SYMPOSIUM ON OPERATORS, ASYMPTOTICS, WAVES

Maxwell's equations with hypersingularities at a negative index material conical tip

Lucas Chesnel[1]

Collaboration with A.S. Bonnet-BenDhia[2] and M. Rihani[3].

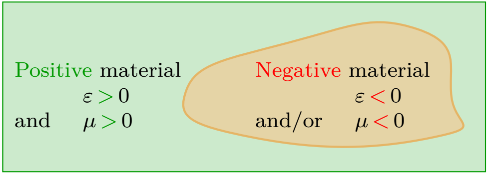
[1]IDEFIX, Inria-Ensta Paris-EDF, Ensta Paris, France
[2]POEMS, Cnrs-Ensta Paris-Inria, Ensta Paris, France
[3]CMAP, École Polytechnique, France

BATH, 25/07/2023

Goal and motivation

We study 3D time harmonic Maxwell's equations in presence of an inclusion of negative material:

$$\left|\begin{array}{l} \textbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0 \text{ in } \Omega \\ \textbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = \boldsymbol{J} \text{ in } \Omega \\ + \text{ PEC boundary cond.:} \\ \boldsymbol{E} \times \nu = 0 \text{ on } \partial\Omega \\ \mu\boldsymbol{H} \cdot \nu = 0 \text{ on } \partial\Omega \end{array}\right.$$

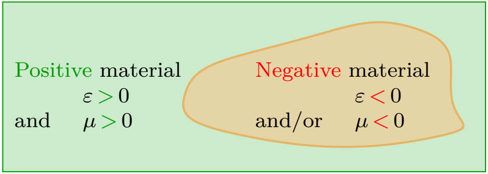


- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.
- Artificial metamaterials have been realized which can be modelled for certain frequencies by $\varepsilon < 0$ and $\mu < 0$.

Goal and motivation

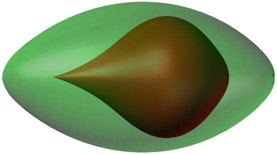
We study 3D time harmonic Maxwell's equations in presence of an inclusion of negative material:

$$\left|\begin{array}{l} \mathbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0 \text{ in } \Omega \\ \mathbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = \boldsymbol{J} \text{ in } \Omega \\ + \text{ PEC boundary cond.:} \\ \boldsymbol{E} \times \nu = 0 \text{ on } \partial\Omega \\ \mu\boldsymbol{H} \cdot \nu = 0 \text{ on } \partial\Omega \end{array}\right.$$



- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.
- Artificial metamaterials have been realized which can be modelled for certain frequencies by $\varepsilon < 0$ and $\mu < 0$.

Particular motivation: non smooth gold nanoparticles.



Difficulty: usual results do not apply, singularities at the tip are amplified.

Outline of the talk

1. Positive coefficients
2. Sign-changing coefficients - non critical ε, μ
3. Scalar problems
4. Sign-changing coefficients - critical ε, non critical μ
5. Sign-changing coefficients - critical ε, μ

1 Positive coefficients

2 Sign-changing coefficients - non critical ε, μ

3 Scalar problems

4 Sign-changing coefficients - critical ε, non critical μ

5 Sign-changing coefficients - critical ε, μ

Classical case

- Let us first consider the classical case where $\varepsilon,\, \mu \geq c > 0$ in Ω.
- We focus our attention on the electric problem

$$(\mathscr{P}) \left| \begin{array}{rcll} \mathbf{curl}\, \mu^{-1} \mathbf{curl}\, \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} & = & i\omega \boldsymbol{J} & \text{in } \Omega \\ \boldsymbol{E} \times \nu & = & 0 & \text{in } \partial\Omega \end{array} \right.$$

where $\boldsymbol{J} \in \mathbf{L}^2(\Omega) := \mathrm{L}^2(\Omega)^3$ is such that $\operatorname{div} \boldsymbol{J} = 0$ in Ω.

Classical case

- Let us first consider the classical case where $\varepsilon,\,\mu \geq c > 0$ in Ω.
- We focus our attention on the electric problem

$$(\mathscr{P})\left|\begin{array}{rcll}\mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E}-\omega^2\varepsilon\boldsymbol{E} & = & i\omega\boldsymbol{J} & \text{in } \Omega\\ \boldsymbol{E}\times\nu & = & 0 & \text{in } \partial\Omega\end{array}\right.$$

where $\boldsymbol{J}\in\mathbf{L}^2(\Omega):=\mathrm{L}^2(\Omega)^3$ is such that $\operatorname{div}\boldsymbol{J}=0$ in Ω.

$$(\mathscr{P})\Leftrightarrow\boxed{(\mathscr{P}_{\mathbf{H}})\left|\begin{array}{l}\text{Find } \boldsymbol{E}\in\mathbf{H}_N(\mathbf{curl}\,) \text{ such that for all } \boldsymbol{E}'\in\mathbf{H}_N(\mathbf{curl}\,)\\ \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{E}'-\omega^2\varepsilon\boldsymbol{E}\cdot\boldsymbol{E}'\,dx=i\omega\int_\Omega\boldsymbol{J}\cdot\boldsymbol{E}'\,dx,\end{array}\right.}$$

where $\mathbf{H}_N(\mathbf{curl}\,):=\{\boldsymbol{u}\in\mathbf{L}^2(\Omega)\,|\,\mathbf{curl}\,\boldsymbol{u}\in\mathbf{L}^2(\Omega) \text{ and } \boldsymbol{u}\times\nu=0 \text{ on } \partial\Omega\}$.

Classical case 1/3

- Let us first consider the classical case where $\varepsilon,\, \mu \geq c > 0$ in Ω.
- We focus our attention on the electric problem

$$(\mathscr{P}) \left| \begin{array}{rcll} \mathbf{curl}\, \mu^{-1} \mathbf{curl}\, \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} & = & i\omega \boldsymbol{J} & \text{in } \Omega \\ \boldsymbol{E} \times \nu & = & 0 & \text{in } \partial\Omega \end{array} \right.$$

where $\boldsymbol{J} \in \mathbf{L}^2(\Omega) := \mathrm{L}^2(\Omega)^3$ is such that $\mathrm{div}\, \boldsymbol{J} = 0$ in Ω.

$$(\mathscr{P}) \Leftrightarrow \boxed{(\mathscr{P}_{\mathbf{H}}) \left| \begin{array}{l} \text{Find } \boldsymbol{E} \in \mathbf{H}_N(\mathbf{curl}\,) \text{ such that for all } \boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}\,) \\ \displaystyle\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{E}' - \omega^2 \varepsilon \boldsymbol{E} \cdot \boldsymbol{E}'\, dx = i\omega \int_\Omega \boldsymbol{J} \cdot \boldsymbol{E}'\, dx, \end{array} \right.}$$

where $\mathbf{H}_N(\mathbf{curl}\,) := \{\boldsymbol{u} \in \mathbf{L}^2(\Omega) \,|\, \mathbf{curl}\, \boldsymbol{u} \in \mathbf{L}^2(\Omega) \text{ and } \boldsymbol{u} \times \nu = 0 \text{ on } \partial\Omega\}$.

Difficulty: $\nabla(\mathrm{H}^1_0) \subset \ker \mathbf{curl}\,\cdot$ and the embedding $\mathbf{H}_N(\mathbf{curl}\,) \subset \mathbf{L}^2(\Omega)$ is not compact which prevents using Fredholm alternative.

Classical case 2/3

Use the divergence free condition and work in the space

$$\mathbf{X}_N(\varepsilon) := \{\boldsymbol{u} \in \mathbf{H}_N(\mathbf{curl}\,) \,|\, \mathrm{div}\,(\varepsilon \boldsymbol{u}) = 0 \text{ in } \Omega\}$$

Use the divergence free condition and work in the space

$$\mathbf{X}_N(\varepsilon) := \{\boldsymbol{u} \in \mathbf{H}_N(\mathbf{curl}\,) \,|\, \mathrm{div}\,(\varepsilon \boldsymbol{u}) = 0 \text{ in } \Omega\}$$

$$(\boldsymbol{H} \in \mathbf{X}_T(\mu) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,) \,|\, \mathrm{div}\,(\mu\, \boldsymbol{u}) = 0,\ \mu \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}\ ).$$

- This leads to the problem

$$(\mathscr{P}_{\mathbf{X}}) \left| \begin{array}{l} \text{Find } \boldsymbol{E} \in \mathbf{X}_N(\varepsilon) \text{ such that for all } \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) \\ \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\boldsymbol{E}' - \omega^2 \varepsilon \boldsymbol{E} \cdot \boldsymbol{E}'\, dx = i\omega \int_\Omega \boldsymbol{J} \cdot \boldsymbol{E}'\, dx. \end{array}\right.$$

Classical case

Use the divergence free condition and work in the space

$$\mathbf{X}_N(\varepsilon) := \{\boldsymbol{u} \in \mathbf{H}_N(\mathbf{curl}\,) \,|\, \mathrm{div}\,(\varepsilon \boldsymbol{u}) = 0 \text{ in } \Omega\}$$

$$(\boldsymbol{H} \in \mathbf{X}_T(\mu) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,) \,|\, \mathrm{div}\,(\mu\, \boldsymbol{u}) = 0,\ \mu \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}\ ).$$

- This leads to the problem

$$(\mathscr{P}_{\mathbf{X}}) \left| \begin{array}{l} \text{Find } \boldsymbol{E} \in \mathbf{X}_N(\varepsilon) \text{ such that for all } \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) \\ \displaystyle\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{E}' - \omega^2 \varepsilon \boldsymbol{E} \cdot \boldsymbol{E}'\, dx = i\omega \int_\Omega \boldsymbol{J} \cdot \boldsymbol{E}'\, dx. \end{array} \right.$$

PROPOSITION: When ε, $\mu \geq c > 0$:

- the embedding $\mathbf{X}_N(\varepsilon) \subset \mathbf{L}^2(\Omega)$ is compact (Weber 80);

- $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \displaystyle\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, dx$ is coercive in $\mathbf{X}_N(\varepsilon)$;

so that $(\mathscr{P}_{\mathbf{X}})$ satisfies the Fredholm alternative (uniqueness $\Rightarrow$ existence).

