# T-coercivity for the Maxwell problem with sign-changing coefficients

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**Abstract.** In this paper, we study the time-harmonic Maxwell problem with sign-changing permittivity and/or permeability, set in a domain of $\mathbb{R}^3$. We prove, using the T-coercivity approach, that the well-posedness of the two canonically associated scalar problems, with Dirichlet and Neumann boundary conditions, implies the well-posedness of the Maxwell problem. This allows us to give simple and sharp criteria, obtained in the study of the scalar cases, to ensure that the Maxwell transmission problem between a classical dielectric material and a negative metamaterial is well-posed.

**Key words.** Maxwell's equations, interface problem, metamaterial, compact embedding, sign-changing coefficients.

**AMS subject classification.** 35A15, 35M99, 35Q61, 78A25, 78A48, 78M10, 78M30.

## 1 Introduction

We investigate the time-harmonic Maxwell problem in a composite material surrounded by a perfect conductor. A composite material is modeled by non constant electric permittivity $\varepsilon$ and magnetic permeability $\mu$. It is well-known that some materials, like metals at optical frequencies, are almost dissipationless and have a dielectric permittivity whose real part is negative. More surprising is the possibility of realizing materials, called negative metamaterials, which exhibit both negative real valued permittivity and permeability in some appropriate range of frequencies. The association of classical dielectrics and such negative materials has very exciting potential applications such as plasmonic waveguides, perfect lenses [36, 26, 32], photonic traps, subwavelength cavities [20] ... From a mathematical point of view, the change of sign of the coefficients $\varepsilon$ and/or $\mu$ in the medium raises a lot of original questions for the corresponding electromagnetic model, both for the mathematical analysis and the numerical simulation [31, 33, 21]. Indeed, standard theorems proving the well-posedness of the problem and the convergence of conventional numerical methods are no-longer valid in such situations. Consequently, and generally speaking, the questions we have to address are the following. Can we extend the classical theory to configurations with sign-changing coefficients? And if not, is there a new functional framework in which well-posedness and stability properties can be recovered?

For 2D configurations, the corresponding electromagnetic model reduces to a scalar problem involving the operators $-\mathrm{div}\,(\sigma\nabla\cdot)$ with Dirichlet or Neumann boundary condition, $\sigma$ being equal to $\varepsilon^{-1}$ or $\mu^{-1}$. Those scalar problems have been thoroughly investigated [7, 40, 9, 29, 3, 14, 5, 12, 15] and sharp results have been recently obtained thanks to the simple variational method of the T-coercivity. This technique consists in constructing explicit operators which realize the so-called *inf-sup Banach-Nečas-Babuška* condition [2]. One of its main interests is that it can be used to justify the convergence of finite element methods. It is necessary to emphasize that this approach is nothing else but a reformulation of the *inf-sup* condition and all the work lies in the definition of the operator T. The above scalar problems are proved to be of Fredholm type in the classical functional framework if the contrasts (ratios of the values of $\sigma$ across the interface between the dielectric and the negative material) are outside some critical interval, which always contains the value $-1$. This interval reduces to $\{-1\}$ if (and only if) the interface is smooth (see also [18, 30, 25] for approaches relying on integral equations). For a contrast equal to $-1$, the problems are severely ill-posed (not Fredholm) in $\mathrm{H}^1$. The influence of corners in the interface, noticed for instance in [35, 37], has been clarified in [19, 10, 34]. When the interface has a corner, depending on the value of the contrast in $\sigma$, the scalar problems can be ill-posed (not Fredholm) in $\mathrm{H}^1$, even for contrasts different from $-1$, because of the

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onset of "strong" singularities at the corner. Well-posedness can be recovered by working in a new functional framework in which a radiation condition at the corner is imposed [6].

For scalar problems, the theory is now quite mature. We wish to obtain such results for Maxwell problems, and first, the results of well-posedness using variational techniques (for results obtained with volume and surface integral equations, we refer the reader to [16, 24, 17]). These variational methods are interesting because they allow one to consider rather general configurations: non smooth interface between the positive and the negative material and $\mathrm{L}^\infty$ coefficients $\varepsilon$, $\varepsilon^{-1}$, $\mu$, $\mu^{-1}$. However, it appears that the geometric approach followed for studying the scalar problems is difficult to apply because of the nature of functional spaces used for Maxwell problems. Therefore, we will proceed differently. Again, we will use the $\mathtt{T}$-coercivity technique but in a different form. We will prove that one can construct $\mathtt{T}$-coercivity operators as soon as the associated 3D scalar problems are well-posed. This will provide very simple criteria (those of the scalar problems) to ensure that Maxwell problems are well-posed. When the contrasts in $\varepsilon$ and/or $\mu$ lie inside the critical intervals, the definition of a new functional framework taking into account the gradients of the strong singularities, is still an open question.

The outline of the paper is the following. The definition of the problem and the notations are introduced in Section 2. In Section 3, we give equivalent formulations of the problem, using some classical functional spaces for the study of Maxwell problems: $\mathbf{V}_N(\varepsilon;\Omega)$ for the electric field and $\mathbf{V}_T(\mu;\Omega)$ for the magnetic field; some divergence free condition is included in their definition. Section 4 expresses the main idea of the paper: how to build a $\mathtt{T}$-coercivity operator for the Maxwell problems when the associated scalar problems are well-posed. Then, we use these results and a technique due to [23] to prove some result of compact embedding of $\mathbf{V}_N(\varepsilon;\Omega)$ and $\mathbf{V}_T(\mu;\Omega)$ in $\mathbf{L}^2(\Omega) := \mathrm{L}^2(\Omega)^3$, extending [8] where additional assumptions on $\varepsilon,\mu$ and on the geometry were needed (in particular, the interface has to be smooth). Again, let us underline that these results are not classical in the literature when the coefficients $\varepsilon$ and $\mu$ change sign. In Section 6, we state the main theorem of this work, summing up the previous results: electric and magnetic Maxwell transmission problems are well-posed as soon as the associated 3D scalar problems are well-posed. We illustrate this result on a series of canonical geometries. Finally, we present some generalizations in Section 8. First, we are interested in configurations where the scalar problems are well-posed in the Fredholm sense with a non-trivial kernel. Second, we consider the case of a non-simply connected domain whose boundary is not connected[4]. This study covers the case of non simply connected domains with connected boundary and the case of simply connected domains with non connected boundary.

## 2 Setting of the problem

Let $\Omega$ be a domain in $\mathbb{R}^3$, *i.e.* an open, connected and bounded subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. For some $\omega \neq 0$ ($\omega \in \mathbb{C}$), the time harmonic Maxwell's equations are given by

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = \boldsymbol{J} \quad \text{in } \Omega. \tag{1}$$

Above, $\boldsymbol{E}$ and $\boldsymbol{H}$ are respectively the electric and magnetic components of the electromagnetic field. The source term $\boldsymbol{J}$ is the current density. We suppose that the medium $\Omega$ is surrounded by a perfect conductor and we impose the boundary conditions

$$\boldsymbol{E}\times\boldsymbol{n} = 0 \quad \text{and} \quad \mu\boldsymbol{H}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$. We assume that the dielectric permittivity $\varepsilon$ and the magnetic permeability $\mu$ are real valued functions which belong to $\mathrm{L}^\infty(\Omega)$, with $\varepsilon^{-1}$, $\mu^{-1} \in \mathrm{L}^\infty(\Omega)$. Let us introduce some classical spaces in the study of Maxwell's equations:

$$\begin{array}{lcl}
\mathbf{L}^2(\Omega) & = & \mathrm{L}^2(\Omega)^3 \\
\mathbf{H}(\mathbf{curl}\,;\Omega) & = & \{\boldsymbol{u}\in\mathbf{L}^2(\Omega)\,|\,\mathbf{curl}\,\boldsymbol{u}\in\mathbf{L}^2(\Omega)\} \\
\mathbf{H}_N(\mathbf{curl}\,;\Omega) & = & \{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl}\,;\Omega)\,|\,\boldsymbol{u}\times\boldsymbol{n}=0 \text{ on } \partial\Omega\} \\
\mathbf{V}_N(\xi;\Omega) & = & \{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl}\,;\Omega)\,|\,\mathrm{div}\,(\xi\,\boldsymbol{u})=0,\ \boldsymbol{u}\times\boldsymbol{n}=0 \text{ on } \partial\Omega\} \\
\mathbf{V}_T(\xi;\Omega) & = & \{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl}\,;\Omega)\,|\,\mathrm{div}\,(\xi\,\boldsymbol{u})=0,\ \xi\boldsymbol{u}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\}\,,
\end{array}$$

where $\xi$ refers to a function of $\mathrm{L}^\infty(\Omega)$ such that $\xi^{-1}\in\mathrm{L}^\infty(\Omega)$. For simplicity, the current density $\boldsymbol{J}$ will be chosen in $\mathbf{L}^2(\Omega)$ with $\mathrm{div}\,\boldsymbol{J} = 0$[5]. We denote indistinctly $(\cdot,\cdot)$ the inner products of $\mathrm{L}^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ and $\|\cdot\|$ the associated norms. The spaces $\mathbf{H}(\mathbf{curl}\,;\Omega)$, $\mathbf{H}_N(\mathbf{curl}\,;\Omega)$, $\mathbf{V}_N(\xi;\Omega)$ and $\mathbf{V}_T(\xi;\Omega)$ are endowed with the inner product

$$(\cdot,\cdot)_{\mathbf{curl}} = (\cdot,\cdot) + (\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot).$$

[4]The Figure 2 at the end of this paper gives an example of such a geometry.

[5]The case $\mathrm{div}\,\boldsymbol{J}\neq 0$ can be handled similarly with the tools that we propose, see Remark 6.2 below.

Let us recall some well-known properties for the particular spaces $\mathbf{V}_N(1;\,\Omega)$ and $\mathbf{V}_T(1;\,\Omega)$ (cf. [38, 1]).
– The embeddings of $\mathbf{V}_N(1;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ and of $\mathbf{V}_T(1;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ are compact.
– Furthermore, when $\partial\Omega$ is connected (resp. when $\Omega$ is simply connected), the map $(\boldsymbol{u},\boldsymbol{v}) \mapsto (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})$ defines an inner product on $\mathbf{V}_N(1;\,\Omega)$ (resp. on $\mathbf{V}_T(1;\,\Omega)$) and the associated norm is equivalent to the canonical norm $\boldsymbol{u} \mapsto (\boldsymbol{u},\boldsymbol{u})^{1/2}_{\mathbf{curl}}$.

Classically, we prove that if $\{\boldsymbol{E},\boldsymbol{H}\}$ satisfies (1)-(2), $\boldsymbol{E}$ and $\boldsymbol{H}$ are respectively solutions of the problems

$$\left|\begin{array}{lcll} \text{Find } \boldsymbol{E} \in \mathbf{H}(\mathbf{curl}\,;\,\Omega) \text{ such that:} & & & \\ \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} & = & i\omega\boldsymbol{J} & \text{in } \Omega \\ \boldsymbol{E}\times\boldsymbol{n} & = & 0 & \text{on } \partial\Omega \end{array}\right. , \qquad (3)$$

$$\left|\begin{array}{lcll} \text{Find } \boldsymbol{H} \in \mathbf{H}(\mathbf{curl}\,;\,\Omega) \text{ such that:} & & & \\ \mathbf{curl}\,\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} - \omega^2\mu\boldsymbol{H} & = & \mathbf{curl}\,\varepsilon^{-1}\boldsymbol{J} & \text{in } \Omega \\ \mu\boldsymbol{H}\cdot\boldsymbol{n} & = & 0 & \text{on } \partial\Omega \\ \varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J})\times\boldsymbol{n} & = & 0 & \text{on } \partial\Omega \end{array}\right. . \qquad (4)$$

As already announced in the introduction, we want to find criteria for $\varepsilon$ and $\mu$ to ensure that problems (3) and (4) are well-posed in the Fredholm sense. Classically, for the study of Maxwell's equations, our strategy will consist in working in the space $\mathbf{V}_N(\varepsilon;\,\Omega)$ for the electric field and in the space $\mathbf{V}_T(\mu;\,\Omega)$ for the magnetic field. Indeed, for example, if $\boldsymbol{E}$ satisfies (3) and if $\omega \neq 0$, then $\operatorname{div}(\varepsilon\boldsymbol{E}) = 0$, so $\boldsymbol{E}$ belongs to the space $\mathbf{V}_N(\varepsilon;\,\Omega)$. Therefore, the Fredholm property for the problem (3) will rely on two arguments: first the compact embedding of $\mathbf{V}_N(\varepsilon;\,\Omega)$ in $\mathbf{L}^2(\Omega)$, secondly, the isomorphism property for the principal part $\mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\cdot : \mathbf{V}_N(\varepsilon;\,\Omega) \to \mathbf{V}_N(\varepsilon;\,\Omega)^*$. In a symmetric way, the Fredholm property for the magnetic field relies on the compact embedding of $\mathbf{V}_T(\mu;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ and on the isomorphism property for the principal part $\mathbf{curl}\,\varepsilon^{-1}\mathbf{curl}\,\cdot : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)^*$.

# 3 Equivalent formulations

Let us first give equivalent formulations to problem (1)-(2) in the spaces $\mathbf{V}_N(\varepsilon;\,\Omega)$ and $\mathbf{V}_T(\mu;\,\Omega)$.

