On the breathing of spectral bands in periodic quantum waveguides with inflating resonators

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Abstract

We are interested in the lower part of the spectrum $\sigma(A^\varepsilon)$ of the Dirichlet Laplacian in a thin waveguide Π^ε obtained by repeating periodically a pattern, itself constructed by scaling an inner field geometry Ω by a small factor $\varepsilon > 0$. The Floquet-Bloch theory ensures that $\sigma(A^\varepsilon)$ has a band-gap structure. Due to the Dirichlet boundary conditions, the bands all move to $+\infty$ as $O(\varepsilon^{-2})$ when $\varepsilon \to 0^+$. Concerning their widths, the results depend on the dimension of the so-called space of almost standing waves in Ω that we denote by $\mathrm{X}_\dagger$. Generically, i.e. for most Ω, there holds $\mathrm{X}_\dagger = \{0\}$ and the lower part of $\sigma(A^\varepsilon)$ is very sparse, made of bands of length at most $O(\varepsilon)$ as $\varepsilon \to 0^+$. For certain Ω however, we have $\dim \mathrm{X}_\dagger = 1$ and then there are bands of length $O(1)$ which allow for wave propagation in Π^ε. We study the behaviour of the spectral bands when perturbing Ω around a particular $\Omega_\star$ where $\dim \mathrm{X}_\dagger = 1$. We show a breathing phenomenon of $\sigma(A^\varepsilon)$: when inflating Ω around $\Omega_\star$, the spectral bands rapidly expand before shrinking. In the process, a band dives below the normalized threshold, stops breathing and becomes extremely short as Ω continues to inflate.

Keywords: Quantum waveguide, thin periodic lattice, threshold resonance, spectral bands.

1 Setting of the problem

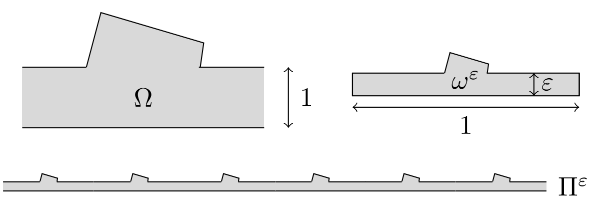

Figure 1: Geometries of Ω (top left), ω^ε (top right) and Π^ε (bottom).

Let $\Omega \subset \mathbb{R}^2$ be a waveguide which coincides with the strip $\mathbb{R} \times (-1/2; 1/2)$ outside of a bounded region (Figure 1 top left). For $\varepsilon > 0$, we consider the unit cell

$$\omega^\varepsilon := \{z = (x, y) \in \mathbb{R}^2 \,|\, z/\varepsilon \in \Omega \text{ and } |x| < 1/2\}$$

and set $\partial\omega^\varepsilon_\pm := \{\pm 1/2\} \times (-\varepsilon/2; \varepsilon/2)$. Finally we define the periodic waveguide

$$\Pi^\varepsilon := \{z \in \mathbb{R}^2 \,|\, (x - m, y) \in \omega^\varepsilon \cup \partial\omega^\varepsilon_+,\ m \in \mathbb{Z}\}.$$

We assume that Ω, ω^ε and Π^ε are connected with Lipschitz boundaries. In Π^ε, we consider the spectral problem for the Dirichlet Laplacian

$$\left|\begin{array}{rcll} -\Delta u^\varepsilon &=& \lambda^\varepsilon u^\varepsilon & \text{in } \Pi^\varepsilon \\ u^\varepsilon &=& 0 & \text{on } \partial\Pi^\varepsilon. \end{array}\right. \qquad (1)$$

We denote by A^ε the unbounded selfadjoint operator of $\mathrm{L}^2(\Pi^\varepsilon)$, with domain $\mathcal{D}(A^\varepsilon) \subset \mathrm{H}^1_0(\Pi^\varepsilon) := \{\varphi \in \mathrm{H}^1(\Pi^\varepsilon) \,|\, \varphi = 0 \text{ on } \partial\Pi^\varepsilon\}$, associated with (1). Since the geometry is periodic, the Floquet-Bloch theory ensures that the spectrum of A^ε has a band/gap structure:

$$\sigma(A^\varepsilon) = \bigcup_{p \in \mathbb{N}^* := \{1,2,\dots\}} \Upsilon^\varepsilon_p \qquad (2)$$

where the Υ^ε_p are compact segments. Our goal is to study the behaviour of $\sigma(A^\varepsilon)$ as $\varepsilon \to 0^+$.