Classical case

- Well-posedness of the initial problem comes from the following result:

PROP.: Assume that $\varepsilon \geq c > 0$. Then $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$ iff $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$.

Proof. $(\mathscr{P}_{\mathbf{H}}) \Rightarrow (\mathscr{P}_{\mathbf{X}})$ is direct.

$\Leftarrow$ Assume that $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$. For $\boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}\,)$, let $\varphi \in \mathrm{H}^1_0(\Omega)$ be s.t.

$$\operatorname{div}(\varepsilon \nabla \varphi) = \operatorname{div}(\varepsilon \boldsymbol{E}').$$

Classical case

- Well-posedness of the initial problem comes from the following result:

PROP.: Assume that $\varepsilon \geq c > 0$. Then $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$ iff $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$.

`Proof.` $(\mathscr{P}_{\mathbf{H}}) \Rightarrow (\mathscr{P}_{\mathbf{X}})$ is direct.
$\Leftarrow$ Assume that $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$. For $\boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}\,)$, let $\varphi \in \mathrm{H}^1_0(\Omega)$ be s.t.

$$\mathrm{div}\,(\varepsilon \nabla \varphi) = \mathrm{div}\,(\varepsilon \boldsymbol{E}').$$

Then we have $\boldsymbol{E}' - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$ so that we can write

$$\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,(\boldsymbol{E}' - \nabla\varphi) - \omega^2 \varepsilon \boldsymbol{E} \cdot (\boldsymbol{E}' - \nabla\varphi)\,dx = i\omega \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{E}' - \nabla\varphi)\,dx.$$

- Well-posedness of the initial problem comes from the following result:

Prop.: Assume that $\varepsilon \geq c > 0$. Then $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$ iff $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$.

`Proof.` $(\mathscr{P}_{\mathbf{H}}) \Rightarrow (\mathscr{P}_{\mathbf{X}})$ is direct.

$\Leftarrow$ Assume that $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$. For $\boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}\,)$, let $\varphi \in \mathrm{H}^1_0(\Omega)$ be s.t.

$$\mathrm{div}\,(\varepsilon\nabla\varphi) = \mathrm{div}\,(\varepsilon\boldsymbol{E}').$$

Then we have $\boldsymbol{E}' - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$ so that we can write

$$\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{E}' - \omega^2\varepsilon\boldsymbol{E}\cdot\boldsymbol{E}'\,dx = i\omega\int_\Omega \boldsymbol{J}\cdot\boldsymbol{E}'\,dx.$$

- Well-posedness of the initial problem comes from the following result:

PROP.: Assume that $\varepsilon \geq c > 0$. Then $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$ iff $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$.

Proof. $(\mathscr{P}_{\mathbf{H}}) \Rightarrow (\mathscr{P}_{\mathbf{X}})$ is direct.

$\Leftarrow$ Assume that $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$. For $\boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}\,)$, let $\varphi \in \mathrm{H}^1_0(\Omega)$ be s.t.

$$\mathrm{div}\,(\varepsilon \nabla \varphi) = \mathrm{div}\,(\varepsilon \boldsymbol{E}').$$

Then we have $\boldsymbol{E}' - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$ so that we can write

$$\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{E}' - \omega^2 \varepsilon \boldsymbol{E} \cdot \boldsymbol{E}' \, dx = i\omega \int_\Omega \boldsymbol{J} \cdot \boldsymbol{E}' \, dx.$$

This implies that $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$. □

1. Positive coefficients
2. Sign-changing coefficients - non critical ε, μ
3. Scalar problems
4. Sign-changing coefficients - critical ε, non critical μ
5. Sign-changing coefficients - critical ε, μ

Sign-changing coefficients

- Now we allow for a possible change of sign of ε and/or μ in Ω.

Introduce the **scalar** operator $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that

$$(A_\varepsilon \varphi, \varphi')_{\mathrm{H}^1_0(\Omega)} = \int_\Omega \varepsilon \nabla \varphi \cdot \nabla \overline{\varphi'}\, dx, \qquad \forall \varphi, \varphi' \in \mathrm{H}^1_0(\Omega).$$

Working as above, one shows:

PROPOSITION: Assume that A_ε is an isomorphism. Then $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$ iff $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$.

Sign-changing coefficients

- Now we allow for a possible change of sign of ε and/or μ in Ω.

Introduce the **scalar** operator $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that

$$(A_\varepsilon\varphi, \varphi')_{\mathrm{H}^1_0(\Omega)} = \int_\Omega \varepsilon\nabla\varphi \cdot \nabla\overline{\varphi'}\,dx, \qquad \forall\varphi, \varphi' \in \mathrm{H}^1_0(\Omega).$$

Working as above, one shows:

PROPOSITION: Assume that A_ε is an isomorphism. Then $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$ iff $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$.

PROPOSITION: Assume that A_ε is an isomorphism. Then we have

$$\|\boldsymbol{u}\|_\Omega \le C\,\|\mathbf{curl}\,\boldsymbol{u}\|_\Omega, \qquad \forall \boldsymbol{u} \in \mathbf{X}_N(\varepsilon).$$

Thus $\mathbf{X}_N(\varepsilon)$ endowed with $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_\Omega$ is a Hilbert space.

`Proof.` Write $\boldsymbol{u} = \nabla\varphi + \mathbf{curl}\,\boldsymbol{\psi}$ with $\varphi \in \mathrm{H}^1_0(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{X}_T(1)$. Then use that $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\psi} = \Delta\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{u}$ and $A_\varepsilon\varphi = \mathrm{div}\,(\varepsilon\mathbf{curl}\,\boldsymbol{\psi})$. □

Sign-changing coefficients

How to study $(\mathscr{P}_{\mathbf{X}})$ now?

$$(\mathscr{P}_{\mathbf{X}}) \left| \begin{array}{l} \text{Find } \boldsymbol{E} \in \mathbf{X}_N(\varepsilon) \text{ such that for all } \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) : \\ \underbrace{\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \overline{\boldsymbol{E}'}}_{a(\boldsymbol{E},\boldsymbol{E}')} - \omega^2 \underbrace{\int_\Omega \varepsilon \boldsymbol{E} \cdot \overline{\boldsymbol{E}'}}_{c(\boldsymbol{E},\boldsymbol{E}')} = \underbrace{\int_\Omega \boldsymbol{F} \cdot \overline{\boldsymbol{E}'}}_{\ell(\boldsymbol{E}')}, \end{array} \right.$$

Difficulties:

When μ changes sign, $a(\cdot,\cdot)$ is not coercive.

When ε changes sign, is the embedding $\mathbf{X}_N(\varepsilon) \subset \mathbf{L}^2(\Omega)$ compact?

$\mathbb{T}$-coercivity in the vector case 1/2

If $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N(\varepsilon)$, we have

$$\begin{array}{rrcll} & a(\boldsymbol{E},\boldsymbol{E}') - \omega^2 c(\boldsymbol{E},\boldsymbol{E}') & = & \ell(\boldsymbol{E}'), & \forall \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) \\ \Leftrightarrow & a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') - \omega^2 c(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') & = & \ell(\mathbb{T}\boldsymbol{E}'), & \forall \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon). \end{array}$$

The key idea is to construct $\mathbb{T} \in \mathbf{X}_N(\varepsilon) \to \mathbf{X}_N(\varepsilon)$ such that $a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') = \int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, (\overline{\mathbb{T}\boldsymbol{E}'})$ is coercive in $\mathbf{X}_N(\varepsilon)$.

$\mathbb{T}$-coercivity in the vector case 1/2

If $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N(\varepsilon)$, we have

$$\begin{array}{rcll} a(\boldsymbol{E},\boldsymbol{E}') - \omega^2 c(\boldsymbol{E},\boldsymbol{E}') & = & \ell(\boldsymbol{E}'), & \forall \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) \\ \Leftrightarrow \quad a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') - \omega^2 c(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') & = & \ell(\mathbb{T}\boldsymbol{E}'), & \forall \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon). \end{array}$$

The key idea is to construct $\mathbb{T} \in \mathbf{X}_N(\varepsilon) \to \mathbf{X}_N(\varepsilon)$ such that $a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') = \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,(\overline{\mathbb{T}\boldsymbol{E}'})$ is coercive in $\mathbf{X}_N(\varepsilon)$.

To present the construction, set $\mathrm{H}^1_\#(\Omega) := \{\varphi \in \mathrm{H}^1(\Omega)\,|\,\int_\Omega \varphi\,dx = 0\}$.

Introduce the **scalar** operator $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ such that

$$(A_\mu\varphi,\varphi')_{\mathrm{H}^1_\#(\Omega)} = \int_\Omega \mu\nabla\varphi\cdot\nabla\overline{\varphi'}\,dx, \qquad \forall \varphi,\varphi' \in \mathrm{H}^1_\#(\Omega).$$

$\mathbb{T}$-coercivity in the vector case 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$. We would like to have

$$\mathbf{curl}\,(\mathbb{T}\boldsymbol{E}) = \mu \mathbf{curl}\,\boldsymbol{E}$$

to get

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1} \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,(\overline{\mathbb{T}\boldsymbol{E})})\,dx = \int_\Omega |\mathbf{curl}\,\boldsymbol{E}|^2\,dx.$$

But this is impossible in general (take the divergence)!

$\mathbb{T}$-coercivity in the vector case 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_{\#}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi \cdot \nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E} \cdot \nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_{\#}(\Omega).$$

$\mathbb{T}$-coercivity in the vector case 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu \nabla\psi \cdot \nabla\psi' \, dx = \int_\Omega \mu \mathbf{curl}\,\boldsymbol{E} \cdot \nabla\psi' \, dx, \quad \forall \psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

$\mathbb{T}$-coercivity in the vector case 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

❶ Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu \nabla\psi \cdot \nabla\psi' \, dx = \int_\Omega \mu \mathbf{curl}\,\boldsymbol{E} \cdot \nabla\psi' \, dx, \quad \forall \psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

❷ Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

$\mathbb{T}$-coercivity in the vector case 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

3. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'\,dx = \int_\Omega \varepsilon\boldsymbol{u}\cdot\nabla\varphi'\,dx, \quad \forall\varphi' \in \mathrm{H}^1_0(\Omega).$$

$\mathbb{T}$-coercivity in the vector case 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_{\#}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_{\#}(\Omega).$$

> ☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

3. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'\,dx = \int_\Omega \varepsilon\boldsymbol{u}\cdot\nabla\varphi'\,dx, \quad \forall\varphi' \in \mathrm{H}^1_0(\Omega).$$

> ☛ Ok when A_ε is an isom.