## 3.1 Problem for the electric field

For the study of the electric field, we introduce the Sobolev space with Dirichlet boundary condition $\mathrm{H}^1_0(\Omega) := \{\varphi \in \mathrm{H}^1(\Omega)\,|\,\varphi = 0 \text{ on } \partial\Omega\}$. On $\mathrm{H}^1_0(\Omega)$, we use the norm $\|\cdot\|_{\mathrm{H}^1_0(\Omega)} = \|\nabla\cdot\|$. With the Riesz representation theorem, we define the bounded operators $A^\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ and $\mathscr{A}_N(\omega) : \mathbf{V}_N(\varepsilon;\,\Omega) \to \mathbf{V}_N(\varepsilon;\,\Omega)$, $\omega \in \mathbb{C}$, such that

$$\begin{aligned} &(\nabla(A^\varepsilon\varphi), \nabla\varphi') = (\varepsilon\nabla\varphi, \nabla\varphi'), \qquad \forall\varphi,\varphi' \in \mathrm{H}^1_0(\Omega) \\ &(\mathscr{A}_N(\omega)\boldsymbol{E}, \boldsymbol{E}')_{\mathbf{curl}} = (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\varepsilon\boldsymbol{E}, \boldsymbol{E}'), \qquad \forall\boldsymbol{E},\boldsymbol{E}' \in \mathbf{V}_N(\varepsilon;\,\Omega). \end{aligned} \qquad (5)$$

**Theorem 3.1** *Assume that* $\omega \neq 0$.
*1) If* $\{\boldsymbol{E},\boldsymbol{H}\}$ *satisfies (1)-(2) then* $\boldsymbol{E}$ *is a solution of the problem*

$$\left|\begin{array}{lcl} \textit{Find } \boldsymbol{E} \in \mathbf{V}_N(\varepsilon;\,\Omega) \textit{ such that for all } \boldsymbol{E}' \in \mathbf{V}_N(\varepsilon;\,\Omega)\textit{:} & & \\ (\mathscr{A}_N(\omega)\boldsymbol{E}, \boldsymbol{E}')_{\mathbf{curl}} & = & i\omega\,(\boldsymbol{J}, \boldsymbol{E}'). \end{array}\right. \qquad (6)$$

*2) Assume that* $A^\varepsilon$ *is an isomorphism. If* $\boldsymbol{E}$ *satisfies (6) then the pair* $\{\boldsymbol{E}, (i\,\omega\mu)^{-1}\mathbf{curl}\,\boldsymbol{E}\}$ *satisfies (1)-(2).*

**Proof.** 1) If $\{\boldsymbol{E},\boldsymbol{H}\}$ satisfies (1)-(2), then $\boldsymbol{E}$ is a solution of (3). On the other hand, since $\omega \neq 0$, there holds $\operatorname{div}(\varepsilon\boldsymbol{E}) = 0$. This allows us to show that $\boldsymbol{E}$ verifies (6).

2) Now, let us prove that if $\boldsymbol{E} \in \mathbf{V}_N(\varepsilon;\,\Omega) \subset \mathbf{H}_N(\mathbf{curl}\,;\,\Omega)$ satisfies (6) then $\boldsymbol{E}$ is a solution of the problem

$$\left|\begin{array}{l} \text{Find } \boldsymbol{E} \in \mathbf{H}_N(\mathbf{curl}\,;\,\Omega) \text{ such that for all } \boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}\,;\,\Omega)\text{:} \\ (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\varepsilon\boldsymbol{E}, \boldsymbol{E}') = i\omega\,(\boldsymbol{J}, \boldsymbol{E}'). \end{array}\right. \qquad (7)$$

If $A^\varepsilon$ is an isomorphism, then for all $\boldsymbol{E}'$ in $\mathbf{H}_N(\mathbf{curl}\,;\,\Omega)$ we can build $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $(\varepsilon\nabla\varphi, \nabla\varphi') = (\varepsilon\boldsymbol{E}', \nabla\varphi')$ for all $\varphi' \in \mathrm{H}^1_0(\Omega)$. The element $\boldsymbol{E}' - \nabla\varphi$ belongs to $\mathbf{V}_N(\varepsilon;\,\Omega)$. Taking $\boldsymbol{E}' - \nabla\varphi$ as a test-field in (6) and observing that $(\varepsilon\boldsymbol{E}, \nabla\varphi) = 0$ and $(\boldsymbol{J}, \nabla\varphi) = 0$ (recall that $\operatorname{div}\boldsymbol{J} = 0$), one obtains

$$(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\varepsilon\boldsymbol{E}, \boldsymbol{E}') = i\omega\,(\boldsymbol{J}, \boldsymbol{E}').$$

But (3) and (7) are equivalents. Therefore, if $\boldsymbol{E}$ satisfies (6) then $\boldsymbol{E}$ is a solution of (3). There just remains to notice that in this case, the pair $\{\boldsymbol{E}, (i\,\omega\mu)^{-1}\mathbf{curl}\,\boldsymbol{E}\}$ satisfies (1)-(2). ■

**Remark 3.2** *In Section 7, we will give examples of configurations, depending on the values of $\varepsilon$ and on the geometry of the domain, where $A^\varepsilon$ is an isomorphism and where $\varepsilon$ changes sign on $\Omega$. However, we refer the reader to [3] (see also [18]) for a general study of this question.*

### 3.2 Problem for the magnetic field

For the study of the magnetic field, we introduce the space

$$\mathrm{H}^1_\#(\Omega) := \left\{ \varphi \in \mathrm{H}^1(\Omega) \mid \int_\Omega \varphi = 0 \right\}.$$

Since $\Omega$ is connected, the map $(\varphi, \varphi') \mapsto (\nabla\varphi, \nabla\varphi')$ is an inner product on $\mathrm{H}^1_\#(\Omega)$. The associated norm is denoted $\|\cdot\|_{\mathrm{H}^1_\#(\Omega)}$. Define the bounded operators $A^\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ and $\mathscr{A}_T(\omega) : \mathbf{V}_T(\mu;\Omega) \to \mathbf{V}_T(\mu;\Omega)$, $\omega \in \mathbb{C}$, such that

$$\begin{aligned} &(\nabla(A^\mu\varphi), \nabla\varphi') = (\mu\nabla\varphi, \nabla\varphi'), \qquad \forall \varphi,\varphi' \in \mathrm{H}^1_\#(\Omega) \\ &(\mathscr{A}_T(\omega)\boldsymbol{H}, \boldsymbol{H}')_{\mathbf{curl}} = (\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}, \mathbf{curl}\,\boldsymbol{H}') - \omega^2(\mu\boldsymbol{H}, \boldsymbol{H}'), \qquad \forall \boldsymbol{H},\boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega). \end{aligned} \tag{8}$$

Adapting the proof of Theorem 3.1, one obtains the

**Theorem 3.3** *Assume that $\omega \neq 0$.*
*1) If $\{\boldsymbol{E}, \boldsymbol{H}\}$ satisfies (1)-(2) then $\boldsymbol{H}$ is a solution of the problem*

$$\left| \begin{array}{l} \textit{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega) \textit{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega)\textit{:} \\ (\mathscr{A}_T(\omega)\boldsymbol{H}, \boldsymbol{H}')_{\mathbf{curl}} = (\varepsilon^{-1}\boldsymbol{J}, \mathbf{curl}\,\boldsymbol{H}'). \end{array} \right. \tag{9}$$

*2) Assume that $A^\mu$ is an isomorphism. If $\boldsymbol{H}$ satisfies (9) then the pair $\{i\,(\omega\varepsilon)^{-1}(\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J}), \boldsymbol{H}\}$ satisfies (1)-(2).*

**Remark 3.4** *Again, in Section 7, we will provide examples of configurations, depending on the values of $\mu$ and on the geometry of the domain, where $A^\mu$ is an isomorphism and where $\mu$ changes sign on $\Omega$.*

## 4 T-coercivity operators for the Maxwell problem

The proof of the following lemma contains the main idea of the paper: we explain how to build T-coercivity operators for the Maxwell problem.

**Lemma 4.1** *Let $\Omega$ be a simply connected domain such that $\partial\Omega$ is connected. Assume that $A^\varepsilon$ and $A^\mu$ are isomorphisms. Then:*
- *There exists a bounded operator $\mathbb{T}^\varepsilon$ of $\mathbf{V}_N(\varepsilon;\Omega)$ such that, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}_N(\varepsilon;\Omega)$,*

$$(\mu^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\mathbb{T}^\varepsilon\boldsymbol{v}) = (\mu^{-1}\mathbf{curl}\,\mathbb{T}^\varepsilon\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}). \tag{10}$$

- *There exists a bounded operator $\mathbb{T}^\mu$ of $\mathbf{V}_T(\mu;\Omega)$ such that, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}_T(\mu;\Omega)$,*

$$(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\mathbb{T}^\mu\boldsymbol{v}) = (\varepsilon^{-1}\mathbf{curl}\,\mathbb{T}^\mu\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}). \tag{11}$$

**Proof.** Below, we focus our attention on the construction of the operator $\mathbb{T}^\varepsilon$. Consider $\boldsymbol{v} \in \mathbf{V}_N(\varepsilon;\Omega)$.
i) Introduce $\varphi$ the unique element of $\mathrm{H}^1_\#(\Omega)$ such that

$$(\mu\nabla\varphi, \nabla\varphi') = (\mu\,\mathbf{curl}\,\boldsymbol{v}, \nabla\varphi'), \qquad \forall \varphi' \in \mathrm{H}^1_\#(\Omega).$$

The function $\varphi$ is well-defined since we have assumed that $A^\mu$ is an isomorphism.
ii) Remark next that $\mu(\mathbf{curl}\,\boldsymbol{v} - \nabla\varphi)$ is a divergence free element of $\mathbf{L}^2(\Omega)$ such that $\mu(\mathbf{curl}\,\boldsymbol{v} - \nabla\varphi)\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. Since $\Omega$ is simply connected and since $\partial\Omega$ is connected, according to theorem 3.17 in [1] (see also theorem 3.6 in [22]), there exists a unique potential $\boldsymbol{\psi} \in \mathbf{V}_N(1;\Omega)$ such that $\mathbf{curl}\,\boldsymbol{\psi} = \mu(\mathbf{curl}\,\boldsymbol{v} - \nabla\varphi)$.
iii) Consider $\zeta$ the unique element of $\mathrm{H}^1_0(\Omega)$ such that

$$(\varepsilon\nabla\zeta, \nabla\zeta') = (\varepsilon\boldsymbol{\psi}, \nabla\zeta'), \qquad \forall \zeta' \in \mathrm{H}^1_0(\Omega).$$

The function $\zeta$ is well-defined since we have assumed that $A^\varepsilon$ is an isomorphism.
iv) Finally, define the bounded operator $\mathbb{T}^\varepsilon : \mathbf{V}_N(\varepsilon;\Omega) \to \mathbf{V}_N(\varepsilon;\Omega)$ such that $\mathbb{T}^\varepsilon\boldsymbol{v} = \boldsymbol{\psi} - \nabla\zeta$ for $\boldsymbol{v} \in$

$\mathbf{V}_N(\varepsilon;\,\Omega)$.
For all $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}_N(\varepsilon;\,\Omega)$, we then compute

$$\begin{aligned}(\mu^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\mathbb{T}^\varepsilon \boldsymbol{v}) &= (\mu^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,(\boldsymbol{\psi} - \nabla\zeta)) \\ &= (\mu^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{\psi}) \\ &= (\mu^{-1}\mathbf{curl}\,\boldsymbol{u}, \mu(\mathbf{curl}\,\boldsymbol{v} - \nabla\varphi)) \;=\; (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}),\end{aligned}$$

because $\boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Omega$. Observing that there also holds $(\mu^{-1}\mathbf{curl}\,\mathbb{T}^\varepsilon \boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})$, we are led to (10). Using the same approach, one can construct a bounded operator $\mathbb{T}^\mu : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)$ such that (11) holds. This ends the proof. ■

From this lemma, in order to prove that $\mathscr{A}_N(0) : \mathbf{V}_N(\varepsilon;\,\Omega) \to \mathbf{V}_N(\varepsilon;\,\Omega)$ (resp. $\mathscr{A}_T(0) : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)$) is an isomorphism when $A^\varepsilon$ and $A^\mu$ are isomorphisms, we still need to show that $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})$ defines an inner product on $\mathbf{V}_N(\varepsilon;\,\Omega)$ (resp. $\mathbf{V}_T(\mu;\,\Omega)$). This is the goal of the next section.

# 5 Compactness results

For $\xi \in \mathrm{L}^\infty(\Omega)$ such that $\xi^{-1} \in \mathrm{L}^\infty(\Omega)$, define

$$\begin{aligned}\mathbf{X}_N(\xi;\,\Omega) &:= \left\{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\,\Omega)\,|\,\mathrm{div}\,(\xi\,\boldsymbol{u}) \in \mathrm{L}^2(\Omega),\ \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\right\};\\ \mathbf{X}_T(\xi;\,\Omega) &:= \left\{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\,\Omega)\,|\,\mathrm{div}\,(\xi\,\boldsymbol{u}) \in \mathrm{L}^2(\Omega),\ \xi\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\right\}.\end{aligned}$$

Theses spaces are equipped with the norm $\boldsymbol{u} \mapsto (\|\boldsymbol{u}\|^2 + \|\mathrm{div}\,(\xi\,\boldsymbol{u})\|^2 + \|\mathbf{curl}\,\boldsymbol{u}\|^2)^{1/2}$. In this paragraph, we prove that $\mathbf{X}_N(\varepsilon;\,\Omega)$ and $\mathbf{X}_T(\mu;\,\Omega)$ are compactly embedded in $\mathbf{L}^2(\Omega)$ when $A^\varepsilon$ and $A^\mu$ are isomorphisms, extending the classical theorems of [38, 23, 28] (for another approach, based on the study of the regularity of fields, in 2D, when $\varepsilon$, $\mu$ change sign, see [13]). This constitutes a more general result than the one we actually need for our study, namely the compact embedding of $\mathbf{V}_N(\varepsilon;\,\Omega)$ and $\mathbf{V}_T(\mu;\,\Omega)$ in $\mathbf{L}^2(\Omega)$. We start by studying the space of electric fields.

