ell_\pm) \\ v(0) = v(\ell_\pm) = 0 & \end{array} \right.$$

obtained by considering the restriction of (2) and (3) to $L^\varepsilon_\pm$ play a central role. Let us fix $\ell_+$ (resp. $\ell_-$) coinciding with a resonant length of $(\mathscr{P}^+_D)$ (resp. $(\mathscr{P}^-_N)$). In other words, we take

$$k\ell_+ = m\pi \quad \text{and} \quad k\ell_- = (m + 1/2)\pi \qquad (5)$$

for some $m \in \mathbb{N}$. Then we can prove that $L^\varepsilon_+$ (resp. $L^\varepsilon_-$) has no influence at order $\varepsilon^0$ on the $u^\varepsilon_{Ni}$ (resp. $u^\varepsilon_{Di}$). This key remark allows us to decouple the action of the ligaments. At the end of the asymptotic procedure, we obtain expansions where the features of the geometry appear explicitly. For example for $u^\varepsilon_{D1}$, we get the following statement:

**Proposition** *There is $\ell'_+(\varepsilon)$ such that as $\varepsilon \to 0$,*

$$\left| \begin{array}{l} u^\varepsilon_{D1} = w^+_1 + w^-_1 + ak\gamma + o(1) \ \textit{in } \Pi_- \\ u^\varepsilon_{D1} = O(1) \ \textit{in } L^\varepsilon_- \\ u^\varepsilon_{D1} = \varepsilon^{-1} a \sin(ks) + O(1) \ \textit{in } L^\varepsilon_+ \\ r^{\varepsilon D}_{11} = \dfrac{2\beta_1 \cos(\pi y_+)^2/\beta_2 - 1}{2\beta_1 \cos(\pi y_+)^2/\beta_2 + 1} + o(1) \\ r^{\varepsilon D}_{12} = \dfrac{2\beta_1 \cos(\pi y_+)\sqrt{2\beta_1/\beta_2}}{2\beta_1 \cos(\pi y_+)^2/\beta_2 + 1} + o(1), \end{array} \right.$$

*where $a \in \mathbb{R}i$, $\gamma$ is a certain Green function centered at $A_+$ and $s$ is the curvilinear abscissa.*

Finally, thanks to similar formulas for $u^\varepsilon_{D2}$, $u^\varepsilon_{Ni}$, we can find values of $\ell'_\pm(\varepsilon)$ and $y_\pm$ such that as $\varepsilon \to 0$, the expansions (4) for $R^\varepsilon_N$, $R^\varepsilon_D$ are valid. This leads to the numerics of Figure 3. We emphasize that in the method, it is crucial to work around the resonance lengths to get effect of order $\varepsilon^0$ with ligaments of width $\varepsilon$.

## 4 Numerics

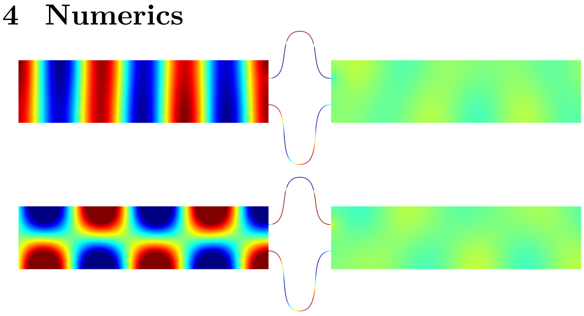

Figure 2: $\Re e\, u^\varepsilon_1$ (top) and $\Re e\, u^\varepsilon_2$ (bottom) for $\varepsilon = 0.01$. Here the lengths of the ligaments are close to the critical values (5) but not particularly selected to get mode conversion.

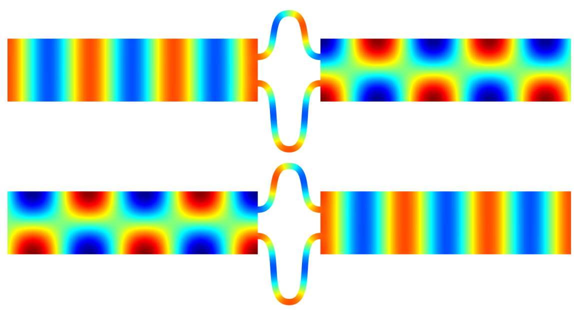

Figure 3: $\Re e\, u^\varepsilon_1$ (top) and $\Re e\, u^\varepsilon_2$ (bottom) for $\varepsilon = 0.1$. The features of the ligaments have been tuned to get mode conversion.

### References

[1] L. Chesnel, J. Heleine, S.A. Nazarov, Design of a mode converter using thin resonant ligaments, *Comm. Math. Sci.* **20**-2 (2022), pp. 425–445.