$\mathbb{T}$-coercivity in the vector case 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu \nabla\psi \cdot \nabla\psi' \, dx = \int_\Omega \mu \mathbf{curl}\,\boldsymbol{E} \cdot \nabla\psi' \, dx, \quad \forall \psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

3. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \, dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla\varphi' \, dx, \quad \forall \varphi' \in \mathrm{H}^1_0(\Omega).$$

☛ Ok when A_ε is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$.

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu \nabla\psi \cdot \nabla\psi' \, dx = \int_\Omega \mu \mathbf{curl}\,\boldsymbol{E} \cdot \nabla\psi' \, dx, \quad \forall \psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

3. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \, dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla\varphi' \, dx, \quad \forall \varphi' \in \mathrm{H}^1_0(\Omega).$$

☛ Ok when A_ε is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1} \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,(\overline{\mathbb{T}\boldsymbol{E}})\, dx$$

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu \nabla\psi \cdot \nabla\psi'\,dx = \int_\Omega \mu\,\mathbf{curl}\,\boldsymbol{E} \cdot \nabla\psi'\,dx, \quad \forall \psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

3. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi'\,dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla\varphi'\,dx, \quad \forall \varphi' \in \mathrm{H}^1_0(\Omega).$$

☛ Ok when A_ε is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{u}}\,dx$$

$\mathbb{T}$-coercivity in the vector case

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

3. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'\,dx = \int_\Omega \varepsilon\boldsymbol{u}\cdot\nabla\varphi'\,dx, \quad \forall\varphi' \in \mathrm{H}^1_0(\Omega).$$

Ok when A_ε is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. There holds:

$$a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot(\overline{\mathbf{curl}\,\boldsymbol{E} - \nabla\psi})\,dx$$

$\mathbb{T}$-coercivity in the vector case

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

3. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'\,dx = \int_\Omega \varepsilon\boldsymbol{u}\cdot\nabla\varphi'\,dx, \quad \forall\varphi' \in \mathrm{H}^1_0(\Omega).$$

☛ Ok when A_ε is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\,dx = \int_\Omega |\mathbf{curl}\,\boldsymbol{E}|^2\,dx.$$

$\mathbb{T}$-coercivity in the vector case 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

Lemma. Suppose that

$$A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega) \text{ is an isomorphism}$$

$$A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \text{ is an isomorphism.}$$

Then, there exists $\mathbb{T} : \mathbf{X}_N(\varepsilon) \to \mathbf{X}_N(\varepsilon)$ such that, for all $\boldsymbol{E}$, $\boldsymbol{E}'$

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}') = a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}') = \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{E}'}\,dx$$

(this implies in particular that $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N(\varepsilon)$).

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{u}}\,dx = \int_\Omega |\mathbf{curl}\,\boldsymbol{E}|^2\,dx.$$

Compact embedding and final result

THEOREM. Assume that $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism. Then the embedding of $\mathbf{X}_N(\varepsilon)$ in $\mathbf{L}^2(\Omega)$ is compact.

`Proof.` 1) $\operatorname{div}(\varepsilon \boldsymbol{u}) = 0 \quad \Rightarrow \quad \varepsilon \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad$ with $\boldsymbol{\psi} \in \mathbf{X}_T(1)$.

2) Then we get $\mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\, \boldsymbol{\psi}) = \mathbf{curl}\, \boldsymbol{u}$.

3) When $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isom, there is $\mathbb{T} : \mathbf{X}_T(1) \to \mathbf{X}_T(1)$ s.t.

$$\|\mathbf{curl}\, \boldsymbol{\psi}\|^2_\Omega = \int_\Omega \varepsilon^{-1}\mathbf{curl}\, \boldsymbol{\psi} \cdot \mathbf{curl}\,(\mathbb{T}\boldsymbol{\psi})\, dx = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot (\mathbb{T}\boldsymbol{\psi})\, dx. \quad \square$$

Compact embedding and final result

THEOREM. Assume that $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism. Then the embedding of $\mathbf{X}_N(\varepsilon)$ in $\mathbf{L}^2(\Omega)$ is compact.

`Proof.` 1) $\mathrm{div}\,(\varepsilon \boldsymbol{u}) = 0 \quad \Rightarrow \quad \varepsilon \boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}$ with $\boldsymbol{\psi} \in \mathbf{X}_T(1)$.

2) Then we get $\mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{\psi}) = \mathbf{curl}\,\boldsymbol{u}$.

3) When $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isom, there is $\mathbb{T} : \mathbf{X}_T(1) \to \mathbf{X}_T(1)$ s.t.

$$\|\mathbf{curl}\,\boldsymbol{\psi}\|^2_\Omega = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{\psi} \cdot \mathbf{curl}\,(\mathbb{T}\boldsymbol{\psi})\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot (\mathbb{T}\boldsymbol{\psi})\,dx. \quad \square$$

- This yields the final result (Bonnet-BenDhia, Chesnel, Ciarlet 14'):

THEOREM. Suppose that

$$A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega) \text{ is an isomorphism}$$

$$A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \text{ is an isomorphism.}$$

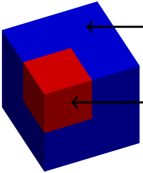
Then, the problem for the electric field is well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

Comments and example

- We have a similar result for the magnetic problem.
- These results extend to:

- situations where A_ε, A_μ are Fredholm of index zero with a non zero kernel;
- situations where Ω is not simply connected/$\partial\Omega$ is not connected.

Example of the Fichera's cube:



Proposition. Assume that

$$\frac{\varepsilon_-}{\varepsilon_+} \notin \left[-7; -\frac{1}{7}\right] \qquad \text{and} \qquad \frac{\mu_-}{\mu_+} \notin \left[-7; -\frac{1}{7}\right]. \quad *$$

Then, the problems for the electric and magnetic fields are well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

* *Note that 7 is the ratio of the blue volume over the red volume. This interval may not be optimal.*

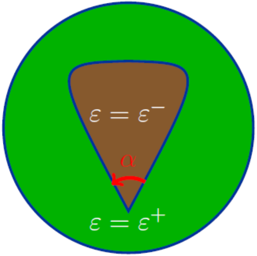
1. Positive coefficients
2. Sign-changing coefficients - non critical ε, μ
3. Scalar problems

The properties of the Maxwell's problem depend on the features of the scalar operators A_ε, A_μ. Let us study them.

4. Sign-changing coefficients - critical ε, non critical μ
5. Sign-changing coefficients - critical ε, μ

2D scalar problem - general picture

- Recall that $(A_\varepsilon \varphi, \varphi')_{\mathrm{H}^1_0(\Omega)} = \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\overline{\varphi'}\, dx, \qquad \forall \varphi, \varphi' \in \mathrm{H}^1_0(\Omega).$



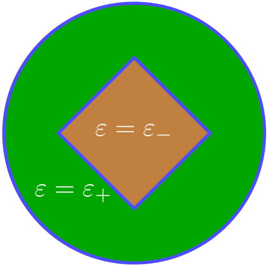
Features of A_ε depend on the angle α and on the contrast $\kappa := \varepsilon_-/\varepsilon_+$:

- If $\kappa \notin I_c := \left[-\frac{2\pi-\alpha}{\alpha}; -\frac{\alpha}{2\pi-\alpha} \right]$, A_ε is Fredholm of index zero.
- If $\kappa \in I_c$, A_ε is not Fredholm (its range is not close in $\mathrm{H}^1_0(\Omega)$).

- We call I_c the critical interval.

2D scalar problem - general picture

- Recall that $(A_\varepsilon \varphi, \varphi')_{\mathrm{H}^1_0(\Omega)} = \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\overline{\varphi'}\, dx, \qquad \forall \varphi, \varphi' \in \mathrm{H}^1_0(\Omega).$



For $\alpha = \pi/2$,
$I_c = [-3; -1/3]$.

Features of A_ε depend on the angle α and on the contrast $\kappa := \varepsilon_-/\varepsilon_+$:

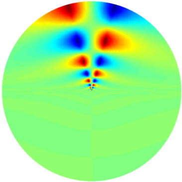
- If $\kappa \notin I_c := \left[-\frac{2\pi-\alpha}{\alpha}; -\frac{\alpha}{2\pi-\alpha} \right]$, A_ε is Fredholm of index zero.
- If $\kappa \in I_c$, A_ε is not Fredholm (its range is not close in $\mathrm{H}^1_0(\Omega)$).

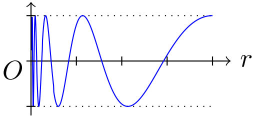
- We call I_c the critical interval.

2D scalar problem - inside the critical interval

- For $\kappa \in I_c \setminus \{-1\}$, Fredholmness in $\mathrm{H}^1_0(\Omega)$ is lost due to the existence of propagating singularities:

$$\left| \begin{array}{l} s^{\pm}(x) = r^{\pm i\eta}\Phi(\theta),\ \eta \in \mathbb{R} \setminus \{0\} \\ \mathrm{div}\,(\varepsilon \nabla s^{\pm}) = 0. \end{array} \right.$$



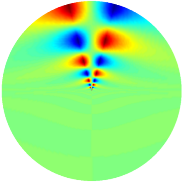


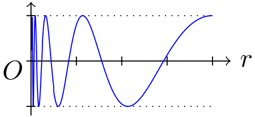
We have $s^{\pm} \in \mathrm{L}^2(\Omega)$ but $s^{\pm} \notin \mathrm{H}^1(\Omega)$.
Energy accumulates at the corner, $s^{\pm}$ are called black-hole singularities.

2D scalar problem - inside the critical interval

► For $\kappa \in I_c \setminus \{-1\}$, Fredholmness in $\mathrm{H}_0^1(\Omega)$ is lost due to the existence of propagating singularities:

$$\left| \begin{array}{l} s^{\pm}(x) = r^{\pm i\eta}\Phi(\theta),\ \eta \in \mathbb{R} \setminus \{0\} \\ \operatorname{div}(\varepsilon \nabla s^{\pm}) = 0. \end{array} \right.$$





We have $s^{\pm} \in \mathrm{L}^2(\Omega)$ but $s^{\pm} \notin \mathrm{H}^1(\Omega)$.
Energy accumulates at the corner, $s^{\pm}$ are called black-hole singularities.