**Theorem 5.1** *Let $\Omega$ be a simply connected domain such that $\partial\Omega$ is connected. Assume that $A^\varepsilon$ is an isomorphism. Then the embedding of $\mathbf{X}_N(\varepsilon;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact.*

**Proof.** Let $(\boldsymbol{u}_n)$ be a bounded sequence of $\mathbf{X}_N(\varepsilon;\,\Omega)$. Define $f_n := \mathrm{div}\,(\varepsilon\boldsymbol{u}_n)$ and $\boldsymbol{F}_n := \mathbf{curl}\,\boldsymbol{u}_n$. The sequences $(f_n)$ and $(\boldsymbol{F}_n)$ are respectively bounded in $\mathrm{L}^2(\Omega)$ and in $\mathbf{L}^2(\Omega)$. Since $A^\varepsilon$ is an isomorphism, there exists, for all $n \in \mathbb{N}$, $\varphi_n \in \mathrm{H}^1_0(\Omega)$ such that $\mathrm{div}\,(\varepsilon\nabla\varphi_n) = \mathrm{div}\,(\varepsilon\boldsymbol{u}_n)$. Then, we notice that $\varepsilon(\boldsymbol{u}_n - \nabla\varphi_n)$ is a divergence free element of $\mathbf{L}^2(\Omega)$. Since $\partial\Omega$ is connected, there exists (see [1], theorem 3.12) $\boldsymbol{w}_n \in \mathbf{V}_T(1;\,\Omega)$ such that $\mathbf{curl}\,\boldsymbol{w}_n = \varepsilon(\boldsymbol{u}_n - \nabla\varphi_n)$. Thus, for all $n \in \mathbb{N}$, one has $\boldsymbol{u}_n = \nabla\varphi_n + \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{w}_n$. Let us show now we can extract sequences from $(\nabla\varphi_n)$ and $(\mathbf{curl}\,\boldsymbol{w}_n)$ which converge in $\mathbf{L}^2(\Omega)$.

Since $A^\varepsilon$ is an isomorphism, $(\varphi_n)$ and $(A^\varepsilon\varphi_n)$ remain bounded in $\mathrm{H}^1_0(\Omega)$. But $\mathrm{H}^1_0(\Omega)$ is compactly embedded in $\mathrm{L}^2(\Omega)$. Therefore, we can extract a subsequence from $(\varphi_n)$ (still denoted $(\varphi_n)$) such that $(A^\varepsilon\varphi_n)$ converges in $\mathrm{L}^2(\Omega)$. Introduce $\varphi_{nm} = \varphi_n - \varphi_m$ and $f_{nm} = f_n - f_m$. By linearity, there holds: $-(\varepsilon\nabla\varphi_{nm}, \nabla\varphi') = (f_{nm}, \varphi')$, for all $\varphi' \in \mathrm{H}^1_0(\Omega)$. Taking $\varphi' = A^\varepsilon\varphi_{nm}$, one obtains

$$\|(A^\varepsilon)^{-1}\|^{-2}\;\|\varphi_{nm}\|^2_{\mathrm{H}^1_0(\Omega)} \le |(\nabla(A^\varepsilon\varphi_{nm}),\, \nabla(A^\varepsilon\varphi_{nm}))| = |(\varepsilon\nabla\varphi_{nm},\, \nabla(A^\varepsilon\varphi_{nm}))| = |(f_{nm}, A^\varepsilon\varphi_{nm})|\,.$$

This shows that $(\nabla\varphi_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega)$, and so, that it converges.

Now, let us work on the sequence $(\mathbf{curl}\,\boldsymbol{w}_n)$. We know that $\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|$ defines a norm on $\mathbf{V}_T(1;\,\Omega)$. Consequently, the sequence $(\boldsymbol{w}_n)$ is bounded in $\mathbf{V}_T(1;\,\Omega)$. By the compact embedding of $\mathbf{V}_T(1;\,\Omega)$ in $\mathbf{L}^2(\Omega)$, we can extract a subsequence, still denoted $(\boldsymbol{w}_n)$, which converges in $\mathbf{L}^2(\Omega)$. According to Lemma 4.1 (remark that $A^\mu$ with $\mu = 1$ is indeed an isomorphism), there exists an operator $\mathbb{T}^1$ of $\mathbf{V}_T(1;\,\Omega)$ such that

$$\left|(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{w}, \mathbf{curl}\,\mathbb{T}^1\boldsymbol{w})\right| = \|\mathbf{curl}\,\boldsymbol{w}\|^2\,, \qquad \forall\boldsymbol{w} \in \mathbf{V}_T(1;\,\Omega).$$

Since $\mathbb{T}^1$ is continuous, the sequence $(\mathbb{T}^1\boldsymbol{w}_n)$ is bounded in $\mathbf{V}_T(1;\,\Omega)$. So, we can extract a subsequence from $(\boldsymbol{w}_n)$, still denoted $(\boldsymbol{w}_n)$, such that $(\mathbb{T}^1\boldsymbol{w}_n)$ converges in $\mathbf{L}^2(\Omega)$. Since $\mathbf{curl}\,\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{w}_n = \boldsymbol{F}_n$ in $\Omega$ and $(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{w}_n) \times \boldsymbol{n} = 0$ on $\partial\Omega$, one has $(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{w}_{nm}, \mathbf{curl}\,\boldsymbol{w}') = (\boldsymbol{F}_{nm}, \boldsymbol{w}')$, for all $\boldsymbol{w}' \in \mathbf{V}_T(1;\,\Omega)$ with $\boldsymbol{w}_{nm} = \boldsymbol{w}_n - \boldsymbol{w}_m$ and $\boldsymbol{F}_{nm} = \boldsymbol{F}_n - \boldsymbol{F}_m$. Testing with $\boldsymbol{w}' = \mathbb{T}^1\boldsymbol{w}_{nm}$ leads to:

$$\|\mathbf{curl}\,\boldsymbol{w}_{nm}\|^2 = \left|(\boldsymbol{F}_{nm}, \mathbb{T}^1\boldsymbol{w}_{nm})\right|.$$

This estimate proves that $(\mathbf{curl}\,\boldsymbol{w}_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega)$. Consequently, it converges. ■

**Corollary 5.2** *Let $\Omega$ be a simply connected domain such that $\partial\Omega$ is connected. Assume that $A^\varepsilon$ is an isomorphism. Then, there exists a constant $C > 0$ such that*

$$\|\boldsymbol{u}\|^2 \leq C\, \|\mathbf{curl}\,\boldsymbol{u}\|^2, \qquad \forall \boldsymbol{u} \in \mathbf{V}_N(\varepsilon;\, \Omega). \tag{12}$$

*Thus, the map $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})$ defines an inner product on $\mathbf{V}_N(\varepsilon;\, \Omega)$ and the associated norm is equivalent to the canonical norm $\boldsymbol{u} \mapsto (\boldsymbol{u}, \boldsymbol{u})^{1/2}_{\mathbf{curl}}$.*

**Proof.** To prove this corollary, it is sufficient to show (12). Let us proceed by contradiction assuming there exists a sequence $(\boldsymbol{u}_n)$ of elements of $\mathbf{V}_N(\varepsilon;\, \Omega)$ such that

$$\forall n \in \mathbb{N},\ \|\boldsymbol{u}_n\| = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \|\mathbf{curl}\,\boldsymbol{u}_n\| = 0.$$

According to Theorem 5.1, we can extract a sequence from $(\boldsymbol{u}_n)$ (still denoted $(\boldsymbol{u}_n)$) which converges to $\boldsymbol{u}$ in $\mathbf{L}^2(\Omega)$. By construction, we have $\|\boldsymbol{u}\| = 1$. Then, one can check easily that $(\boldsymbol{u}_n)$ converges to $\boldsymbol{u}$ in $\mathbf{H}_N(\mathbf{curl}\,;\, \Omega)$ with $\mathbf{curl}\,\boldsymbol{u} = 0$ a.e. in $\Omega$. Since $\partial\Omega$ is connected, one deduces (see [11], chapter 2, theorem 8) that there exists a scalar potential $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} = \nabla\varphi$ in $\Omega$. Finally, we notice that $\operatorname{div}(\varepsilon\boldsymbol{u}) = 0$ and so $A^\varepsilon\varphi = 0$. This implies $\varphi = 0$ so $\boldsymbol{u} = 0$. This leads to a contradiction because we must have $\|\boldsymbol{u}\| = 1$. ■

Analogously, we can prove successively the following results for the space of magnetic fields.

**Theorem 5.3** *Let $\Omega$ be a simply connected domain such that $\partial\Omega$ is connected. Assume that $A^\mu$ is an isomorphism. Then, the embedding of $\mathbf{X}_T(\mu;\, \Omega)$ in $\mathbf{L}^2(\Omega)$ is compact.*

**Corollary 5.4** *Let $\Omega$ be a simply connected domain such that $\partial\Omega$ is connected. Assume that $A^\mu$ is an isomorphism. Then, there exists a constant $C > 0$ such that*

$$\|\boldsymbol{u}\|^2 \leq C\, \|\mathbf{curl}\,\boldsymbol{u}\|^2, \qquad \forall \boldsymbol{u} \in \mathbf{V}_T(\mu;\, \Omega).$$

*Thus, the map $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})$ defines an inner product on $\mathbf{V}_T(\mu;\, \Omega)$ and the associated norm is equivalent to the canonical norm $\boldsymbol{u} \mapsto (\boldsymbol{u}, \boldsymbol{u})^{1/2}_{\mathbf{curl}}$.*

# 6 Well-posedness of Maxwell's equations

We now have all the tools to prove the main result of this paper.

**Theorem 6.1** *Let $\Omega$ be a simply connected domain such that $\partial\Omega$ is connected. Assume that $A^\varepsilon$ and $A^\mu$ are isomorphisms. Then, the following results hold.*

- *The operator for the electric field $\mathscr{A}_N(\omega) : \mathbf{V}_N(\varepsilon;\, \Omega) \to \mathbf{V}_N(\varepsilon;\, \Omega)$, defined in (5), is an isomorphism for all $\omega \in \mathbb{C}\backslash S$, where $S \subset \mathbb{R}$ is a discrete set.*
- *The operator for the magnetic field $\mathscr{A}_T(\omega) : \mathbf{V}_T(\mu;\, \Omega) \to \mathbf{V}_T(\mu;\, \Omega)$, defined in (8), is an isomorphism for all $\omega \in \mathbb{C}\backslash S$, where $S \subset \mathbb{R}$ is a discrete set.*
- *Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\backslash S$, where $S \subset \mathbb{R}$ is a discrete set.*

**Proof.** Let us begin with the first point. Lemma 4.1 ensures the existence of a bounded map $\mathbb{T}^\varepsilon : \mathbf{V}_N(\varepsilon;\, \Omega) \to \mathbf{V}_N(\varepsilon;\, \Omega)$ such that, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}_N(\varepsilon;\, \Omega)$,

$$(\mathscr{A}_N(0)(\mathbb{T}^\varepsilon\boldsymbol{u}), \boldsymbol{v})_{\mathbf{curl}} = (\mu^{-1}\mathbf{curl}\,\mathbb{T}^\varepsilon\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}).$$

According to Corollary 5.2, $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})$ defines an inner product on $\mathbf{V}_N(\varepsilon;\, \Omega)$. Consequently, the operator $\mathscr{A}_N(0) \circ \mathbb{T}^\varepsilon$ is an isomorphism of $\mathbf{V}_N(\varepsilon;\, \Omega)$. Since $\mathscr{A}_N(0)$ is selfadjoint, we deduce that $\mathscr{A}_N(0)$ and $\mathbb{T}^\varepsilon$ are isomorphisms. On the other hand, Theorem 5.1 guarantees that $\mathbf{V}_N(\varepsilon;\, \Omega)$ is compactly embedded in $\mathbf{L}^2(\Omega)$. As a consequence, $\mathscr{A}_N(\omega) - \mathscr{A}_N(0)$ is a compact operator of $\mathbf{V}_N(\varepsilon;\, \Omega)$ for all $\omega \in \mathbb{C}$. The analytic Fredholm theorem then allows us to conclude. The second point can be proven in the same way while one shows the third statement thanks to Theorems 3.1 and 3.3. ■

**Remark 6.2** *If in Eq. (1) one considers $\boldsymbol{J}$ such that $\operatorname{div}\boldsymbol{J} \neq 0$, it follows that $\operatorname{div}(\varepsilon\boldsymbol{E}) = (i\omega)^{-1}\operatorname{div}\boldsymbol{J} \neq 0$. However, if one assumes that $A^\varepsilon$ is an isomorphism, one can solve the problem "Find $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $(\varepsilon\nabla\varphi, \nabla\varphi') = (i\omega)^{-1}(\boldsymbol{J}, \nabla\varphi')$, for all $\varphi' \in \mathrm{H}^1_0(\Omega)$". Then, one can proceed exactly as before with $\{\boldsymbol{J} - i\omega\varepsilon\nabla\varphi, \boldsymbol{E} - \nabla\varphi\}$ replacing $\{\boldsymbol{J}, \boldsymbol{E}\}$ in Eq. (1).*

**Remark 6.3** *One can also prove the following "reciprocal" assertions.*
*- If there exists an isomorphism $\mathbb{T}^1$ of $\mathbf{V}_N(1;\Omega)$ such that $(\mu^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\mathbb{T}^1\boldsymbol{u}) \geq \|\mathbf{curl}\,\boldsymbol{u}\|^2$ for all $\boldsymbol{u} \in \mathbf{V}_N(1;\Omega)$, then $A^\mu$ is an isomorphism.*
*- If there exists an isomorphism $\mathbb{T}^1$ of $\mathbf{V}_T(1;\Omega)$ such that $(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\mathbb{T}^1\boldsymbol{u}) \geq \|\mathbf{curl}\,\boldsymbol{u}\|^2$ for all $\boldsymbol{u} \in \mathbf{V}_T(1;\Omega)$, then $A^\varepsilon$ is an isomorphism.*

# 7 Illustrations

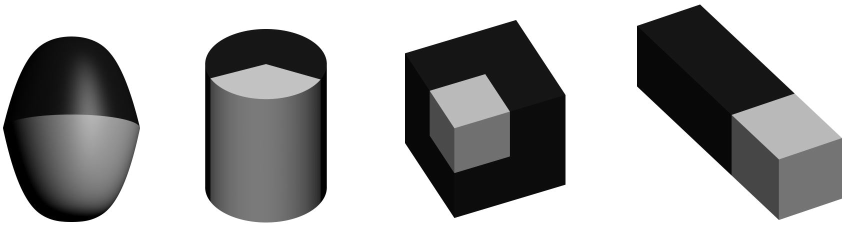

Figure 1: Canonical geometries: symmetric domain, prismatic edge, Fichera's corner, non symmetric cavity.