2 Near field problem and first result

The analysis developed in particular in [2] shows that the asymptotic behaviour of the Υ^ε_p with respect to ε depends on the features of the Dirichlet Laplacian A^Ω in Ω. Its continuous spectrum occupies the ray $[\pi^2; +\infty)$. To set ideas, we assume that A^Ω has exactly $N_\bullet \in \mathbb{N} := \{0, 1, 2, \dots\}$ eigenvalues (counted with multiplicity) in its discrete spectrum, that we denote by

$$0 < \mu_1 < \mu_2 \leq \mu_3 \leq \dots \leq \mu_{N_\bullet} < \pi^2. \qquad (3)$$

Of particular importance in the study are the features of the inner field problem with a spectral parameter coinciding with the bottom of the continuous spectrum of A^Ω:

$$\left|\begin{array}{rcll} \Delta W + \pi^2 W &=& 0 & \text{in } \Omega \\ W &=& 0 & \text{on } \partial\Omega. \end{array}\right. \qquad (4)$$

To simplify the exposition, assume that the only solution of (4) in $L^2(\Omega)$, i.e. which decays at infinity, is the null function. Denote by $X_\dagger$ the space of bounded solutions of (4), the so-called space of almost standing waves in Ω. Using techniques of dimension reduction on the spectral problem depending on the Floquet parameter obtained when applying the Floquet-Bloch transform to (1), we show the statement:

Theorem 1 *For $p \in \mathbb{N}^*$, let $\Upsilon_p^\varepsilon = [a_{p-}^\varepsilon; a_{p+}^\varepsilon]$, with $a_{p-}^\varepsilon \le a_{p+}^\varepsilon$, be the spectral band in (2). There are some constants $c_{p-} < c_{p+}$, C_p, β_p, $\varepsilon_p > 0$ and $\delta_p > 1$ such that we have*

For $p = 1, \dots, N_\bullet$:
$$|a_{p\pm}^\varepsilon - (\varepsilon^{-2}\mu_p + \varepsilon^{-2}e^{-\beta_p/\varepsilon}c_{p\pm})| \le C_p\, e^{-\delta_p\beta_p/\varepsilon};$$

For $p = N_\bullet + m$, $m \in \mathbb{N}^*$:

i) if $X_\dagger = \{0\}$,
$$|a_{p\pm}^\varepsilon - (\varepsilon^{-2}\pi^2 + m^2\pi^2 + \varepsilon c_{p\pm})| \le C_p\, \varepsilon^{\delta_p};$$

ii) if $\dim X_\dagger = 1$,
$$|a_{p\pm}^\varepsilon - (\varepsilon^{-2}\pi^2 + c_{p\pm})| \le C_p\, \varepsilon^{\delta_p};$$

iii) if $\dim X_\dagger = 2$,
$$|a_{p\pm}^\varepsilon - (\varepsilon^{-2}\pi^2 + (m-1)^2\pi^2 + \varepsilon c_{p\pm})| \le C_p\, \varepsilon^{\delta_p}.$$

Each estimate above is valid for all $\varepsilon \in (0; \varepsilon_p]$ and the μ_p are the ones introduced in (3).