► To recover Fredholmness, we have to modify the functional framework and take into account these singularities:

The corner is like infinity for scattering problems: a radiation condition must be imposed to select the outgoing behaviour s^{out}.

2D scalar problem - inside the critical interval

- We incorporate the radiation condition in the space by setting

$$\mathrm{V}^{\mathrm{out}} := \mathrm{span}(\mathfrak{s}^{\mathrm{out}}) \oplus \mathrm{V}^1_{-\beta}(\Omega)$$

where
- $\mathfrak{s}^{\mathrm{out}} := \chi s^{\mathrm{out}}$ (localization);
- $\mathrm{V}^1_{-\beta}(\Omega)$ is a weighted Sobolev space of functions which decay at O

- Define the operator $A^{\mathrm{out}}_{\varepsilon} : \mathrm{V}^{\mathrm{out}} \to \mathrm{V}^1_{\beta}(\Omega)^*$ such that

$$\langle A^{\mathrm{out}}_{\varepsilon}\varphi, \psi\rangle = \fint_{\Omega} \varepsilon \nabla\varphi \cdot \nabla\overline{\psi}\, dx$$

2D scalar problem - inside the critical interval

- We incorporate the radiation condition in the space by setting

$$\mathrm{V}^{\mathrm{out}} := \mathrm{span}(\mathfrak{s}^{\mathrm{out}}) \oplus \mathrm{V}^1_{-\beta}(\Omega)$$

where $\mathfrak{s}^{\mathrm{out}} := \chi s^{\mathrm{out}}$ (localization);
$\mathrm{V}^1_{-\beta}(\Omega)$ is a weighted Sobolev space of functions which decay at O

- Define the operator $A^{\mathrm{out}}_{\varepsilon} : \mathrm{V}^{\mathrm{out}} \to \mathrm{V}^1_{\beta}(\Omega)^*$ such that

$$\langle A^{\mathrm{out}}_{\varepsilon}\varphi, \psi\rangle = \fint_{\Omega} \varepsilon\nabla\varphi \cdot \nabla\overline{\psi}\, dx$$

where, for all $\varphi = c_{\varphi}\, \mathfrak{s}^{\mathrm{out}} + \tilde{\varphi}$, $\psi \in \mathrm{V}^1_{\beta}(\Omega)$,

$$\fint_{\Omega} \varepsilon\nabla\varphi \cdot \nabla\overline{\psi}\, dx := -\int_{\Omega} c_{\varphi}\mathrm{div}\,(\varepsilon\nabla\mathfrak{s}^{\mathrm{out}})\,\overline{\psi}\, dx + \int_{\Omega} \varepsilon\nabla\tilde{\varphi} \cdot \nabla\overline{\psi}\, dx.$$

2D scalar problem - inside the critical interval

- We incorporate the radiation condition in the space by setting

$$\mathrm{V}^{\mathrm{out}} := \mathrm{span}(\mathfrak{s}^{\mathrm{out}}) \oplus \mathrm{V}^1_{-\beta}(\Omega)$$

where | $\mathfrak{s}^{\mathrm{out}} := \chi s^{\mathrm{out}}$ (localization);
$\mathrm{V}^1_{-\beta}(\Omega)$ is a weighted Sobolev space of functions which decay at O

- Define the operator $A^{\mathrm{out}}_{\varepsilon} : \mathrm{V}^{\mathrm{out}} \to \mathrm{V}^1_{\beta}(\Omega)^*$ such that

$$\langle A^{\mathrm{out}}_{\varepsilon}\varphi, \psi\rangle = ⨍_{\Omega} \varepsilon\nabla\varphi\cdot\nabla\overline{\psi}\,dx$$

where, for all $\varphi = c_{\varphi}\,\mathfrak{s}^{\mathrm{out}} + \tilde{\varphi}$, $\psi \in \mathrm{V}^1_{\beta}(\Omega)$,

$$⨍_{\Omega} \varepsilon\nabla\varphi\cdot\nabla\overline{\psi}\,dx := -\int_{\Omega} c_{\varphi}\mathrm{div}\,(\varepsilon\nabla\mathfrak{s}^{\mathrm{out}})\,\overline{\psi}\,dx + \int_{\Omega}\varepsilon\nabla\tilde{\varphi}\cdot\nabla\overline{\psi}\,dx.$$

THEOREM: Assume that $\kappa \in I_c \setminus \{-1\}$. Then the operator $A^{\mathrm{out}}_{\varepsilon}$ is Fredholm of index zero. (Bonnet-BenDhia, Chesnel, Claeys 13')

`Tools of the proof.` Kondratiev approach (Mellin transform) (Kondratiev 67) + spaces with detached asymptotics (Nazarov, Plamenevski 94). □

3D scalar problem

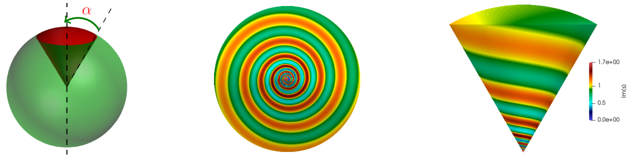
- Consider the conical tip, the simplest singular geometry in 3D. Now propagating singularities are of the form

$$s^{\pm}(x) = r^{\pm i\eta - 1/2}\Phi(\theta, \psi), \qquad \eta \in \mathbb{R} \setminus \{0\}$$

3D scalar problem

- Consider the conical tip, the simplest singular geometry in 3D. Now propagating singularities are of the form

$$s^{\pm}(x) = r^{\pm i\eta - 1/2}\Phi(\theta, \psi), \qquad \eta \in \mathbb{R} \setminus \{0\}$$

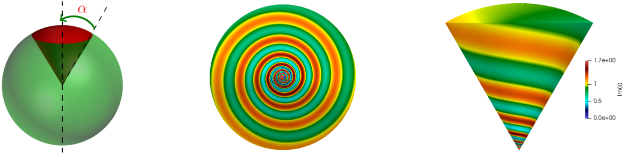


For the **circular conical tip**, they exist iff $\boldsymbol{\kappa \in (-1; -a_\alpha)}$ (but not for $\kappa < -1$!) for a certain explicit a_α (Li, Shipman 19, Li, Perfekt, Shipman 22).

3D scalar problem

- Consider the conical tip, the simplest singular geometry in 3D. Now propagating singularities are of the form

$$s^{\pm}(x) = r^{\pm i\eta - 1/2}\Phi(\theta, \psi), \qquad \eta \in \mathbb{R} \setminus \{0\}$$



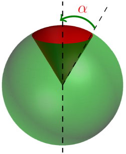
For the **circular conical tip**, they exist iff $\boldsymbol{\kappa \in (-1; -a_\alpha)}$ (but not for $\kappa < -1$!) for a certain explicit a_α (Li, Shipman 19, Li, Perfekt, Shipman 22).

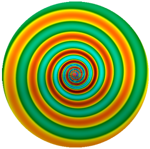
- Contrary to 2D, in 3D we can have $N > 1$ singularities $s_1^{\pm}, \dots, s_N^{\pm}$. Moreover $N \to +\infty$ when $\kappa \to -1^+$ or $\alpha \to 0^+$.

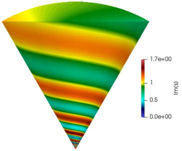
3D scalar problem

- Consider the conical tip, the simplest singular geometry in 3D. Now propagating singularities are of the form

$$s^{\pm}(x) = r^{\pm i\eta - 1/2}\Phi(\theta,\psi), \qquad \eta \in \mathbb{R}\setminus\{0\}$$







For the **circular conical tip**, they exist iff $\boldsymbol{\kappa} \in (\mathbf{-1; -a_{\alpha}})$ (but not for $\kappa < -1$!) for a certain explicit a_{α} (Li, Shipman 19, Li, Perfekt, Shipman 22).

- Contrary to 2D, in 3D we can have $N > 1$ singularities $s_1^{\pm},\dots,s_N^{\pm}$. Moreover $N \to +\infty$ when $\kappa \to -1^+$ or $\alpha \to 0^+$.

The solution to $\mathrm{div}\,(\varepsilon\nabla\varphi) = f$ must be searched in

$$\left|\begin{array}{ll} \mathrm{H}^1_0(\Omega) & \text{when } \boldsymbol{\kappa} \notin [\mathbf{-1; -a_{\alpha}}]; \\ \mathrm{V}^{\mathrm{out}} := \mathrm{span}(\mathfrak{s}_1^{\mathrm{out}},\dots,\mathfrak{s}_N^{\mathrm{out}}) \oplus \mathrm{V}^1_{-\beta}(\Omega) & \text{when } \boldsymbol{\kappa} \in (\mathbf{-1; -a_{\alpha}}). \end{array}\right.$$

Remark

- Propagating singularities are exactly the ones responsible for the existence of essential spectrum for the Neumann-Poincaré operator in non smooth domains:

→ Li, Shipman 19, Li, Perfekt, Shipman 22, De León-Contreras, Perfekt 22,...

1 Positive coefficients

2 Sign-changing coefficients - non critical ε, μ

3 Scalar problems

4 Sign-changing coefficients - critical ε, non critical μ

How to address the Maxwell's problem when one of the two scalars problems is well-posed only in the new framework?

5 Sign-changing coefficients - critical ε, μ

A new framework for electric fields

- Assume that the negative material has a conical tip and that there are N propagating singularities $\mathfrak{s}_1^{\text{out}}, \dots, \mathfrak{s}_N^{\text{out}}$ for the operator $\operatorname{div}(\varepsilon\nabla\cdot)$.
- Assume that μ is such that $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ is an isomorphism.
- Instead of working in $\mathbf{X}_N(\varepsilon)$, we look for a solution in

$$\mathbf{X}_N^{\text{out}}(\varepsilon) := \{\boldsymbol{u} = \sum_{n=1}^{N} c_n \nabla \mathfrak{s}_n^{\text{out}} + \tilde{\boldsymbol{u}},\ c_n \in \mathbb{C},\ \tilde{\boldsymbol{u}} \in \mathbf{V}^0_{-\beta}(\Omega)\,|\\ \mathbf{curl}\,\boldsymbol{u} \in \mathbf{L}^2(\Omega), \operatorname{div}(\varepsilon\boldsymbol{u}) = 0 \text{ in } \Omega \text{ and } \boldsymbol{u}\times\nu = 0 \text{ on } \partial\Omega\}$$

Here $\mathbf{V}^0_{-\beta}(\Omega) := \{\boldsymbol{u}\,|\,r^{-\beta}\boldsymbol{u} \in \mathbf{L}^2(\Omega)\}$, $\beta > 0$.