We apply Theorem 6.1 in a few simple geometries. We focus on situations where the medium consists of two different materials. To model this problem, we assume that $\Omega$ is divided into two sub-domains $\Omega_1$ and $\Omega_2$ with $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ et $\Omega_1 \cap \Omega_2 = \emptyset$. We denote $\Sigma := \partial\Omega_1 \setminus \partial\Omega = \partial\Omega_2 \setminus \partial\Omega$. Let us introduce $\varepsilon_1$ and $\mu_1$ (resp. $\varepsilon_2$ and $\mu_2$) two elements of $\mathrm{L}^\infty(\Omega_1)$ (resp. $\mathrm{L}^\infty(\Omega_2)$). Define the functions $\varepsilon$ and $\mu$ such that $\varepsilon|_{\Omega_k} = \varepsilon_k$ and $\mu|_{\Omega_k} = \mu_k$ for $k = 1, 2$. We assume that $\Omega_1$ is filled with a positive material and that $\Omega_2$ is filled with a possibly negative material (for $\varepsilon$ and/or $\mu$). For that, we make the following assumptions:
– there exists a constant $C$ s.t. $\varepsilon_1 \geq C > 0$ and $\mu_1 \geq C > 0$ a.e. in $\Omega_1$;
– there exists a constant $C$ s.t. $\varepsilon_2 \geq C > 0$ a.e. in $\Omega_2$ or $\varepsilon_2 \leq -C < 0$ a.e. in $\Omega_2$ ;
– there exists a constant $C$ s.t. $\mu_2 \geq C > 0$ a.e. in $\Omega_2$ or $\mu_2 \leq -C < 0$ a.e. in $\Omega_2$.

In particular, notice that $\varepsilon^{-1} \in \mathrm{L}^\infty(\Omega)$ and $\mu^{-1} \in \mathrm{L}^\infty(\Omega)$. Then, we define

$$\sigma_1^+ := \sup_{\Omega_1} \sigma_1, \quad \sigma_2^+ := \sup_{\Omega_2} |\sigma_2|, \quad \sigma_1^- := \inf_{\Omega_1} \sigma_1 \quad \text{and} \quad \sigma_2^- := \inf_{\Omega_2} |\sigma_2|, \quad \text{for } \sigma = \varepsilon, \mu.$$

Generally speaking, if $v$ is a measurable function on $\Omega$, we use the notation $v_k := v|_{\Omega_k}$, $k = 1, 2$. For the first three examples, to obtain criteria on $\varepsilon$, $\mu$ ensuring that $A^\varepsilon$ and $A^\mu$ are isomorphisms, we use the geometric version of the $\mathtt{T}$-coercivity as in [3]. More precisely, thanks to simple geometric transformations, we construct isomorphisms $\mathtt{T}^\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ (resp. $\mathtt{T}^\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$) such that $A^\varepsilon \circ \mathtt{T}^\varepsilon$ (resp. $A^\mu \circ \mathtt{T}^\mu$) is an isomorphism. Observe that this indeed implies that $A^\varepsilon$ (resp. $A^\mu$) is an isomorphism.

**Remark 7.1** *We wish to emphasize that in this section, our goal is just to give a flavour of the results existing for the scalar problems (which lead directly to results for the Maxwell problem) as well as an idea of how to prove them. General statements cannot be presented without introducing rather heavy notations to specify the geometry of the domain and the features of the parameters $\varepsilon$, $\mu$. For a more complete description of the properties of $A^\varepsilon$ and $A^\mu$, we refer the reader to [3].*

## 7.1 Symmetric domain

Let $\Omega$ be a symmetric domain, in the sense that $\Omega_1$ and $\Omega_2$ can be mapped from one to the other with the help of a reflection symmetry. Without loss of generality, we assume that the interface $\Sigma$ is included in the plane $z = 0$ (see Figure 1, left, for an example). Consider the operators $R_1$ and $R_2$ respectively defined by $(R_1\varphi_1)(x,y,z) = \varphi_1(x,y,-z)$ and $(R_2\varphi_2)(x,y,z) = \varphi_2(x,y,-z)$ for $\varphi \in \mathrm{H}^1(\Omega)$, where $(x,y,z)$ denote the cartesian coordinates. Define the operators $\mathtt{T}_1$ and $\mathtt{T}_2$ such that:

$$\mathtt{T}_1\varphi = \begin{cases} \varphi_1 & \text{in } \Omega_1 \\ -\varphi_2 + 2R_1\varphi_1 & \text{in } \Omega_2 \end{cases} ; \quad \mathtt{T}_2\varphi = \begin{cases} \varphi_1 - 2R_2\varphi_2 & \text{in } \Omega_1 \\ -\varphi_2 & \text{in } \Omega_2 \end{cases} .$$

By construction, $\mathtt{T}_1\varphi$ and $\mathtt{T}_2\varphi$ belong to $\mathrm{H}^1(\Omega)$. As $\mathtt{T}_1 \circ \mathtt{T}_1 = \mathtt{T}_2 \circ \mathtt{T}_2 = \mathrm{Id}$, we deduce that $\mathtt{T}_1$ and $\mathtt{T}_2$ are isomorphisms of $\mathrm{H}^1(\Omega)$. The restrictions $\mathtt{T}_1^\varepsilon$ and $\mathtt{T}_2^\varepsilon$ of $\mathtt{T}_1$ and $\mathtt{T}_2$ to $\mathrm{H}^1_0(\Omega)$ are isomorphisms of $\mathrm{H}^1_0(\Omega)$.
Let us introduce the linear form $\gamma : \mathrm{H}^1(\Omega) \to \mathbb{R}$ such that $\gamma(\varphi) = \int_\Omega \varphi / \int_\Omega 1$. Note that $\ker(\gamma) = \mathrm{H}^1_\#(\Omega)$.

Then, we define the operators $\mathtt{T}_1^\mu$ and $\mathtt{T}_2^\mu$ such that, for all $\varphi \in \mathrm{H}^1_\#(\Omega)$, $\mathtt{T}_1^\mu\varphi = \mathtt{T}_1\varphi - \gamma(\mathtt{T}_1\varphi)$ and $\mathtt{T}_2^\mu\varphi = \mathtt{T}_2\varphi - \gamma(\mathtt{T}_2\varphi)$. Notice that $\mathtt{T}_1^\mu\varphi$ and $\mathtt{T}_2^\mu\varphi$ are elements of $\mathrm{H}^1_\#(\Omega)$. Moreover, we have

$$\begin{array}{rclcl} \mathtt{T}_1^\mu(\mathtt{T}_1^\mu\varphi) & = & \mathtt{T}_1^\mu(\mathtt{T}_1\varphi - \gamma(\mathtt{T}_1\varphi)) & = & \mathtt{T}_1(\mathtt{T}_1\varphi - \gamma(\mathtt{T}_1\varphi)) - \gamma(\mathtt{T}_1(\mathtt{T}_1\varphi - \gamma(\mathtt{T}_1\varphi))) \\ & & & = & \varphi - \mathtt{T}_1(\gamma(\mathtt{T}_1\varphi)) - \gamma(\varphi - \mathtt{T}_1(\gamma(\mathtt{T}_1\varphi))) \\ & & & = & \varphi - \mathtt{T}_1(\gamma(\mathtt{T}_1\varphi)) + \gamma(\mathtt{T}_1(\gamma(\mathtt{T}_1\varphi))) \;=\; \varphi. \end{array}$$

Thus, $\mathtt{T}_1^\mu \circ \mathtt{T}_1^\mu = \mathrm{Id}$. In the same way, we find $\mathtt{T}_2^\mu \circ \mathtt{T}_2^\mu = \mathrm{Id}$. Hence $\mathtt{T}_1^\mu$ and $\mathtt{T}_2^\mu$ are isomorphisms of $\mathrm{H}^1_\#(\Omega)$.

**Proposition 7.2** (Symmetric domain)
*Assume that $\varepsilon$ satisfies $\varepsilon \geq C > 0$ a.e. in $\Omega$ or $\max(\varepsilon_1^-/\varepsilon_2^+, \varepsilon_2^-/\varepsilon_1^+) > 1$.*
*Assume that $\mu$ satisfies $\mu \geq C > 0$ a.e. in $\Omega$ or $\max(\mu_1^-/\mu_2^+, \mu_2^-/\mu_1^+) > 1$.*
*Then, Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\backslash S$ where $S \subset \mathbb{R}$ is a discrete set.*

**Proof.** Apply Theorem 6.1. To check that $A^\varepsilon$ and $A^\mu$ are isomorphisms, use the following table [3].

| For | $\varepsilon \geq C > 0$ | $\varepsilon_1^-/\varepsilon_2^+ > 1$ | $\varepsilon_2^-/\varepsilon_1^+ > 1$ |
|---|---|---|---|
| Take $\mathtt{T}^\varepsilon$ equal to | Id | $\mathtt{T}_1^\varepsilon$ | $\mathtt{T}_2^\varepsilon$ |

| For | $\mu \geq C > 0$ | $\mu_1^-/\mu_2^+ > 1$ | $\mu_2^-/\mu_1^+ > 1$ |
|---|---|---|---|
| Take $\mathtt{T}^\mu$ equal to | Id | $\mathtt{T}_1^\mu$ | $\mathtt{T}_2^\mu$ |

■

In the case where $\varepsilon$ and $\mu$ are constant on each side of the interface, the statement of Proposition 7.2 can be further simplified.

**Proposition 7.3** (Symmetric domain: piecewise constant coefficients)
*Assume that $\varepsilon_1$, $\varepsilon_2$, $\mu_1$ and $\mu_2$ are constant numbers. Then, if $\varepsilon_2/\varepsilon_1, \mu_2/\mu_1 \in \mathbb{R}^*\backslash\{-1\}$, Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\backslash S$ where $S \subset \mathbb{R}$ is a discrete set.*

## 7.2 Prismatic edge

Consider the geometry of Figure 1, middle-left. Introduce the cylindrical coordinates $(r, \theta, z)$ centered on the edge, so that the cartesian coordinates are mapped as $(x, y, z) = (r\cos\theta, r\sin\theta, z)$. Let $H > 0$ denote the height of the cylinder, $R > 0$ its radius. Given $0 < \alpha < 2\pi$, define

$$\begin{array}{l} \Omega_1 := \{(r\cos\theta, r\sin\theta, z)\,|\,0 < r < R,\ 0 < \theta < \alpha,\ 0 < z < H\}\ ; \\ \Omega_2 := \{(r\cos\theta, r\sin\theta, z)\,|\,0 < r < R,\ \alpha < \theta < 2\pi,\ 0 < z < H\}\ . \end{array}$$

Introduce the two operators $R_1$ and $R_2$ such that $(R_1\varphi_1)(r, \theta, z) = \varphi_1(r, \frac{\alpha}{\alpha - 2\pi}(\theta - 2\pi), z)$ and $(R_2\varphi_2)(r, \theta, z) = \varphi_2(r, \frac{\alpha - 2\pi}{\alpha}\theta + 2\pi, z)$ for $\varphi \in \mathrm{H}^1(\Omega)$.
Proceeding as for the case of the symmetric domain, one obtains the

**Proposition 7.4** (Prismatic edge)
*Define $I_\alpha := \max(\frac{\alpha}{2\pi - \alpha}, \frac{2\pi - \alpha}{\alpha})$.*
*Assume that $\varepsilon$ satisfies $\varepsilon \geq C > 0$ a.e. in $\Omega$ or $\max(\varepsilon_1^-/\varepsilon_2^+, \varepsilon_2^-/\varepsilon_1^+) > I_\alpha$.*
*Assume that $\mu$ satisfies $\mu \geq C > 0$ a.e. in $\Omega$ or $\max(\mu_1^-/\mu_2^+, \mu_2^-/\mu_1^+) > I_\alpha$.*
*Then, Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\backslash S$ where $S \subset \mathbb{R}$ is a discrete set.*