Let us comment these results. First, as already mentioned, when $\varepsilon \to 0^+$, the whole spectrum of A^ε goes to $+\infty$ as ε^{-2}. Besides, the first $N_\bullet$ spectral bands of A^ε become extremely short, in $O(e^{-c/\varepsilon})$ for some $c > 0$ which depends on the band. Concerning the next spectral bands Υ_p^ε, $p = N_\bullet + m$ with $m \in \mathbb{N}^*$, the behaviour depends on the dimension of $X_\dagger$. When the latter is zero (the generic situation) or two (cases *i*) and *iii*)), the spectral bands are of length $O(\varepsilon)$. Moreover, between Υ_p^ε and $\Upsilon_{p+1}^\varepsilon$, there is a gap, that is, a segment of spectral parameters λ^ε such that waves cannot propagate, whose length tends to $(2m+1)\pi^2$ (resp. $(2m-1)\pi^2$) in case *i*) (resp. *iii*)). In other words, for these two cases, the propagation of waves in the thin lattice Π^ε is hampered and occurs only for very narrow (closed) intervals of frequencies. When the dimension of $X_\dagger$ is one (case *ii*)), the situation is very different. Indeed, asymptotically the spectral band Υ_p^ε is of length $c_{p+} - c_{p-}$, with in general $c_{p+} > c_{p-}$. As a consequence, waves can propagate in Π^ε for much larger intervals of frequencies than in cases *i*) and *iii*).

3 Breathing of the spectral bands

Now assume that the inner field geometry $\Omega = \Omega(H)$ depends smoothly on a parameter H. We denote by $X_\dagger(H)$ the corresponding space of almost standing waves. Consider some $H_\star$ such that $\dim X_\dagger(H_\star) = 1$ (see [1, Prop. 7.1] for the proof of existence of such geometries) and work with $H = H_\star + \varepsilon\rho$, $\rho \in \mathbb{R}$. Let $\Upsilon_p^{\rho,\varepsilon}$ stand for the spectral bands of the operator A^ε defined in the Π^ε constructed from $\Omega(H_\star + \varepsilon\rho)$. Computing an asymptotic expansion of the $\Upsilon_p^{\rho,\varepsilon}$ as $\varepsilon \to 0^+$, we get results depending on the parameter ρ. By varying $\rho \in \mathbb{R}$, this provides a model describing the transition of $\sigma(A^\varepsilon)$ when inflating the inner field geometry around $\Omega(H_\star)$. With this model, we proved that the spectral bands above the normalized threshold $\varepsilon^{-2}\pi^2$ first expand and then shrink (see [1, Thm. 6.1]). This is what we call the breathing phenomenon of the spectrum of A^ε. In the process, in $\sigma(A^\varepsilon)$ a band dives below $\varepsilon^{-2}\pi^2$, stops breathing and becomes extremely short as the inner field geometry continues to inflate. The numerics of Figure 2, obtained by computing with a finite element method the spectrum of (1), illustrates this phenomenon (here $\varepsilon = 0.05$).

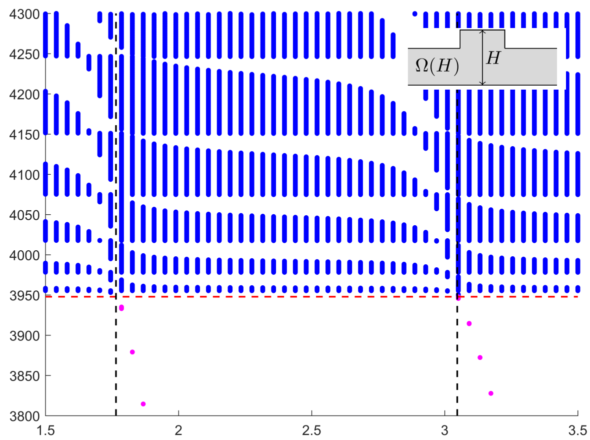

Figure 2: Spectrum of A^ε with respect to $H \in [1.5; 3.5]$. The horizontal red dashed line corresponds to $\varepsilon^{-2}\pi^2$. The vertical dashed lines mark the values of $H_\star$ such that $\dim X_\dagger(H_\star) = 1$.

References

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