A new framework for electric fields

- Assume that the negative material has a conical tip and that there are N propagating singularities $\mathfrak{s}_1^{\text{out}}, \dots, \mathfrak{s}_N^{\text{out}}$ for the operator $\operatorname{div}(\varepsilon\nabla\cdot)$.
- Assume that μ is such that $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ is an isomorphism.
- Instead of working in $\mathbf{X}_N(\varepsilon)$, we look for a solution in

$$\mathbf{X}_N^{\text{out}}(\varepsilon) := \{\boldsymbol{u} = \sum_{n=1}^{N} c_n \nabla \mathfrak{s}_n^{\text{out}} + \tilde{\boldsymbol{u}},\, c_n \in \mathbb{C},\, \tilde{\boldsymbol{u}} \in \mathbf{V}^0_{-\beta}(\Omega)\,|\, \mathbf{curl}\,\boldsymbol{u} \in \mathbf{L}^2(\Omega), \operatorname{div}(\varepsilon\boldsymbol{u}) = 0 \text{ in } \Omega \text{ and } \boldsymbol{u} \times \nu = 0 \text{ on } \partial\Omega\}$$

Here $\mathbf{V}^0_{-\beta}(\Omega) := \{\boldsymbol{u}\,|\,r^{-\beta}\boldsymbol{u} \in \mathbf{L}^2(\Omega)\}$, $\beta > 0$.

- Note that $\mathbf{X}_N^{\text{out}}(\varepsilon) \not\subset \mathbf{L}^2(\Omega)$ (infinite energy!). More precisely, the fields are singular but the curls are not.

PROPOSITION: When $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}^1_\beta(\Omega)^*$ is an isomorphism, we have

$$|\boldsymbol{c}| + \|\tilde{\boldsymbol{u}}\|_{\mathbf{V}^0_{-\beta}(\Omega)} \le C\,\|\mathbf{curl}\,\boldsymbol{u}\|_\Omega, \qquad \forall \boldsymbol{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon).$$

Thus $\mathbf{X}_N^{\text{out}}(\varepsilon)$ endowed with $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_\Omega$ is a Hilbert space.

A new functional framework

- Then we consider the problem

$$(\mathscr{P}_{\mathbf{X}^{\text{out}}}) \left| \begin{array}{l} \text{Find } \boldsymbol{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \text{ such that for all } \boldsymbol{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \\ \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,dx - \omega^2 ⨍_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = i\omega\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{v}}\,dx \end{array}\right.$$

with $⨍_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = c_{\boldsymbol{u}}\overline{c_{\boldsymbol{v}}}\int_\Omega \operatorname{div}(\varepsilon\nabla\overline{s^+})s^+\,dx + \int_\Omega \varepsilon\tilde{\boldsymbol{u}}\cdot\overline{\tilde{\boldsymbol{v}}}\,dx.$

A new functional framework

- Then we consider the problem

$$(\mathscr{P}_{\mathbf{X}^{\text{out}}}) \left| \begin{array}{l} \text{Find } \boldsymbol{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \text{ such that for all } \boldsymbol{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \\ \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,dx - \omega^2 \fint_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = i\omega\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{v}}\,dx \end{array}\right.$$

with $\displaystyle\fint_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = c_{\boldsymbol{u}}\overline{c_{\boldsymbol{v}}}\int_\Omega \operatorname{div}(\varepsilon\nabla\overline{s^+})s^+\,dx + \int_\Omega \varepsilon\tilde{\boldsymbol{u}}\cdot\overline{\tilde{\boldsymbol{v}}}\,dx.$

Proposition: When $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}_\beta^1(\Omega)^*$ is an isomorphism, $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}^{\text{out}}})$ iff $\boldsymbol{E}$ solves the initial problem.

A new functional framework

- Then we consider the problem

$$(\mathscr{P}_{\mathbf{X}^{\text{out}}}) \left| \begin{array}{l} \text{Find } \boldsymbol{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \text{ such that for all } \boldsymbol{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \\ \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,dx - \omega^2 \fint_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = i\omega\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{v}}\,dx \end{array}\right.$$

with $\displaystyle\fint_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = c_{\boldsymbol{u}}\overline{c_{\boldsymbol{v}}}\int_\Omega \text{div}\,(\varepsilon\nabla\overline{s^+})s^+\,dx + \int_\Omega \varepsilon\tilde{\boldsymbol{u}}\cdot\overline{\tilde{\boldsymbol{v}}}\,dx.$

Proposition: When $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}_\beta^1(\Omega)^*$ is an isomorphism, $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}^{\text{out}}})$ iff $\boldsymbol{E}$ solves the initial problem.

- To study $(\mathscr{P}_{\mathbf{X}^{\text{out}}})$, next we construct a $\mathbb{T}$-coercivity operator in $\mathbf{X}_N^{\text{out}}(\varepsilon)$.

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_{\#}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_{\#}(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_{\#}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu \nabla\psi \cdot \nabla\psi' \, dx = \int_\Omega \mu \mathbf{curl}\,\boldsymbol{E} \cdot \nabla\psi' \, dx, \quad \forall \psi' \in \mathrm{H}^1_{\#}(\Omega).$$

Ok when A_μ is an isom.

2. Since $\operatorname{div}(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\mathrm{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\mathrm{out}}(\varepsilon)$.

❶ Introduce $\psi \in \mathrm{H}^1_{\#}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_{\#}(\Omega).$$

☛ Ok when A_μ is an isom.

❷ Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

❸ Introduce $\varphi \in \mathrm{V}^{\mathrm{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N^{\mathrm{out}}(\varepsilon)$. To proceed, solve

$$A_\varepsilon^{\mathrm{out}}\varphi = -\mathrm{div}\,(\varepsilon\boldsymbol{u}).$$

☛ Ok when $A_\varepsilon^{\mathrm{out}}$ is an isom.

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi \cdot \nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$A_\varepsilon^{\text{out}}\varphi = -\mathrm{div}\,(\varepsilon\boldsymbol{u}).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. There holds:

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\,\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\operatorname{div}(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$A_\varepsilon^{\text{out}}\varphi = -\operatorname{div}(\varepsilon\boldsymbol{u}).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,(\overline{\mathbb{T}\boldsymbol{E}})\,dx$$

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$A_\varepsilon^{\text{out}}\varphi = -\mathrm{div}\,(\varepsilon\boldsymbol{u}).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\,dx$$

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$A_\varepsilon^{\text{out}}\varphi = -\mathrm{div}\,(\varepsilon\boldsymbol{u}).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. There holds:

$$a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot(\overline{\mathbf{curl}\,\boldsymbol{E} - \nabla\psi})\,dx$$

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$A_\varepsilon^{\text{out}}\varphi = -\mathrm{div}\,(\varepsilon\boldsymbol{u}).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\,dx = \int_\Omega |\mathbf{curl}\,\boldsymbol{E}|^2\,dx.$$

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\mathrm{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\mathrm{out}}(\varepsilon)$.

Lemma. When

$$A_\varepsilon^{\mathrm{out}} : \mathrm{V}^{\mathrm{out}} \to \mathrm{V}^1_\beta(\Omega)^* \text{ is an isomorphism}$$

$$A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \text{ is an isomorphism,}$$

there exists $\mathbb{T} : \mathbf{X}_N^{\mathrm{out}}(\varepsilon) \to \mathbf{X}_N^{\mathrm{out}}(\varepsilon)$ such that, for all $\boldsymbol{E}$, $\boldsymbol{E}'$

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}') = a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}') = \int_\Omega \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \overline{\boldsymbol{E}'}\, dx$$

(this implies in particular that $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N^{\mathrm{out}}(\varepsilon)$).

is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \overline{\boldsymbol{u}}\, dx = \int_\Omega |\mathbf{curl}\, \boldsymbol{E}|^2\, dx.$$

Compact embedding and final result

THEOREM. Assume that $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}_\beta^1(\Omega)^*$ is an isomorphism. If $\left(\boldsymbol{u}_k = \sum_{n=1}^N c_k^n \nabla \mathfrak{s}_n^{\text{out}} + \tilde{\boldsymbol{u}}_k\right)$ is bounded in $\mathbf{X}_N^{\text{out}}(\varepsilon)$, up to a subsequence, $(\boldsymbol{c}_k)$, $(\tilde{\boldsymbol{u}}_k)$ converge in $\mathbb{C}^N$, $\mathbf{V}_{-\beta}^0(\Omega)$ respectively.

Compact embedding and final result

Theorem. Assume that $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}^1_\beta(\Omega)^*$ is an isomorphism. If $\big(\boldsymbol{u}_k = \sum_{n=1}^N c_k^n \nabla \mathfrak{s}_n^{\text{out}} + \tilde{\boldsymbol{u}}_k\big)$ is bounded in $\mathbf{X}_N^{\text{out}}(\varepsilon)$, up to a subsequence, $(\boldsymbol{c}_k)$, $(\tilde{\boldsymbol{u}}_k)$ converge in $\mathbb{C}^N$, $\mathbf{V}^0_{-\beta}(\Omega)$ respectively.

`Proof.` 1) Helmholtz decompo. $\Rightarrow \boldsymbol{u}_k = \sum_{n=1}^N c_k^n \nabla \mathfrak{s}_n^{\text{out}} + \nabla \varphi_k + \mathbf{curl}\, \boldsymbol{\psi}_k$.

2) $-\boldsymbol{\Delta} \boldsymbol{\psi}_k = \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{\psi}_k = \mathbf{curl}\, \boldsymbol{u}_k \Rightarrow (\mathbf{curl}\, \boldsymbol{u}_k)$ converges in $\mathbf{V}^0_{-\beta}(\Omega)$.