**Proposition 7.5** (Prismatic edge: piecewise constant coefficients)
*Assume that $\varepsilon_1$, $\varepsilon_2$, $\mu_1$ and $\mu_2$ are constant numbers. Define $I_\alpha := \max(\frac{\alpha}{2\pi - \alpha}, \frac{2\pi - \alpha}{\alpha})$. Then, if $\varepsilon_2/\varepsilon_1, \mu_2/\mu_1 \in \mathbb{R}^*\backslash[-I_\alpha; -1/I_\alpha]$, Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\backslash S$ where $S \subset \mathbb{R}$ is a discrete set.*

### 7.3 Fichera corner

Consider the geometry of Figure 1, middle-right. More precisely, define $\Omega := (-1;1)^3$, $\Omega_1 := (0;1)^3$ and $\Omega_2 := \Omega\setminus\overline{\Omega_1}$. Introduce the operators $R_1$, $R_2$, such that, for $\varphi \in \mathrm{H}^1(\Omega)$,

$$
\begin{array}{rcl}
(R_1\varphi_1)(x,y,z) & = & \left\{\begin{array}{lll}
\varphi_1(-x,y,z) & \text{in} & \Omega_2^1 := (-1;0)\times(0;1)^2\\
\varphi_1(x,-y,z) & \text{in} & \Omega_2^2 := (0;1)\times(-1;0)\times(0;1)\\
\varphi_1(x,y,-z) & \text{in} & \Omega_2^3 := (0;1)^2\times(-1;0)\\
\varphi_1(-x,-y,z) & \text{in} & \Omega_2^4 := (-1;0)^2\times(0;1)\\
\varphi_1(-x,y,-z) & \text{in} & \Omega_2^5 := (-1;0)\times(0;1)\times(-1;0)\\
\varphi_1(x,-y,-z) & \text{in} & \Omega_2^6 := (0;1)\times(-1;0)^2\\
\varphi_1(-x,-y,-z) & \text{in} & \Omega_2^7 := (-1;0)^3
\end{array}\right. ;\\[2ex]
(R_2\varphi_2)(x,y,z) & = & \varphi_2^1(-x,y,z)+\varphi_2^2(x,-y,z)+\varphi_2^3(x,y,-z)\\
 & & -\varphi_2^4(-x,-y,z)-\varphi_2^5(-x,y,-z)-\varphi_2^6(x,-y,-z)\\
 & & +\varphi_2^7(-x,-y,-z).
\end{array}
$$

Above, for $\ell = 1\dots 7$, $\varphi_2^\ell$ is the restriction of $\varphi_2$ to $\Omega_2^\ell$.
Again, proceeding as for the case of the symmetric domain, one obtains the

**Proposition 7.6** (Fichera's corner)
*Assume that $\varepsilon$ satisfies $\varepsilon \ge C > 0$ a.e. in $\Omega$ or $\max(\varepsilon_1^-/\varepsilon_2^+, \varepsilon_2^-/\varepsilon_1^+) > 7$.
Assume that $\mu$ satisfies $\mu \ge C > 0$ a.e. in $\Omega$ or $\max(\mu_1^-/\mu_2^+, \mu_2^-/\mu_1^+) > 7$.
Then, Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\setminus S$ where $S \subset \mathbb{R}$ is a discrete set.*

**Proposition 7.7** (Fichera's corner: piecewise constant coefficients)
*Assume that $\varepsilon_1$, $\varepsilon_2$, $\mu_1$ and $\mu_2$ are constant numbers. Then, if $\varepsilon_2/\varepsilon_1, \mu_2/\mu_1 \in \mathbb{R}^*\setminus[-7;-1/7]$, Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\setminus S$ where $S \subset \mathbb{R}$ is a discrete set.*

### 7.4 Non symmetric cavity

Let us consider the non symmetric cavity of Figure 1. More precisely, define $\Omega := \{(x,y,z) \in (-a;b)\times(0;1)\times(0;1)\}$, $\Omega_1 := (-a;0)\times(0;1)\times(0;1)$ and $\Omega_2 := (0;b)\times(0;1)\times(0;1)$ with $a > 0$ and $b > 0$. The interface $\Sigma$ is then equal to $\{0\}\times(0;1)\times(0;1)$. Assume that $\varepsilon_1$, $\varepsilon_2$, $\mu_1$ and $\mu_2$ are constant numbers.

For this particular geometry, we know (see [3]) that the operator $A^\varepsilon$ (resp. $A^\mu$) is Fredholm of index 0 (see Definition 8.1 below) if and only if $\varepsilon_2/\varepsilon_1 \neq -1$ (resp. $\mu_2/\mu_1 \neq -1$). To apply Theorem 6.1, we need $A^\varepsilon$ and $A^\mu$ to be isomorphisms. Therefore, it is necessary to study the question of the injectivity of $A^\varepsilon$ and $A^\mu$. Let us start with $A^\varepsilon$. Consider $\varphi$ an element of $\mathrm{H}^1_0(\Omega)$ such that $A^\varepsilon\varphi = 0$. The pair $(\varphi_1,\varphi_2)$ satisfies the equations

$$
\begin{array}{rclcrcll}
\Delta\varphi_1 & = & 0 & \text{in } \Omega_1; & \varphi_1-\varphi_2 & = & 0 & \text{on } \Sigma;\\
\Delta\varphi_2 & = & 0 & \text{in } \Omega_2; & \varepsilon_1\partial_x\varphi_1-\varepsilon_2\partial_x\varphi_2 & = & 0 & \text{on } \Sigma.
\end{array}
$$

Decomposing $\varphi_1$ and $\varphi_2$ in Fourier series (the family $\{(y,z)\mapsto \sin(m\pi y)\sin(n\pi z)\}_{m,n=1}^{\infty}$ is a basis of $L^2((0;1)\times(0;1))$) and writing the transmission conditions on $\Sigma$, one finds that $A^\varepsilon : \mathrm{H}^1_0(\Omega)\to\mathrm{H}^1_0(\Omega)$ is injective if and only if $\varepsilon_2/\varepsilon_1$ is not an element of

$$
\mathscr{S}_\varepsilon := \{-\tanh(\sqrt{m^2+n^2}\pi b)/\tanh(\sqrt{m^2+n^2}\pi a),\,(m,n)\in\mathbb{N}^*\times\mathbb{N}^*\}. \tag{13}
$$

Following the same approach, exchanging the "sin" by "cos" to satisfy the Neumann condition, one can prove that $A^\mu : \mathrm{H}^1_\#(\Omega)\to\mathrm{H}^1_\#(\Omega)$ is injective if and only if $\mu_2/\mu_1$ is not an element of

$$
\mathscr{S}_\mu := \{-\tanh(\sqrt{m^2+n^2}\pi a)/\tanh(\sqrt{m^2+n^2}\pi b),\,(m,n)\in\mathbb{N}^*\times\mathbb{N}^*\} = \{1/\rho,\,\rho\in\mathscr{S}_\varepsilon\}. \tag{14}
$$

**Remark 7.8** *The map $g : z\mapsto -\tanh(z\pi b)/\tanh(z\pi a)$ is continuous, strictly decreasing if $a > b$ and strictly increasing if $a < b$. Besides, we have $\lim_{z\to+\infty} g(z) = -1$. As a consequence, $-1$ is an accumulation point of both sets $\mathscr{S}_\varepsilon$ and $\mathscr{S}_\mu$.*

**Remark 7.9** *For this particular 3D geometry, we obtain a result specific to 2D configurations (see [5]): the problem with Dirichlet boundary condition for the coefficient $\varepsilon$ is well-posed if and only if the problem with Neumann boundary condition is well-posed for the coefficient $\mu := \varepsilon^{-1}$.*

We deduce the

**Proposition 7.10** (Non symmetric cavity: piecewise constant coefficients)
*Assume that* $\varepsilon_1$, $\varepsilon_2$, $\mu_1$ *and* $\mu_2$ *are constant numbers. Assume that* $\varepsilon_2/\varepsilon_1 \in \mathbb{R}^*\backslash\{\,\mathscr{S}_\varepsilon \cup \{-1\}\,\}$ *and* $\mu_2/\mu_1 \in \mathbb{R}^*\backslash\{\,\mathscr{S}_\mu \cup \{-1\}\,\}$, *with* $\mathscr{S}_\varepsilon$ *and* $\mathscr{S}_\mu$ *respectively defined in (13) and (14). Then, Maxwell's equations (1)-(2) are uniquely solvable for all* $\omega \in \mathbb{C}^*\backslash S$ *where* $S \subset \mathbb{R}$ *is a discrete set.*

# 8 Relaxing the main hypotheses

To prove the previous results, we rely extensively on two types of hypotheses. On the one hand, we assume that $A^\varepsilon$ and $A^\mu$ are isomorphisms. On the other hand, the domain $\Omega$ is supposed to be simply connected, with a connected boundary. We would like now to relax these assumptions.

Concerning the hypotheses on the geometry, the difficulty is well-known (see for instance [11]). For instance, if the boundary $\partial\Omega$ is not connected, the space $\mathbf{V}_N(\varepsilon;\,\Omega)$ contains non-trivial curl free fields $\nabla\varphi$, so that $\boldsymbol{u} \mapsto (\boldsymbol{u},\boldsymbol{u})^{1/2}_{\mathbf{curl}}$ is not a norm on $\mathbf{V}_N(\varepsilon;\,\Omega)$ anymore. The same occurs for $\mathbf{V}_T(\mu;\,\Omega)$ when $\Omega$ is not simply connected.
At first glance, relaxing the assumptions on $A^\varepsilon$ and $A^\mu$ has similar consequences. For example, suppose that there is a non-trivial function $\tilde{\varphi}$ in the kernel of $A^\varepsilon$. In such a situation, the non-trivial curl free field $\nabla\tilde{\varphi}$ belongs to $\mathbf{V}_N(\varepsilon;\,\Omega)$. Once more, $\boldsymbol{u} \mapsto (\boldsymbol{u},\boldsymbol{u})^{1/2}_{\mathbf{curl}}$ is not a norm on $\mathbf{V}_N(\varepsilon;\,\Omega)$.

However, we observe a fundamental difference between the scalar potentials which are built in the two cases: $\varphi \notin \mathrm{H}^1_0(\Omega)$, whereas $\tilde{\varphi} \in \mathrm{H}^1_0(\Omega)$. As a consequence, the field $\tilde{\boldsymbol{E}} = \nabla\tilde{\varphi}$ verifies

$$(\varepsilon\tilde{\boldsymbol{E}}, \boldsymbol{E}') = 0, \qquad \forall \boldsymbol{E}' \in \mathbf{V}_N(\varepsilon;\,\Omega),$$

which is not true for $\boldsymbol{E} = \nabla\varphi$. So, $\tilde{\boldsymbol{E}}$ is a solution of the homogeneous problem ($\boldsymbol{J} = 0$) for the electric field (6) stated in $\mathbf{V}_N(\varepsilon;\,\Omega)$ but not to the homogeneous problem (3) stated in $\mathbf{H}_N(\mathbf{curl}\,;\,\Omega)$. In other words, when the scalar problems have non-trivial kernels, Theorems 3.1 and 3.3 (equivalence with the original Maxwell's problem) are no longer true.

Summing up, we see that the difficulties which occur when relaxing either hypotheses on the geometry or hypotheses on the scalar problem present some similarities (existence of admissible fields which are both divergence free and curl free) but also some fundamental differences (formulations in $\mathbf{V}_N(\varepsilon;\,\Omega)$ and $\mathbf{V}_T(\mu;\,\Omega)$ are no longer equivalent to the original problem when the scalar problem have non-trivial kernels).
Since the non-injectivity of the scalar problems is a difficulty which is specific to the presence of sign-changing coefficients $\varepsilon$ and/or $\mu$, it will be treated first in Subsection 8.1. Then, in Subsection 8.2, we shall check that the usual treatment for non-trivial geometries can be extended to sign-changing coefficients.

## 8.1 Extension to non injective scalar problems

We have introduced the bounded operators

$$\begin{array}{lll} & A^\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega) \quad \text{s.t. } (\nabla(A^\varepsilon\varphi), \nabla\varphi') = (\varepsilon\nabla\varphi,\, \nabla\varphi'), & \forall \varphi, \varphi' \in \mathrm{H}^1_0(\Omega); \\ \text{and} & A^\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \quad \text{s.t. } (\nabla(A^\mu\varphi), \nabla\varphi') = (\mu\nabla\varphi,\, \nabla\varphi'), & \forall \varphi, \varphi' \in \mathrm{H}^1_\#(\Omega). \end{array}$$

Theorem 6.1 indicates that Maxwell's equations (1)-(2) are well-posed in the Fredholm sense when $A^\varepsilon$ and $A^\mu$ are isomorphisms. In this section, we wish to consider situations where the physical parameters $\varepsilon$, $\mu$ and the geometry are such that

$$(\mathscr{H}^\varepsilon) \quad |\ \text{The operator } A^\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega) \text{ is Fredholm of index 0 and non injective;}$$

$$(\mathscr{H}^\mu) \quad |\ \text{The operator } A^\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \text{ is Fredholm of index 0 and non injective.}$$

For ease of reading, we recall the definition of a Fredholm operator (see [39, 27]).

**Definition 8.1** *Let* X *and* Y *be two Banach spaces, and let* $L : \mathrm{X} \to \mathrm{Y}$ *be a continuous linear map. The operator* $L$ *is said to be a Fredholm operator if and only if the following two conditions are fulfilled*

*i)* $\dim(\ker L) < \infty$ *and* $\operatorname{range} L$ *is closed;*

*ii)* $\dim(\operatorname{coker} L) < \infty$ *where* $\operatorname{coker} L := \big(\mathrm{Y}/\operatorname{range} L\big)$.

*Besides, the index of a Fredholm operator* $L$ *is defined by* $\operatorname{ind} L = \dim(\ker L) - \dim(\operatorname{coker} L)$.

Regarding the non symmetric cavity described in §7.4, this corresponds to considering the case where $\kappa_\varepsilon \in \mathscr{S}_\varepsilon$ and $\kappa_\mu \in \mathscr{S}_\mu$. For the sake of brevity, we will focus on configurations where $A^\varepsilon$ and $A^\mu$ both have a kernel non reduced to zero. When only one of these two operators is not injective, the study of the Maxwell's equations can be easily inferred from the one we present below.

Let us introduce $\{\lambda_i^\varepsilon\}_{i=1}^{N^\varepsilon}$ a basis of ker $A^\varepsilon$ such that $(\nabla\lambda_i^\varepsilon, \nabla\lambda_j^\varepsilon) = \delta_{ij}$ and $\{\lambda_i^\mu\}_{i=1}^{N^\mu}$ a basis of ker $A^\mu$ such that $(\nabla\lambda_i^\mu, \nabla\lambda_j^\mu) = \delta_{ij}$. Define the spaces $\mathrm{S}^\varepsilon$ and $\mathrm{S}^\mu$ such that

$$\mathrm{H}^1_0(\Omega) = \ker A^\varepsilon \overset{\perp}{\oplus} \mathrm{S}^\varepsilon \quad \text{and} \quad \mathrm{H}^1_\#(\Omega) = \ker A^\mu \overset{\perp}{\oplus} \mathrm{S}^\mu.$$

Consider the operators

$$\begin{array}{lll} & \tilde{A}^\varepsilon : \mathrm{S}^\varepsilon \to \mathrm{S}^\varepsilon & \text{s.t. } (\nabla(\tilde{A}^\varepsilon\varphi), \nabla\varphi') = (\varepsilon\nabla\varphi, \nabla\varphi'), \qquad \forall\varphi, \varphi' \in \mathrm{S}^\varepsilon; \\ \text{and} & \tilde{A}^\mu : \mathrm{S}^\mu \to \mathrm{S}^\mu & \text{s.t. } (\nabla(\tilde{A}^\mu\varphi), \nabla\varphi') = (\mu\nabla\varphi, \nabla\varphi'), \qquad \forall\varphi, \varphi' \in \mathrm{S}^\mu. \end{array}$$

Classically (see [27]), one has the

**Proposition 8.2** *The operators $\tilde{A}^\varepsilon : \mathrm{S}^\varepsilon \to \mathrm{S}^\varepsilon$ and $\tilde{A}^\mu : \mathrm{S}^\mu \to \mathrm{S}^\mu$ are isomorphisms.*

As mentioned in the introduction of Section 8, Theorems 3.1 and 3.3 do not hold anymore. Indeed, $\mathrm{span}(\nabla\lambda_1^\varepsilon, \dots, \nabla\lambda_{N^\varepsilon}^\varepsilon)$ is included in the kernel of problem (6), stated in $\mathbf{V}_N(\varepsilon;\,\Omega)$, but not in the kernel of the original problem (3), stated in $\mathbf{H}_N(\mathbf{curl}\,;\,\Omega)$. Our objective is therefore to write variational formulations of Maxwell's problems in some spaces different from $\mathbf{V}_N(\varepsilon;\,\Omega)$ and $\mathbf{V}_T(\mu;\,\Omega)$ in order to eliminate these artificial kernels. A way to achieve that aim is to enrich the usual spaces by setting

$$\begin{array}{lll} \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega) & := & \{\boldsymbol{u} \in \mathbf{H}_N(\mathbf{curl}\,;\,\Omega)\,|\,(\varepsilon\boldsymbol{u}, \nabla\varphi) = 0,\ \forall\varphi \in \mathrm{S}^\varepsilon\}\,; \\ \tilde{\mathbf{V}}_T(\mu;\,\Omega) & := & \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\,\Omega)\,|\,(\mu\boldsymbol{u}, \nabla\varphi) = 0,\ \forall\varphi \in \mathrm{S}^\mu\}\,. \end{array} \tag{15}$$

Notice that we have $\mathbf{V}_N(\varepsilon;\,\Omega) \subset \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ and $\mathbf{V}_T(\mu;\,\Omega) \subset \tilde{\mathbf{V}}_T(\mu;\,\Omega)$. Let us clarify the relation between these spaces. For the proof of the following result, we refer the reader to [4, lemma 8.3].

**Lemma 8.3** • *For $i = 1 \dots N^\varepsilon$, there exists $\boldsymbol{\Lambda}_i^\varepsilon \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ such that $(\varepsilon\boldsymbol{\Lambda}_i^\varepsilon, \nabla\lambda_j^\varepsilon) = \delta_{ij}$, for $j = 1 \dots N^\varepsilon$. We deduce*

$$\tilde{\mathbf{V}}_N(\varepsilon;\,\Omega) = \mathbf{V}_N(\varepsilon;\,\Omega) \oplus \mathrm{span}(\boldsymbol{\Lambda}_i^\varepsilon)_{i=1}^{N^\varepsilon}.$$

- *For $i = 1 \dots N^\mu$, there exists $\boldsymbol{\Lambda}_i^\mu \in \tilde{\mathbf{V}}_T(\mu;\,\Omega)$ such that $(\mu\boldsymbol{\Lambda}_i^\mu, \nabla\lambda_j^\mu) = \delta_{ij}$, for $j = 1 \dots N^\mu$. We deduce*

$$\tilde{\mathbf{V}}_T(\mu;\,\Omega) = \mathbf{V}_T(\mu;\,\Omega) \oplus \mathrm{span}(\boldsymbol{\Lambda}_i^\mu)_{i=1}^{N^\mu}.$$

Adapting the proof of Theorem 3.1, we can give equivalent formulations to problem (1)-(2) in the spaces $\tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ and $\tilde{\mathbf{V}}_T(\mu;\,\Omega)$.

**Theorem 8.4** *Assume $(\mathscr{H}^\varepsilon)$ and that $\omega \neq 0$. Let $\tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ be defined as in (15).*
*1) If $\{\boldsymbol{E}, \boldsymbol{H}\}$ satisfies (1)-(2) then $\boldsymbol{E}$ is a solution of the problem*

$$\left|\begin{array}{l} \textit{Find } \boldsymbol{E} \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega) \textit{ such that for all } \boldsymbol{E}' \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)\textit{:} \\ (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\varepsilon\boldsymbol{E}, \boldsymbol{E}') = i\omega\,(\boldsymbol{J}, \boldsymbol{E}'). \end{array}\right. \tag{16}$$

*2) If $\boldsymbol{E}$ satisfies (16) then the pair $\{\boldsymbol{E}, (i\,\omega\mu)^{-1}\mathbf{curl}\,\boldsymbol{E}\}$ satisfies (1)-(2).*

**Theorem 8.5** *Assume $(\mathscr{H}^\mu)$ and that $\omega \neq 0$. Let $\tilde{\mathbf{V}}_T(\mu;\,\Omega)$ be defined as in (15).*
*1) If $\{\boldsymbol{E}, \boldsymbol{H}\}$ satisfies (1)-(2) then $\boldsymbol{H}$ is a solution of the problem*

$$\left|\begin{array}{l} \textit{Find } \boldsymbol{H} \in \tilde{\mathbf{V}}_T(\mu;\,\Omega) \textit{ such that for all } \boldsymbol{H}' \in \tilde{\mathbf{V}}_T(\mu;\,\Omega)\textit{:} \\ (\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}, \mathbf{curl}\,\boldsymbol{H}') - \omega^2(\mu\boldsymbol{H}, \boldsymbol{H}') = (\varepsilon^{-1}\boldsymbol{J}, \mathbf{curl}\,\boldsymbol{H}'). \end{array}\right. \tag{17}$$

*2) If $\boldsymbol{H}$ satisfies (17) then the pair $\{i\,(\omega\varepsilon)^{-1}(\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J}), \boldsymbol{H}\}$ satisfies (1)-(2).*

To study formulations (16) and (17), we need some new compactness results. The latter can be shown working as in the proof of Theorem 5.1 with the help of Lemma 8.3 (see [4, theorem 8.8] for the details).

**Theorem 8.6** *Let $\Omega$ be a simply connected domain such that $\partial\Omega$ is connected.*

- *Assume $(\mathscr{H}^\varepsilon)$. Then the embedding of $\tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact.*
- *Assume $(\mathscr{H}^\mu)$. Then the embedding of $\tilde{\mathbf{V}}_T(\mu;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact.*

Using the Riesz representation theorem, introduce the bounded operators $\tilde{\mathscr{A}}_N(\omega) : \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega) \to \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ and $\tilde{\mathscr{A}}_T(\omega) : \tilde{\mathbf{V}}_T(\mu;\,\Omega) \to \tilde{\mathbf{V}}_T(\mu;\,\Omega)$, $\omega \in \mathbb{C}$, such that for all $\boldsymbol{E}, \boldsymbol{E}' \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ and for all $\boldsymbol{H}, \boldsymbol{H}' \in \tilde{\mathbf{V}}_T(\mu;\,\Omega)$,

$$\begin{aligned}(\tilde{\mathscr{A}}_N(\omega)\boldsymbol{E}, \boldsymbol{E}')_{\mathbf{curl}} &= (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\varepsilon\boldsymbol{E}, \boldsymbol{E}'),\\ (\tilde{\mathscr{A}}_T(\omega)\boldsymbol{H}, \boldsymbol{H}')_{\mathbf{curl}} &= (\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}, \mathbf{curl}\,\boldsymbol{H}') - \omega^2(\mu\boldsymbol{H}, \boldsymbol{H}').\end{aligned}$$

Now, we state the main result when the geometry and the physical coefficients $\varepsilon, \mu$ are such that the scalar problems are well-posed in the Fredholm sense with a non-trivial kernel.

**Theorem 8.7** *Let $\Omega$ be a simply connected domain such that $\partial\Omega$ is connected. Assume ($\mathscr{H}^\varepsilon$) and ($\mathscr{H}^\mu$). Consider $\boldsymbol{J} \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div}\boldsymbol{J} = 0$. Then, the following results hold.*

- *For all $\omega \in \mathbb{C}$, the operator $\tilde{\mathscr{A}}_N(\omega) : \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega) \to \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ is a Fredholm operator of index 0. Moreover, for $\omega \in \mathbb{C}^*$, $\boldsymbol{E} \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ satisfies $(\tilde{\mathscr{A}}_N(\omega)\boldsymbol{E}, \boldsymbol{E}')_{\mathbf{curl}} = i\omega(\boldsymbol{J}, \boldsymbol{E}')$, for all $\boldsymbol{E}' \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$, if and only if the pair $\{\boldsymbol{E}, (i\,\omega\mu)^{-1}\mathbf{curl}\,\boldsymbol{E}\}$ satisfies the Maxwell's equations (1)-(2).*
- *For all $\omega \in \mathbb{C}$, $\tilde{\mathscr{A}}_T(\omega) : \tilde{\mathbf{V}}_T(\mu;\,\Omega) \to \tilde{\mathbf{V}}_T(\mu;\,\Omega)$ is a Fredholm operator of index 0. Moreover, for $\omega \in \mathbb{C}^*$, $\boldsymbol{H} \in \tilde{\mathbf{V}}_T(\mu;\,\Omega)$ satisfies $(\tilde{\mathscr{A}}_T(\omega)\boldsymbol{H}, \boldsymbol{H}')_{\mathbf{curl}} = (\varepsilon^{-1}\boldsymbol{J}, \mathbf{curl}\,\boldsymbol{H}')$, for all $\boldsymbol{H}' \in \tilde{\mathbf{V}}_T(\mu;\,\Omega)$, if and only if the pair $\{i\,(\omega\varepsilon)^{-1}(\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J}), \boldsymbol{H}\}$ satisfies the Maxwell's equations (1)-(2).*

**Proof.** Let us prove that $\tilde{\mathscr{A}}_N(\omega)$ is a Fredholm operator of index 0. For all $\omega \in \mathbb{C}$, using Theorem 8.6, we can prove that $\tilde{\mathscr{A}}_N(\omega) - \tilde{\mathscr{A}}_N(0)$ is a compact operator of $\tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$. Consequently, according to [27, theorem 2.26], $\tilde{\mathscr{A}}_N(\omega)$ is a Fredholm operator of index 0 if and only if $\tilde{\mathscr{A}}_N(0)$ is a Fredholm operator of index 0. In the sequel, we work on $\tilde{\mathscr{A}}_N(0)$. We build a bounded operator $\tilde{\mathbb{T}}^\varepsilon : \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega) \to \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ to restore some positivity up to a compact perturbation. Let us consider $\boldsymbol{u} \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$.
i) First, define $\varphi$ the unique element of $\mathrm{S}^\mu$ such that

$$(\mu\nabla\varphi, \nabla\varphi') = (\mu\,(\mathbf{curl}\,\boldsymbol{u} - \sum_{i=1}^{N^\mu}\beta_i\boldsymbol{\Lambda}_i^\mu), \nabla\varphi'), \qquad \forall\varphi' \in \mathrm{S}^\mu,$$

where $\beta_i = (\mu\,\mathbf{curl}\,\boldsymbol{u}, \nabla\lambda_i^\mu)$. The function $\varphi$ is well-defined since $\tilde{A}^\mu : \mathrm{S}^\mu \to \mathrm{S}^\mu$ is an isomorphism.
ii) Then, notice that $\mu(\mathbf{curl}\,\boldsymbol{u} - \sum_{i=1}^{N^\mu}\beta_i\boldsymbol{\Lambda}_i^\mu - \nabla\varphi)$ is a divergence free element of $\mathbf{L}^2(\Omega)$ such that $\mu(\mathbf{curl}\,\boldsymbol{u} - \sum_{i=1}^{N^\mu}\beta_i\boldsymbol{\Lambda}_i^\mu - \nabla\varphi)\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. Since $\Omega$ is simply connected and since $\partial\Omega$ is connected, according to theorem 3.17 in [1], there exists a unique potential $\boldsymbol{\psi} \in \mathbf{V}_N(1;\,\Omega)$ such that $\mathbf{curl}\,\boldsymbol{\psi} = \mu(\mathbf{curl}\,\boldsymbol{u} - \sum_{i=1}^{N^\mu}\beta_i\boldsymbol{\Lambda}_i^\mu - \nabla\varphi)$.
iii) Consider $\zeta$ the unique element of $\mathrm{S}^\varepsilon$ such that

$$(\varepsilon\nabla\zeta, \nabla\zeta') = (\varepsilon\,\boldsymbol{\psi}, \nabla\zeta'), \qquad \forall\zeta' \in \mathrm{S}^\varepsilon.$$