3) Use that $\operatorname{div}(\varepsilon \nabla \boldsymbol{u}_k) = 0$ and that $A_\varepsilon^{\text{out}}$ is an isomorphism. $\square$

Compact embedding and final result

Theorem. Assume that $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}_\beta^1(\Omega)^*$ is an isomorphism. If $\left(\boldsymbol{u}_k = \sum_{n=1}^N c_k^n \nabla \mathfrak{s}_n^{\text{out}} + \tilde{\boldsymbol{u}}_k\right)$ is bounded in $\mathbf{X}_N^{\text{out}}(\varepsilon)$, up to a subsequence, $(\boldsymbol{c}_k)$, $(\tilde{\boldsymbol{u}}_k)$ converge in $\mathbb{C}^N$, $\mathbf{V}_{-\beta}^0(\Omega)$ respectively.

`Proof.` 1) Helmholtz decompo. $\Rightarrow \boldsymbol{u}_k = \sum_{n=1}^N c_k^n \nabla \mathfrak{s}_n^{\text{out}} + \nabla \varphi_k + \mathbf{curl}\, \boldsymbol{\psi}_k$.

2) $-\boldsymbol{\Delta} \boldsymbol{\psi}_k = \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{\psi}_k = \mathbf{curl}\, \boldsymbol{u}_k \Rightarrow (\mathbf{curl}\, \boldsymbol{u}_k)$ converges in $\mathbf{V}_{-\beta}^0(\Omega)$.

3) Use that $\operatorname{div}(\varepsilon \nabla \boldsymbol{u}_k) = 0$ and that $A_\varepsilon^{\text{out}}$ is an isomorphism. □

- This yields the final result (Bonnet-BenDhia, Chesnel, Rihani 22'):

Theorem. Suppose that

$$A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}_\beta^1(\Omega)^* \text{ is an isomorphism}$$

$$A_\mu : \mathrm{H}_\#^1(\Omega) \to \mathrm{H}_\#^1(\Omega) \text{ is an isomorphism.}$$

Then, the problem $(\mathscr{P}_{\mathbf{X}^{\text{out}}})$ and the initial problem are well-posed for all $\omega \in \mathbb{C} \backslash \mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

1. Positive coefficients
2. Sign-changing coefficients - non critical ε, μ
3. Scalar problems
4. Sign-changing coefficients - critical ε, non critical μ
5. Sign-changing coefficients - critical ε, μ

How to address the Maxwell's problem when the two scalars problems are well-posed only in the new framework?

A new framework for electric fields

- Assume that the negative material has a conical tip and that there are

  N propagating singularities $\mathfrak{s}_1^{\varepsilon,\mathrm{out}}, \dots, \mathfrak{s}_N^{\varepsilon,\mathrm{out}}$ for the operator $\mathrm{div}\,(\varepsilon\nabla\cdot)$;

  M propagating singularities $\mathfrak{s}_1^{\mu,\mathrm{out}}, \dots, \mathfrak{s}_M^{\mu,\mathrm{out}}$ for the operator $\mathrm{div}\,(\mu\nabla\cdot)$.

A new framework for electric fields

- Assume that the negative material has a conical tip and that there are

  N propagating singularities $\mathfrak{s}_1^{\varepsilon,\mathrm{out}}, \dots, \mathfrak{s}_N^{\varepsilon,\mathrm{out}}$ for th

  M propagating singularities $\mathfrak{s}_1^{\mu,\mathrm{out}}, \dots, \mathfrak{s}_M^{\mu,\mathrm{out}}$ for th

Depending on κ_ε, κ_μ, we can have $N \neq M$.

A new framework for electric fields

- Assume that the negative material has a conical tip and that there are

  N propagating singularities $\mathfrak{s}_1^{\varepsilon,\mathrm{out}}, \dots, \mathfrak{s}_N^{\varepsilon,\mathrm{out}}$ for the operator $\mathrm{div}\,(\varepsilon\nabla\cdot)$;

  M propagating singularities $\mathfrak{s}_1^{\mu,\mathrm{out}}, \dots, \mathfrak{s}_M^{\mu,\mathrm{out}}$ for the operator $\mathrm{div}\,(\mu\nabla\cdot)$.

A new framework for electric fields

- Assume that the negative material has a conical tip and that there are

  | N propagating singularities $\mathfrak{s}_1^{\varepsilon,\text{out}},\dots,\mathfrak{s}_N^{\varepsilon,\text{out}}$ for the operator $\text{div}\,(\varepsilon\nabla\cdot)$;
  | M propagating singularities $\mathfrak{s}_1^{\mu,\text{out}},\dots,\mathfrak{s}_M^{\mu,\text{out}}$ for the operator $\text{div}\,(\mu\nabla\cdot)$.

- Instead of working in $\mathbf{X}_N(\varepsilon)$, $\mathbf{X}_N^{\text{out}}(\varepsilon)$, we look for an electric field in

$$\mathscr{X}_N^{\text{out}}(\varepsilon) := \{\boldsymbol{u} = \sum_{n=1}^{N} c_n^{\varepsilon}\nabla\mathfrak{s}_n^{\varepsilon,\text{out}} + \tilde{\boldsymbol{u}} \,|\, \mathbf{curl}\,\boldsymbol{u} = \sum_{m=1}^{M} c_m^{\mu}\mu\nabla\mathfrak{s}_m^{\mu,\text{out}} + \boldsymbol{\psi}_{\boldsymbol{u}},$$
$$\text{div}\,(\varepsilon\boldsymbol{u}) = 0 \text{ in } \Omega,\ \boldsymbol{u}\times\nu = 0 \text{ on } \partial\Omega,\ c_n^{\varepsilon},\ c_m^{\mu}\in\mathbb{C},\ \tilde{\boldsymbol{u}},\ \boldsymbol{\psi}_{\boldsymbol{u}}\in\mathbf{V}^0_{-\beta}(\Omega)\}$$

⚠ Note that both the fields and the curl of fields are singular.

A new framework for electric fields

- Assume that the negative material has a conical tip and that there are

  | N propagating singularities $\mathfrak{s}_1^{\varepsilon,\text{out}},\dots,\mathfrak{s}_N^{\varepsilon,\text{out}}$ for the operator $\text{div}\,(\varepsilon\nabla\cdot)$;
  | M propagating singularities $\mathfrak{s}_1^{\mu,\text{out}},\dots,\mathfrak{s}_M^{\mu,\text{out}}$ for the operator $\text{div}\,(\mu\nabla\cdot)$.

- Instead of working in $\mathbf{X}_N(\varepsilon)$, $\mathbf{X}_N^{\text{out}}(\varepsilon)$, we look for an electric field in

$$\mathscr{X}_N^{\text{out}}(\varepsilon) := \{\boldsymbol{u} = \sum_{n=1}^{N} c_n^{\varepsilon}\nabla\mathfrak{s}_n^{\varepsilon,\text{out}} + \tilde{\boldsymbol{u}}\,|\,\mathbf{curl}\,\boldsymbol{u} = \sum_{m=1}^{M} c_m^{\mu}\mu\nabla\mathfrak{s}_m^{\mu,\text{out}} + \boldsymbol{\psi}_{\boldsymbol{u}},$$
$$\text{div}\,(\varepsilon\boldsymbol{u}) = 0 \text{ in } \Omega,\ \boldsymbol{u}\times\nu = 0 \text{ on } \partial\Omega,\ c_n^{\varepsilon},\ c_m^{\mu}\in\mathbb{C},\ \tilde{\boldsymbol{u}},\ \boldsymbol{\psi}_{\boldsymbol{u}}\in\mathbf{V}_{-\beta}^0(\Omega)\}$$

⚠ Note that both the fields and the curl of fields are singular.

- Then we consider the problem

$$(\mathscr{P}_{\mathscr{X}^{\text{out}}})\ \Big|\ \text{Find } \boldsymbol{u}\in\mathscr{X}_N^{\text{out}}(\varepsilon) \text{ such that for all } v\in\mathbf{X}_N^{\text{out},\beta}(\varepsilon)$$
$$\int_\Omega \mu^{-1}\boldsymbol{\psi}_{\boldsymbol{u}}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,dx - \omega^2 ⨍_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = i\omega\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{v}}\,dx$$

with $\mathbf{X}_N^{\text{out},\beta}(\varepsilon) := \{\boldsymbol{u} = \sum_{n=1}^{N} c_n^{\varepsilon}\nabla\mathfrak{s}_n^{\varepsilon,\text{out}} + \tilde{\boldsymbol{u}}\,|\,\mathbf{curl}\,\boldsymbol{u}\in\mathbf{V}_{\beta}^0(\Omega),$
$\text{div}\,(\varepsilon\boldsymbol{u}) = 0 \text{ in } \Omega,\ \boldsymbol{u}\times\nu = 0 \text{ on } \partial\Omega,\ c_n^{\varepsilon}\in\mathbb{C},\ \tilde{\boldsymbol{u}}\in\mathbf{V}_{-\beta}^0(\Omega)\}$

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$. First, we would like to solve

$$\int_\Omega \mu \nabla \psi \cdot \nabla \psi' \, dx = \int_\Omega \mu \,\mathbf{curl}\, \boldsymbol{E} \cdot \nabla \psi' \, dx, \quad \forall \psi' \in \mathcal{V}^1_\beta(\Omega).$$

But this is impossible because the rhs is not in the good space.

$\rightarrow$ We have to regularise.

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$.

❶ Introduce $\psi \in \mathcal{V}^{\text{out}}$ such that

$$\fint_\Omega \mu \nabla \psi \cdot \nabla \psi' \, dx = \int_\Omega \mu r^{2\beta} \,\mathbf{curl}\, \boldsymbol{E} \cdot \nabla \psi' \, dx, \ \forall \psi' \in \mathcal{V}_\beta^1(\Omega).$$

☛ Ok when A_μ^{out} is an isom.

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\mathrm{out},\beta}(\varepsilon)$.