The function $\zeta$ is well-defined since $\tilde{A}^\varepsilon : \mathrm{S}^\varepsilon \to \mathrm{S}^\varepsilon$ is an isomorphism.
iv) Finally, define the operator $\tilde{\mathbb{T}}^\varepsilon : \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega) \to \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ such that $\tilde{\mathbb{T}}^\varepsilon\boldsymbol{u} = \boldsymbol{\psi} - \nabla\zeta$ and the operator $\tilde{K}^\varepsilon : \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega) \to \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ such that

$$(\tilde{K}^\varepsilon\boldsymbol{u}, \boldsymbol{v})_{\mathbf{curl}} = (\boldsymbol{u}, \boldsymbol{v}) + \sum_{i=1}^{N^\mu}(\mu\,\mathbf{curl}\,\boldsymbol{u}, \nabla\lambda_i^\mu)(\boldsymbol{\Lambda}_i^\mu, \mathbf{curl}\,\boldsymbol{v}), \qquad \forall\boldsymbol{v} \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega).$$

According to Theorem 8.6, we know that the embedding of $\tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact. Consequently, $\tilde{K}^\varepsilon$ is the sum of a compact operator and a finite rank operator. Therefore, it is a compact operator. Now, for all $\boldsymbol{u}, \boldsymbol{v} \in \tilde{\mathbf{V}}_N(\varepsilon;\,\Omega)$, we obtain

$$\begin{aligned}(\tilde{\mathscr{A}}_N(0)(\tilde{\mathbb{T}}^\varepsilon\boldsymbol{u}), \boldsymbol{v})_{\mathbf{curl}} &= (\mu^{-1}\mathbf{curl}\,(\tilde{\mathbb{T}}^\varepsilon\boldsymbol{u}), \mathbf{curl}\,\boldsymbol{v})\\ &= (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) - (\tilde{K}^\varepsilon\boldsymbol{u}, \boldsymbol{v})_{\mathbf{curl}}.\end{aligned}$$

We deduce $\tilde{\mathscr{A}}_T(0) \circ \tilde{\mathbb{T}}^\varepsilon + \tilde{K}^\varepsilon = \mathrm{Id}$. This proves that $\tilde{\mathbb{T}}^\varepsilon$ is a right parametrix for $\tilde{\mathscr{A}}_N(0)$. Thus, the selfadjoint operator $\tilde{\mathscr{A}}_N(0)$ is Fredholm of index 0 (use [27, lemma 2.23]). In the same way, we prove that $\tilde{\mathscr{A}}_T(\omega) : \tilde{\mathbf{V}}_T(\mu;\,\Omega) \to \tilde{\mathbf{V}}_T(\mu;\,\Omega)$ is a Fredholm operator of index 0 for all $\omega \in \mathbb{C}$. Finally, the equivalence with Maxwell's equations (1)-(2) comes from Theorems 8.4 and 8.5. ■

**Remark 8.8** *To apply the analytic Fredholm theorem to prove that Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\backslash S$ where $S \subset \mathbb{R}$ is a discrete set, it remains to show that there exists $\omega \in \mathbb{C}$ such that $\tilde{\mathscr{A}}_N(\omega)$ or $\tilde{\mathscr{A}}_T(\omega)$ is invertible. However, we have not been able to prove this result.*

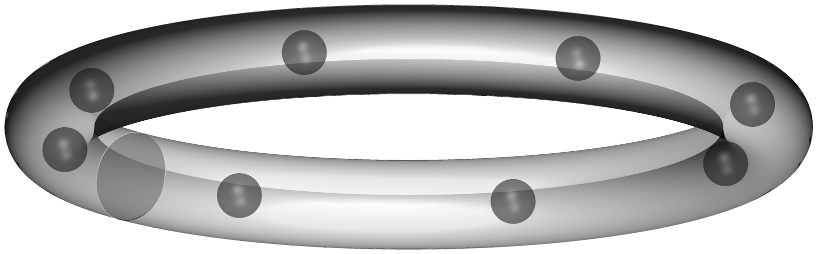

Figure 2: An example of domain which is not simply connected and whose boundary is not connected. The domain is made of the torus without the dark grey inclusions. It is not simply connected because of the toroidal structure. The boundary is not connected because the boundary of the torus and the ones of the spheres are not connected. The grey disk represents a cut $\Sigma_1$ which is such that $\Omega \setminus \Sigma_1$ is simply connected.

## 8.2 Extension to a non-trivial geometry

Classical configurations for Maxwell's equations include non-topologically trivial domains, and/or domains with a non-connected boundary. We study these configurations here. To avoid multiple sub-cases, we focus our work on the case of a non simply connected domain whose boundary is not connected. Figure 2 presents an example of such a geometry. In these geometries, the functions $\boldsymbol{u}$ of $\mathbf{V}_N(\varepsilon;\,\Omega)$ (resp. $\mathbf{V}_T(\mu;\,\Omega)$) may not necessarily be written in the form $\boldsymbol{u} = \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{\psi}$ (resp. $\boldsymbol{u} = \mu^{-1}\mathbf{curl}\,\boldsymbol{\psi}$) where $\boldsymbol{\psi}$ belongs to $\mathbf{V}_T(1;\,\Omega)$ (resp. $\mathbf{V}_N(1;\,\Omega)$). However, imposing more restrictive conditions to the functions of $\mathbf{V}_N(\varepsilon;\,\Omega)$ and $\mathbf{V}_T(\mu;\,\Omega)$, we can recover these results of existence for the potentials.

To introduce the spaces adapted to the study of Maxwell's equations in this kind of domains, we use the notations of [1].

**Notations for domains with a non connected boundary.** We denote $\Gamma_i$, $i = 0\ldots I$, the connected components of the boundary $\partial\Omega$. Since we assume that $\partial\Omega$ is not connected, we have $I \geq 1$. Let us introduce

$$\mathrm{H}^1_\Gamma(\Omega) := \left\{\varphi \in \mathrm{H}^1(\Omega) \,|\, \varphi_{|\Gamma_0} = 0,\ \varphi_{|\Gamma_i} = cst,\ i = 1\ldots I\right\}.$$

We start by characterizing this space. Using a lifting function, we prove the

**Proposition 8.9** *Assume that $A^\varepsilon$ is an isomorphism. Then, for $i = 1\ldots I$, there exists a unique solution $p_i$ of the problem*

$$\left|\begin{array}{lcl} \multicolumn{3}{l}{\textit{Find } p_i \in \mathrm{H}^1_\Gamma(\Omega) \textit{ such that:}} \\ \operatorname{div}(\varepsilon\nabla p_i) & = & 0 \quad \textit{in } \Omega \\ p_i & = & \delta_{ik} \quad \textit{on } \Gamma_k,\ k = 1\ldots I. \end{array}\right.$$

*Then, we have* $\mathrm{H}^1_\Gamma(\Omega) = \mathrm{H}^1_0(\Omega) \oplus \operatorname{span}(p_i)_{i=1}^I$.

Define

$$\hat{\mathbf{V}}_N(\varepsilon;\,\Omega) := \left\{\boldsymbol{u} \in \mathbf{H}_N(\mathbf{curl}\,;\,\Omega) \,|\, (\varepsilon\boldsymbol{u}, \nabla\varphi) = 0,\ \forall\varphi \in \mathrm{H}^1_\Gamma(\Omega)\right\}.$$

Notice that $\hat{\mathbf{V}}_N(\varepsilon;\,\Omega) \subset \mathbf{V}_N(\varepsilon;\,\Omega)$. The following result (see the proof in [4, lemma 8.13]) clarifies the link between these two spaces.

**Lemma 8.10** *Assume that $A^\varepsilon$ is an isomorphism. For $i = 1\ldots I$, there exists $\boldsymbol{P}_i \in \mathbf{V}_N(\varepsilon;\,\Omega)$ such that $(\varepsilon\boldsymbol{P}_i, \nabla p_k) = \delta_{ik}$, for $k = 1\ldots I$. We deduce*

$$\begin{array}{rcl} \mathbf{V}_N(\varepsilon;\,\Omega) & = & \hat{\mathbf{V}}_N(\varepsilon;\,\Omega) \oplus \operatorname{span}(\boldsymbol{P}_i)_{i=1}^I \\ \textit{and} \quad \mathbf{H}_N(\mathbf{curl}\,;\,\Omega) & = & \hat{\mathbf{V}}_N(\varepsilon;\,\Omega) \oplus \operatorname{span}(\boldsymbol{P}_i)_{i=1}^I \oplus \nabla\mathrm{H}^1_0(\Omega). \end{array}$$

**Notations for non simply connected domains.** We will assume that there exist connected open surfaces $\Sigma_j$, $j = 1\ldots J$ called "cuts" such that:

- *i)* each surface $\Sigma_j$ is an open subset of a smooth variety;
- *ii)* the boundary of $\Sigma_j$ is contained in $\partial\Omega$, $j = 1\ldots J$;
- *iii)* the intersection $\overline{\Sigma_j} \cap \overline{\Sigma_k}$ is empty for $j \neq k$;
- *iv)* the open set $\dot{\Omega} := \Omega \setminus \bigcup_{i=1}^J \Sigma_j$ is pseudo-lipschitz [1] and simply connected.

The domain $\Omega$ is said topologically trivial when we can take $J = 0$. The extension operator from $\mathrm{L}^2(\dot{\Omega})$ to $\mathrm{L}^2(\Omega)$ is denoted $\tilde{\cdot}$ whereas $[\cdot]_{\Sigma_j}$ denotes the jump through $\Sigma_j$, $j = 1\ldots J$. In this definition of the jump, we assume that a convention has been established for the sign. We also assume that a unit vector $\boldsymbol{n}$ normal to $\Sigma_j$, $j = 1 \dots J$, is chosen, consistent with the choice of the sign of the jump. Define the space of scalar potentials

$$\Theta(\dot{\Omega}) := \left\{ \varphi \in \mathrm{H}^1(\dot{\Omega}) \,|\, \int_{\Omega} \widetilde{\varphi} = 0 \text{ and } [\varphi]_{\Sigma_j} = cst,\ j = 1 \dots J \right\}.$$

Let us present a result of decomposition of this space.

**Proposition 8.11** *Assume that $A^\mu$ is an isomorphism. Then for $j = 1 \dots J$, there exists a unique solution $q_j$ of the problem*

$$\left| \begin{array}{rcll} \multicolumn{4}{l}{\textit{Find } q_j \in \Theta(\dot{\Omega}) \textit{ such that:}} \\ \operatorname{div}(\mu \nabla q_j) & = & 0 & \textit{in } \dot{\Omega} \\ \mu \partial_{\boldsymbol{n}} q_j & = & 0 & \textit{on } \partial\Omega \\ [q_j]_{\Sigma_k} & = & \delta_{jk}, & k = 1 \dots J \\ [\mu \partial_{\boldsymbol{n}} q_j]_{\Sigma_k} & = & 0, & k = 1 \dots J. \end{array} \right. \qquad (18)$$

*We then have $\Theta(\dot{\Omega}) = \mathrm{H}^1_{\#}(\Omega) \oplus \operatorname{span}(q_j)_{j=1}^J$.*

**Proof.** Since we have assumed that $A^\mu$ is an isomorphism, problem (18) has at most one solution. Let us build it. For $1 \le j \le J$, let $r_j \in \Theta(\dot{\Omega})$ be a function such that $[r_j]_{\Sigma_k} = \delta_{jk}$ for $k = 1 \dots J$. Then, let us define $q_j = r_j - \varphi$ where $\varphi$ is the unique element of $\mathrm{H}^1_{\#}(\Omega)$ such that

$$(\mu \nabla \varphi, \nabla \varphi') = (\mu \widetilde{\nabla r_j}, \nabla \varphi'), \qquad \forall \varphi' \in \mathrm{H}^1_{\#}(\Omega).$$

One easily checks that $q_j$ satisfies problem (18). This allows us to obtain the result of decomposition of the space $\Theta(\dot{\Omega})$. ■

Let us introduce

$$\hat{\mathbf{V}}_T(\mu;\, \Omega) := \left\{ \boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\, \Omega) \,|\, (\varepsilon \boldsymbol{u}, \widetilde{\nabla \varphi}) = 0,\ \forall \varphi \in \Theta(\dot{\Omega}) \right\}.$$

Observe that we have $\hat{\mathbf{V}}_T(\mu;\, \Omega) \subset \mathbf{V}_T(\mu;\, \Omega)$. More precisely, there holds the following decomposition.

**Lemma 8.12** *Assume that $A^\mu$ is an isomorphism. For $j = 1 \dots J$, there exists $\boldsymbol{Q}_j \in \mathbf{V}_T(\mu;\, \Omega)$ such that $(\mu \boldsymbol{Q}_j, \widetilde{\nabla q_k}) = \delta_{jk}$, for $k = 1 \dots J$. We deduce*

$$\begin{array}{rrcl} & \mathbf{V}_T(\mu;\, \Omega) & = & \hat{\mathbf{V}}_T(\mu;\, \Omega) \oplus \operatorname{span}(\boldsymbol{Q}_j)_{j=1}^J \\ \textit{and} & \mathbf{H}(\mathbf{curl}\,;\, \Omega) & = & \hat{\mathbf{V}}_T(\mu;\, \Omega) \oplus \operatorname{span}(\boldsymbol{Q}_j)_{j=1}^J \oplus \nabla \mathrm{H}^1_{\#}(\Omega). \end{array}$$

**Remark 8.13** *Theorem 3.12 in [1] states that every element $\boldsymbol{u}$ of $\hat{\mathbf{V}}_N(\varepsilon;\, \Omega)$ can be written as $\boldsymbol{u} = \varepsilon^{-1} \mathbf{curl}\, \boldsymbol{\psi}$ with $\boldsymbol{\psi}$ belonging to $\hat{\mathbf{V}}_T(1;\, \Omega)$. Similarly, theorem 3.17 of [1] ensures that for every $\boldsymbol{u} \in \hat{\mathbf{V}}_T(\mu;\, \Omega)$, there exists a unique $\boldsymbol{\psi} \in \hat{\mathbf{V}}_N(1;\, \Omega)$ such that $\boldsymbol{u} = \mu^{-1} \mathbf{curl}\, \boldsymbol{\psi}$. In the sequel, we will adapt the proofs of the previous sections using these results of existence of vector potentials.*

Assume that $A^\varepsilon$ and $A^\mu$ are isomorphisms. Remark that Theorems 3.1, 3.3 which prove the equivalence between initial Maxwell's equations and formulations in $\mathbf{V}_N(\varepsilon;\, \Omega)$, $\mathbf{V}_T(\mu;\, \Omega)$ do not require any assumption concerning the topology of the domain. Therefore, they are true for the geometry we are considering. In the sequel, we will work with these formulations set in $\mathbf{V}_N(\varepsilon;\, \Omega)$, $\mathbf{V}_T(\mu;\, \Omega)$.

**Remark 8.14** *Can we work with formulations set in $\hat{\mathbf{V}}_N(\varepsilon;\, \Omega)$, $\hat{\mathbf{V}}_T(\mu;\, \Omega)$? A priori, the electric field which satisfies Maxwell's equations has no reason to belong to the space $\hat{\mathbf{V}}_N(\varepsilon;\, \Omega)$. To see this, we use Lemma 8.10 and we decompose $\boldsymbol{E}$ under the form*

$$\boldsymbol{E} = \hat{\boldsymbol{E}} + \sum_{i=1}^{I} \alpha_i \boldsymbol{P}_i,$$

*with $\hat{\boldsymbol{E}} \in \hat{\mathbf{V}}_N(\varepsilon;\, \Omega)$ and $(\alpha_1, \dots, \alpha_I) \in \mathbb{C}^I$. For $i = 1 \dots I$, testing with $\nabla p_i$ in (6), we find*

$$\alpha_i = (\varepsilon \boldsymbol{E}, \nabla p_i) = (i\omega)^{-1} (\boldsymbol{J}, \nabla p_i) = (i\omega)^{-1} \langle \boldsymbol{J} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i},$$

*where $\langle \cdot, \cdot \rangle_{\Gamma_i}$ denotes the duality product between $\mathrm{H}^{1/2}(\Gamma_i)$ and its dual space. Above, we have used the properties $\operatorname{div} \boldsymbol{J} = 0$ in $\Omega$ and $p_i = \delta_{ik}$ on $\Gamma_k$, $k = 1 \dots I$. Thus, if there exists $0 \le i \le I$ such that $\langle \boldsymbol{J} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} \ne 0$, then $\boldsymbol{E}$ does not belong to $\hat{\mathbf{V}}_N(\varepsilon;\, \Omega)$. But this also proves that to know the field $\boldsymbol{E}$, it is sufficient to determine $\hat{\boldsymbol{E}}$. On the other hand, following the same reasoning, we can check that the magnetic field is always an element of $\hat{\mathbf{V}}_T(\mu;\, \Omega)$, regardless of the source term $\boldsymbol{J}$.*

The following theorem ensures that the compactness results of Theorems 5.1, 5.3 are also valid in the case where $\Omega$ is not simply connected with a non connected boundary. For the proof, which is very similar to the one of Theorem 5.1, we refer the reader to [4, theorem 8.18].

**Theorem 8.15** *Let $\Omega$ be a bounded connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary.*

- *Assume that $A^\varepsilon$ is an isomorphism. Then the embedding of $\mathbf{V}_N(\varepsilon;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact.*
- *Assume that $A^\mu$ is an isomorphism. Then the embedding of $\mathbf{V}_T(\mu;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact.*

Let us state now the main result of this section concerning the well-posedness of Maxwell's equations in non-trivial geometries. The proof follows the same lines as the one of Theorem 8.7. Nevertheless, we chose to present it to point out where the material specific to non-trivial geometries is needed in the analysis.

**Theorem 8.16** *Let $\Omega$ be a bounded connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Consider $\boldsymbol{J} \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div} \boldsymbol{J} = 0$. Assume that $A^\varepsilon$ and $A^\mu$ are isomorphisms. Then, we have the following results.*

- *For all $\omega \in \mathbb{C}$, the operator for the electric field $\mathscr{A}_N(\omega) : \mathbf{V}_N(\varepsilon;\,\Omega) \to \mathbf{V}_N(\varepsilon;\,\Omega)$ defined in (5) is a Fredholm operator of index 0. Moreover, for $\omega \in \mathbb{C}^*$, $\boldsymbol{E} \in \mathbf{V}_N(\varepsilon;\,\Omega)$ satisfies $(\mathscr{A}_N(\omega)\boldsymbol{E}, \boldsymbol{E}')_{\mathbf{curl}} = i\omega(\boldsymbol{J}, \boldsymbol{E}')$, for all $\boldsymbol{E}' \in \mathbf{V}_N(\varepsilon;\,\Omega)$, if and only if the pair $\{\boldsymbol{E}, (i\,\omega\mu)^{-1}\mathbf{curl}\,\boldsymbol{E}\}$ satisfies the Maxwell's equations (1)-(2).*
- *For all $\omega \in \mathbb{C}$, $\mathscr{A}_T(\omega) : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)$ defined in (8) is a Fredholm operator of index 0. Moreover, for $\omega \in \mathbb{C}^*$, $\boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega)$ satisfies $(\mathscr{A}_T(\omega)\boldsymbol{H}, \boldsymbol{H}')_{\mathbf{curl}} = (\varepsilon^{-1}\boldsymbol{J}, \mathbf{curl}\,\boldsymbol{H}')$, for all $\boldsymbol{H}' \in \mathbf{V}_T(\mu;\,\Omega)$, if and only if the pair $\{i\,(\omega\varepsilon)^{-1}(\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J}), \boldsymbol{H}\}$ satisfies the Maxwell's equations (1)-(2).*

**Proof.** Let us prove that $\mathscr{A}_N(\omega)$ is a Fredholm operator of index 0. For all $\omega \in \mathbb{C}$, using Theorem 8.15, one can prove that $\mathscr{A}_N(\omega) - \mathscr{A}_N(0)$ is a compact operator of $\mathbf{V}_N(\varepsilon;\,\Omega)$. So, it is sufficient to show that $\mathscr{A}_N(0)$ is Fredholm of index 0. Again, we are going to build a right parametrix $\mathbb{T}^\varepsilon : \mathbf{V}_N(\varepsilon;\,\Omega) \to \mathbf{V}_N(\varepsilon;\,\Omega)$ for the operator $\mathscr{A}_N(0)$. Consider $\boldsymbol{u} \in \mathbf{V}_N(\varepsilon;\,\Omega)$.
i) First, define $\varphi$ the unique element of $\mathrm{H}^1_{\#}(\Omega)$ such that

$$(\mu\nabla\varphi, \nabla\varphi') = (\mu\,\mathbf{curl}\,\boldsymbol{u}, \nabla\varphi'), \qquad \forall \varphi' \in \mathrm{H}^1_{\#}(\Omega).$$

The function $\varphi$ is well-defined because we have assumed that $A^\mu$ is an isomorphism.
ii) Defining $\beta_j := (\mu\,\mathbf{curl}\,\boldsymbol{w}, \widetilde{\nabla q_j})$ for $j = 1 \dots J$, notice that we have $(\mu(\mathbf{curl}\,\boldsymbol{w} - \sum_{j=1}^J \beta_j \boldsymbol{Q}_j - \nabla\varphi), \widetilde{\nabla\varphi'}) = 0$ for all $\varphi' \in \Theta(\dot{\Omega})$. Therefore, according to theorem 3.17 in [1], there exists a unique potential $\boldsymbol{\psi} \in \hat{\mathbf{V}}_N(1;\,\Omega)$ such that $\mathbf{curl}\,\boldsymbol{\psi} = \mu(\mathbf{curl}\,\boldsymbol{w} - \sum_{j=1}^J \beta_j \boldsymbol{Q}_j - \nabla\varphi)$.
iii) Consider $\zeta$ the unique element of $\mathrm{H}^1_0(\Omega)$ such that

$$(\varepsilon\nabla\zeta, \nabla\zeta') = (\varepsilon\,\boldsymbol{\psi}, \nabla\zeta'), \qquad \forall \zeta' \in \mathrm{H}^1_0(\Omega).$$

The function $\zeta$ is well-defined because we have assumed that $A^\varepsilon$ is an isomorphism.
iv) Finally, let us define the operator $\mathbb{T}^\varepsilon : \mathbf{V}_N(\varepsilon;\,\Omega) \to \mathbf{V}_N(\varepsilon;\,\Omega)$ such that $\mathbb{T}^\varepsilon\boldsymbol{u} = \boldsymbol{\psi} - \nabla\zeta$ and the operator $K^\varepsilon : \mathbf{V}_N(\varepsilon;\,\Omega) \to \mathbf{V}_N(\varepsilon;\,\Omega)$ such that

$$(K^\varepsilon\boldsymbol{u}, \boldsymbol{v})_{\mathbf{curl}} = (\boldsymbol{u}, \boldsymbol{v}) + \sum_{j=1}^{J} (\mu\,\mathbf{curl}\,\boldsymbol{u}, \widetilde{\nabla q_j})(\boldsymbol{Q}_j, \mathbf{curl}\,\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbf{V}_N(\varepsilon;\,\Omega).$$

According to Theorem 8.15, we know that the embedding of $\mathbf{V}_N(\varepsilon;\,\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact. Consequently, $K^\varepsilon$ is the sum of a compact operator and a finite rank operator. Therefore, it is a compact operator. Now, for all $\boldsymbol{v} \in \mathbf{V}_N(\varepsilon;\,\Omega)$, we find

$$\begin{aligned}(\mathscr{A}_N(0)(\mathbb{T}^\varepsilon\boldsymbol{u}), \boldsymbol{v})_{\mathbf{curl}} &= (\mu^{-1}\mathbf{curl}\,(\mathbb{T}^\varepsilon\boldsymbol{u}), \mathbf{curl}\,\boldsymbol{v})\\ &= (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) - (K^\varepsilon\boldsymbol{u}, \boldsymbol{v})_{\mathbf{curl}}\,.\end{aligned}$$

Thus, we have $\mathscr{A}_T(0) \circ \mathbb{T}^\varepsilon + K^\varepsilon = \mathrm{Id}$. This proves that the selfadjoint operator $\mathscr{A}_N(0)$ is Fredholm of index 0. Similarly, we prove that $\mathscr{A}_T(\omega) : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)$ is a Fredholm operator of index 0 for all $\omega \in \mathbb{C}$. Finally, the equivalence with Maxwell's equations (1)-(2) comes from Theorems 3.1 and 3.3. ■

**Remark 8.17** *Again, to apply the analytic Fredholm theorem to prove that Maxwell's equations (1)-(2) are uniquely solvable for all $\omega \in \mathbb{C}^*\backslash S$ where $S \subset \mathbb{R}$ is a discrete set, it remains to show that there exists $\omega \in \mathbb{C}$ such that $\mathscr{A}_N(\omega)$ or $\mathscr{A}_T(\omega)$ is invertible. This results does not seem simple to obtain. However, according to Remark 8.14, we observe that the elements of $\ker \mathscr{A}_N(\omega)$ (resp. $\ker \mathscr{A}_T(\omega)$) always belong to $\hat{\mathbf{V}}_N(\varepsilon;\,\Omega)$ (resp. $\hat{\mathbf{V}}_T(\mu;\,\Omega)$). On the other hand, the map $\boldsymbol{u} \mapsto \|\mathbf{curl}\,\boldsymbol{u}\|$ defines a norm on $\hat{\mathbf{V}}_N(\varepsilon;\,\Omega)$ and on $\hat{\mathbf{V}}_T(\mu;\,\Omega)$. But one ingredient is still missing to achieve the injectivity of $\mathscr{A}_N(\omega)$ or $\mathscr{A}_T(\omega)$.*

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