1. Introduce $\psi \in \mathcal{V}^{\mathrm{out}}$ such that

$$⨏_\Omega \mu \nabla \psi \cdot \nabla \psi' \, dx = \int_\Omega \mu r^{2\beta} \mathbf{curl}\, \boldsymbol{E} \cdot \nabla \psi' \, dx, \; \forall \psi' \in \mathcal{V}^1_\beta(\Omega).$$

☛ Ok when A^{out}_μ is an isom.

2. Since $\mathrm{div}\,(\mu(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, one can prove $\exists \boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ s.t.

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$.

1. Introduce $\psi \in \mathcal{V}^{\text{out}}$ such that

$$\oint_\Omega \mu \nabla \psi \cdot \nabla \psi' \, dx = \int_\Omega \mu r^{2\beta} \mathbf{curl}\, \boldsymbol{E} \cdot \nabla \psi' \, dx, \; \forall \psi' \in \mathcal{V}^1_\beta(\Omega).$$

☛ Ok when A_μ^{out} is an isom.

2. Since $\operatorname{div}\left(\mu(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi)\right) = 0$, one can prove $\exists \boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ s.t.

$$\mathbf{curl}\, \boldsymbol{u} = \mu \left(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi\right) \quad \text{in } \Omega.$$

Non trivial result because $\mu\left(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi\right) \in \mathbf{V}^0_\beta(\Omega) \supset \mathbf{L}^2(\Omega)$.

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$.

1. Introduce $\psi \in \mathcal{V}^{\text{out}}$ such that

$$\int_\Omega \mu \nabla \psi \cdot \nabla \psi' \, dx = \int_\Omega \mu r^{2\beta} \mathbf{curl}\, \boldsymbol{E} \cdot \nabla \psi' \, dx, \ \forall \psi' \in \mathcal{V}_\beta^1(\Omega).$$

☛ Ok when A_μ^{out} is an isom.

2. Since $\text{div}\,(\mu(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, one can prove $\exists \boldsymbol{u} \in \mathbf{V}_{-\beta}^0(\Omega)$ s.t.

$$\mathbf{curl}\, \boldsymbol{u} = \mu\,(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Non trivial result because $\mu\,(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \in \mathbf{V}_\beta^0(\Omega) \supset \mathbf{L}^2(\Omega)$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathscr{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon \nabla \varphi \cdot \nabla \varphi' \, dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla \varphi' \, dx, \ \forall \varphi' \in \mathrm{V}_\beta^1(\Omega).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$.

1. Introduce $\psi \in \mathcal{V}^{\text{out}}$ such that

$$\oint_\Omega \mu \nabla \psi \cdot \nabla \psi' \, dx = \int_\Omega \mu r^{2\beta} \mathbf{curl}\, \boldsymbol{E} \cdot \nabla \psi' \, dx, \ \forall \psi' \in \mathcal{V}_\beta^1(\Omega).$$

☛ Ok when A_μ^{out} is an isom.

2. Since $\operatorname{div}\left(\mu(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi)\right) = 0$, one can prove $\exists \boldsymbol{u} \in \mathbf{V}_{-\beta}^0(\Omega)$ s.t.

$$\mathbf{curl}\, \boldsymbol{u} = \mu\left(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi\right) \quad \text{in } \Omega.$$

Non trivial result because $\mu\left(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi\right) \in \mathbf{V}_\beta^0(\Omega) \supset \mathbf{L}^2(\Omega)$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathscr{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$\oint_\Omega \varepsilon \nabla \varphi \cdot \nabla \varphi' \, dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla \varphi' \, dx, \ \forall \varphi' \in \mathrm{V}_\beta^1(\Omega).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, set $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$.

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\mathrm{out},\beta}(\varepsilon)$.

1. Introduce $\psi \in \mathcal{V}^{\mathrm{out}}$ such that

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu r^{2\beta}\,\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx,\ \forall\psi'\in\mathcal{V}^1_\beta(\Omega).$$

☛ Ok when A_μ^{out} is an isom.

2. Since $\mathrm{div}\,(\mu(r^{2\beta}\,\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, one can prove $\exists\boldsymbol{u}\in\mathbf{V}^0_{-\beta}(\Omega)$ s.t.

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(r^{2\beta}\,\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Non trivial result because $\mu\,(r^{2\beta}\,\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \in \mathbf{V}^0_\beta(\Omega) \supset \mathbf{L}^2(\Omega)$.

3. Introduce $\varphi\in \mathrm{V}^{\mathrm{out}}$ such that $\boldsymbol{u}-\nabla\varphi\in\mathscr{X}_N^{\mathrm{out}}(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'\,dx = \int_\Omega \varepsilon\boldsymbol{u}\cdot\nabla\varphi'\,dx,\ \forall\varphi'\in\mathrm{V}^1_\beta(\Omega).$$

☛ Ok when $A_\varepsilon^{\mathrm{out}}$ is an isom.

4. Finally, set $\mathbb{T}\boldsymbol{E} := \boldsymbol{u}-\nabla\varphi$. One has $\mathbf{curl}\,(\mathbb{T}\boldsymbol{E}) = \mu\nabla\mathfrak{s}^{\mu,\mathrm{out}} + \boldsymbol{\psi}_{\mathbb{T}\boldsymbol{E}}$ and so

$$a(\mathbb{T}\boldsymbol{E},\boldsymbol{E}) = \int_\Omega \mu^{-1}\boldsymbol{\psi}_{\mathbb{T}\boldsymbol{E}}\cdot\mathbf{curl}\,\overline{\boldsymbol{E}}\,dx$$

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$.

1 Introduce $\psi \in \mathcal{V}^{\text{out}}$ such that

$$\int_\Omega \mu \nabla\psi \cdot \nabla\psi' \, dx = \int_\Omega \mu r^{2\beta} \mathbf{curl}\, \boldsymbol{E} \cdot \nabla\psi' \, dx, \ \forall \psi' \in \mathcal{V}_\beta^1(\Omega).$$

☛ Ok when A_μ^{out} is an isom.

2 Since $\operatorname{div}(\mu(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, one can prove $\exists \boldsymbol{u} \in \mathbf{V}_{-\beta}^0(\Omega)$ s.t.

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{ in } \Omega.$$

Non trivial result because $\mu\,(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \in \mathbf{V}_\beta^0(\Omega) \supset \mathbf{L}^2(\Omega)$.

3 Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathscr{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \, dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla\varphi' \, dx, \ \forall \varphi' \in \mathrm{V}_\beta^1(\Omega).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4 Finally, set $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. One has $\mathbf{curl}\,(\mathbb{T}\boldsymbol{E}) = \mu\nabla\mathfrak{s}^{\mu,\text{out}} + \boldsymbol{\psi}_{\mathbb{T}\boldsymbol{E}}$ and so

$$a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}) = \int_\Omega \mu^{-1} \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\overline{\boldsymbol{E}} \, dx$$

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$.

1. Introduce $\psi \in \mathcal{V}^{\text{out}}$ such that

$$\int_\Omega \mu \nabla \psi \cdot \nabla \psi' \, dx = \int_\Omega \mu r^{2\beta} \mathbf{curl}\, \boldsymbol{E} \cdot \nabla \psi' \, dx, \ \forall \psi' \in \mathcal{V}^1_\beta(\Omega).$$

☛ Ok when A_μ^{out} is an isom.

2. Since $\text{div}\,(\mu(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi)) = 0$, one can prove $\exists \boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ s.t.

$$\mathbf{curl}\, \boldsymbol{u} = \mu\,(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi) \quad \text{ in } \Omega.$$

Non trivial result because $\mu\,(r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi) \in \mathbf{V}^0_\beta(\Omega) \supset \mathbf{L}^2(\Omega)$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathscr{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon \nabla \varphi \cdot \nabla \varphi' \, dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla \varphi' \, dx, \ \forall \varphi' \in \mathrm{V}^1_\beta(\Omega).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, set $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. One has $\mathbf{curl}\,(\mathbb{T}\boldsymbol{E}) = \mu\nabla\mathfrak{s}^{\mu,\text{out}} + \boldsymbol{\psi}_{\mathbb{T}\boldsymbol{E}}$ and so

$$a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}) = \int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{E}} \, dx = \int_\Omega (r^{2\beta}\mathbf{curl}\, \boldsymbol{E} - \nabla\psi) \cdot \mathbf{curl}\, \overline{\boldsymbol{E}} \, dx$$

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$.

1. Introduce $\psi \in \mathcal{V}^{\text{out}}$ such that

$$\int_\Omega \mu \nabla\psi \cdot \nabla\psi' \, dx = \int_\Omega \mu r^{2\beta} \mathbf{curl}\, \boldsymbol{E} \cdot \nabla\psi' \, dx, \ \forall \psi' \in \mathcal{V}_\beta^1(\Omega).$$

☛ Ok when A_μ^{out} is an isom.

2. Since $\text{div}\,(\mu(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, one can prove $\exists \boldsymbol{u} \in \mathbf{V}_{-\beta}^0(\Omega)$ s.t.

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{ in } \Omega.$$

Non trivial result because $\mu\,(r^{2\beta}\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \in \mathbf{V}_\beta^0(\Omega) \supset \mathbf{L}^2(\Omega)$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathscr{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \, dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla\varphi' \, dx, \ \forall \varphi' \in \mathrm{V}_\beta^1(\Omega).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, set $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. One has $\mathbf{curl}\,(\mathbb{T}\boldsymbol{E}) = \mu\nabla\mathfrak{s}^{\mu,\text{out}} + \boldsymbol{\psi}_{\mathbb{T}\boldsymbol{E}}$ and so

$$a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}) = \int_\Omega \mu^{-1} \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\overline{\boldsymbol{E}} \, dx = \int_\Omega r^{2\beta} |\mathbf{curl}\,\boldsymbol{E}|^2 \, dx.$$

$\mathbb{T}$-coercivity

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$.

1. Introduce $\psi \in \mathcal{V}^{\text{out}}$ such that

Lemma. When

$$A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}_\beta^1(\Omega)^* \text{ is an isomorphism}$$

$$A_\mu^{\text{out}} : \mathcal{V}_\mu^{\text{out}} \to \mathcal{V}_\beta^1(\Omega)^* \text{ is an isomorphism,}$$

there is $\mathbb{T} : \mathbf{X}_N^{\text{out},\beta}(\varepsilon) \to \mathscr{X}_N^{\text{out}}(\varepsilon)$ such that, for all $\boldsymbol{E}, \boldsymbol{E}' \in \mathbf{X}_N^{\text{out},\beta}(\varepsilon)$

$$a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}') = \int_\Omega r^{2\beta} \,\mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \overline{\boldsymbol{E}'}\, dx.$$

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathscr{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \, dx = \int_\Omega \varepsilon \boldsymbol{u} \cdot \nabla\varphi' \, dx, \ \forall \varphi' \in \mathrm{V}_\beta^1(\Omega).$$

☛ Ok when $A_\varepsilon^{\text{out}}$ is an isom.

4. Finally, set $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. One has $\mathbf{curl}\,(\mathbb{T}\boldsymbol{E}) = \mu\nabla\mathfrak{s}^{\mu,\text{out}} + \boldsymbol{\psi}_{\mathbb{T}\boldsymbol{E}}$ and so

Analysis of the principal part

- With Riesz, define $\mathbb{A}_N^{\text{out}} : \mathscr{X}_N^{\text{out}}(\varepsilon) \to (\mathbf{X}_N^{\text{out},\beta}(\varepsilon))^*$ s.t. for all $\boldsymbol{u}$, $\boldsymbol{v}$

$$\langle \mathbb{A}_N^{\text{out}} \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \mu^{-1} \boldsymbol{\psi}_{\boldsymbol{u}} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, dx.$$

Analysis of the principal part

- With Riesz, define $\mathbb{A}_N^{\text{out}} : \mathscr{X}_N^{\text{out}}(\varepsilon) \to (\mathbf{X}_N^{\text{out},\beta}(\varepsilon))^*$ s.t. for all $\boldsymbol{u}$, $\boldsymbol{v}$

$$\langle \mathbb{A}_N^{\text{out}} \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \mu^{-1} \boldsymbol{\psi}_{\boldsymbol{u}} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, dx.$$

Proposition: When $A_\varepsilon^{\text{out}}$, A_μ^{out} are isomorphisms, $\mathbf{X}_N^{\text{out},\beta}(\varepsilon)$ endowed with $(r^{2\beta}\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_\Omega$ is a Hilbert space.

- Therefore, from the previous lemma, we get $\mathbb{A}_N^{\text{out}}\, \mathbb{T} = \text{Id}$. This shows that $\mathbb{A}_N^{\text{out}}$ is onto.

Analysis of the principal part

- With Riesz, define $\mathbb{A}_N^{\mathrm{out}} : \mathscr{X}_N^{\mathrm{out}}(\varepsilon) \to (\mathbf{X}_N^{\mathrm{out},\beta}(\varepsilon))^*$ s.t. for all $\boldsymbol{u}$, $\boldsymbol{v}$

$$\langle \mathbb{A}_N^{\mathrm{out}} \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \mu^{-1} \boldsymbol{\psi}_{\boldsymbol{u}} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, dx.$$

Proposition: When $A_\varepsilon^{\mathrm{out}}$, A_μ^{out} are isomorphisms, $\mathbf{X}_N^{\mathrm{out},\beta}(\varepsilon)$ endowed with $(r^{2\beta}\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_\Omega$ is a Hilbert space.

- Therefore, from the previous lemma, we get $\mathbb{A}_N^{\mathrm{out}}\, \mathbb{T} = \mathrm{Id}$. This shows that $\mathbb{A}_N^{\mathrm{out}}$ is onto.

- Now if $\boldsymbol{E} \in \ker \mathbb{A}_N^{\mathrm{out}}$, energy considerations ensure that $\mathbf{curl}\, \boldsymbol{E} \in \mathbf{L}^2(\Omega)$. Then we obtain

$$0 = \langle \mathbb{A}_N^{\mathrm{out}} \boldsymbol{E}, \mathbb{T}\boldsymbol{E} \rangle = \int_\Omega r^{2\beta} |\mathbf{curl}\, \boldsymbol{E}|^2\, dx$$

and so $\boldsymbol{E} \equiv 0$. This shows that $\mathbb{A}_N^{\mathrm{out}}$ is injective.

Analysis of the principal part

- With Riesz, define $\mathbb{A}_N^{\text{out}} : \mathscr{X}_N^{\text{out}}(\varepsilon) \to (\mathbf{X}_N^{\text{out},\beta}(\varepsilon))^*$ s.t. for all $\boldsymbol{u}$, $\boldsymbol{v}$

$$\langle \mathbb{A}_N^{\text{out}} \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \mu^{-1} \boldsymbol{\psi}_{\boldsymbol{u}} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, dx.$$

PROPOSITION: When $A_\varepsilon^{\text{out}}$, A_μ^{out} are isomorphisms, $\mathbf{X}_N^{\text{out},\beta}(\varepsilon)$ endowed with $(r^{2\beta}\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_\Omega$ is a Hilbert space.

- Therefore, from the previous lemma, we get $\mathbb{A}_N^{\text{out}}\, \mathbb{T} = \text{Id}$. This shows that $\mathbb{A}_N^{\text{out}}$ is onto.

- Now if $\boldsymbol{E} \in \ker \mathbb{A}_N^{\text{out}}$, energy considerations ensure that $\mathbf{curl}\, \boldsymbol{E} \in \mathbf{L}^2(\Omega)$. Then we obtain

$$0 = \langle \mathbb{A}_N^{\text{out}} \boldsymbol{E}, \mathbb{T}\boldsymbol{E} \rangle = \int_\Omega r^{2\beta} |\mathbf{curl}\, \boldsymbol{E}|^2\, dx$$

and so $\boldsymbol{E} \equiv 0$. This shows that $\mathbb{A}_N^{\text{out}}$ is injective.

THEOREM: When $A_\varepsilon^{\text{out}}$, A_μ^{out} are isomorphisms, $\mathbb{A}_N^{\text{out}}$ is an isomorphism.

Final result

- Additional work is needed to prove the compactness of the operator associated to

$$(\boldsymbol{u},\boldsymbol{v}) \mapsto \fint_{\Omega} \varepsilon \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx$$

and to show the equivalence with the initial problem.

- Finally, we get (Bonnet-BenDhia, Chesnel, Rihani 23):

THEOREM. Suppose that

$$A_{\varepsilon}^{\text{out}} : \mathrm{V}_{\varepsilon}^{\text{out}} \to \mathrm{V}_{\beta}^{1}(\Omega)^* \text{ is an isomorphism}$$

$$A_{\mu}^{\text{out}} : \mathcal{V}_{\mu}^{\text{out}} \to \mathcal{V}_{\beta}^{1}(\Omega)^* \text{ is an isomorphism.}$$

Then, the problem $(\mathscr{P}_{\mathscr{X}_N^{\text{out}}})$ and the initial problem are well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

1. Positive coefficients
2. Sign-changing coefficients - non critical ε, μ
3. Scalar problems
4. Sign-changing coefficients - critical ε, non critical μ
5. Sign-changing coefficients - critical ε, μ

Conclusion

What we obtained

1) When $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$, $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ are isomorphisms, the electric problem is well-posed in the usual space.

→ For the circular conical tip, this corresponds to $\boldsymbol{\kappa_\varepsilon}, \boldsymbol{\kappa_\mu} \notin [\boldsymbol{-1; -a_\alpha}]$.

2) When $A^{\mathrm{out}}_\varepsilon : \mathrm{V}^{\mathrm{out}}_\varepsilon \to \mathrm{V}^1_\beta(\Omega)^*$, $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ are isomorphisms, the electric problem is well-posed in a space of singular fields whose curls are in $\mathbf{L}^2(\Omega)$.

→ For the circular conical tip, case $\boldsymbol{\kappa_\varepsilon} \in (\boldsymbol{-1; -a_\alpha})$, $\boldsymbol{\kappa_\mu} \notin [\boldsymbol{-1; -a_\alpha}]$.

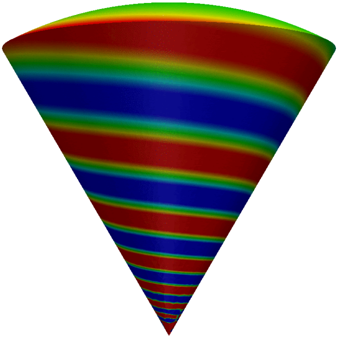
3) When $A^{\mathrm{out}}_\varepsilon : \mathrm{V}^{\mathrm{out}}_\varepsilon \to \mathrm{V}^1_\beta(\Omega)^*$, $A^{\mathrm{out}}_\mu : \mathcal{V}^{\mathrm{out}}_\mu \to \mathcal{V}^1_\beta(\Omega)^*$ are isomorphisms, the electric problem is well-posed in a space where the fields and their curls are singular.

→ For the circular conical tip, case $\boldsymbol{\kappa_\varepsilon}, \boldsymbol{\kappa_\mu} \in (\boldsymbol{-1; -a_\alpha})$.

Conclusion

Comments and open questions

- ♠ We have similar results for the magnetic problem.
- ♠ In cases 2), 3), the problems in the usual spaces are either ill-posed or not equivalent to the initial Maxwell's equations.
- ♠ Outgoing behaviours can be justified in certain situations with the limiting absorption principle.
- ♠ It is not clear how to solve numerically the problems 2), 3).
- ♠ How to study other 3D singular geometries, in particular with edges?
- ♠ Can this be useful to study other problems (elasticity)?



Thank you!

Section 2 A.-S. Bonnet-BenDhia, L. Chesnel, P. Ciarlet Jr., *T-coercivity for the Maxwell problem with sign-changing coefficients*, CPDE, vol. 39, 06:1007-1031, 2014.

Section 4 A.-S. Bonnet-BenDhia, L. Chesnel, M. Rihani, *Maxwell's equations with hyper-singularities at a conical plasmonic tip*, JMPA, vol. 161, 70-110, 2022.

Section 5 A.-S. Bonnet-BenDhia, L. Chesnel, M. Rihani, *Maxwell's equations with hyper-singularities at a negative index material conical tip*, arXiv 2305.01982, 2023.

H.-M. Nguyen, S. Sil, *Limiting absorption principle and well-posedness for the time-harmonic Maxwell equations with anisotropic sign-changing coefficients*, Commun. Math. Phys., vol. 379, 145-176, 2020.

M. Rihani, *Maxwell's equations in presence of negative materials*, PhD thesis, 2022.