Filtrage de Kalman appliqué à la localisation

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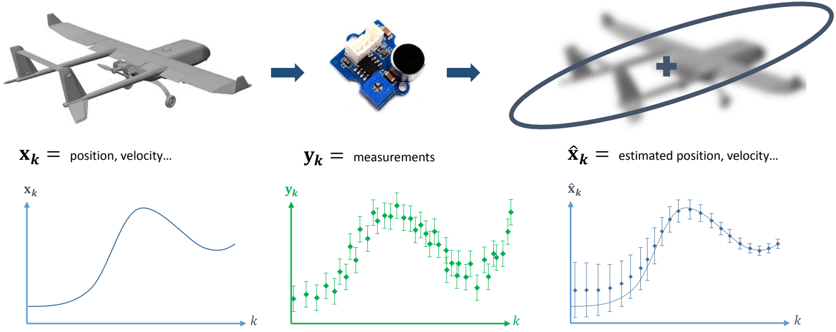
IA712

Motivation

Autonomous systems need to perform accurate state estimation (e.g., self localization, objective assessment...)

... with limited embedded calculation capabilities.

State estimation for navigation



State estimation consists in retrieving the vehicle's state from noisy and incomplete measurements and uncertain evolution model.

What we know

Theoretical state evolution (dynamical model):

$$\begin{aligned} \dot{\mathbf{x}} &= F(\mathbf{x}, \mathbf{u}) \\ \mathbf{x}_k &= f(\mathbf{x}_{k-1}, \mathbf{u}_k) \end{aligned} \tag{1}$$

Theoretical observation equation (sensor model):

$$\mathbf{y}_k = h(\mathbf{x}_k) \tag{2}$$

However, these equations are not totally representative of the actual system, e.g.:

- unexpected wind, friction, unmodeled dynamics...
- sensor noise, unmodeled disturbances...

What we **don't** know (uncertainties)

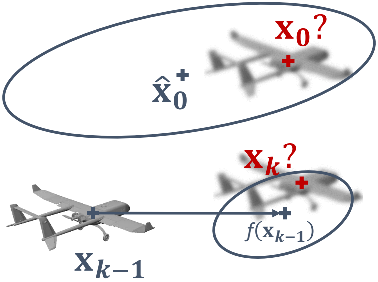
- Initial state uncertainty

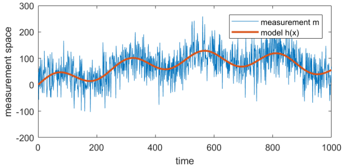
- Process noise (dynamics)

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k) + \mathbf{w}_k$$

- Measurement noise (and potentially some bias)

$$\mathbf{y}_k = h(\mathbf{x}_k) + \mathbf{v}_k$$





A way to model uncertainties: probability distribution functions

- State distribution:

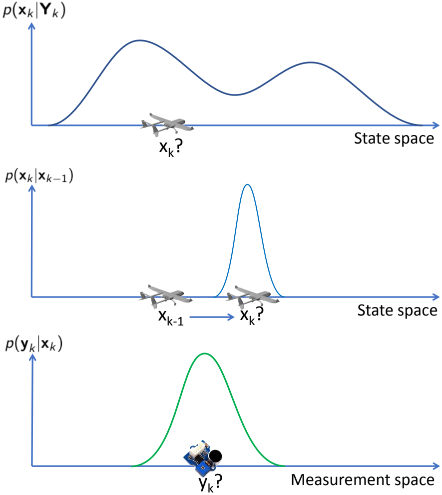
$$p(\mathbf{x}_k|\mathbf{y}_1,\ldots,\mathbf{y}_k) \triangleq p(\mathbf{x}_k|\mathbf{Y}_k)$$

- Process noise distribution

$$p(\mathbf{x}_k|\mathbf{x}_{k-1})$$

- Measurement distribution

$$p(\mathbf{y}_k|\mathbf{x}_k)$$



Optimal filter equations (Bayesian filtering)

State density propagation (dynamics, Chapman-Kolmogorov equation):

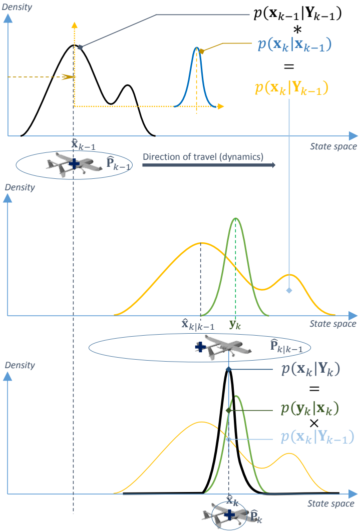
$$p(\mathbf{x}_k|\mathbf{Y}_{k-1}) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{Y}_{k-1})\mathrm{d}\mathbf{x}_{k-1} \qquad (3)$$

State density correction/update (measurements, Bayes rule):

$$p(\mathbf{x}_k|\mathbf{Y}_k) = \frac{p(\mathbf{y}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{Y}_{k-1})}{\int p(\mathbf{y}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{Y}_{k-1})\mathrm{d}\mathbf{x}_k} \qquad (4)$$

(Measurements $\mathbf{y}_i$ and $\mathbf{y}_j$ ($\forall\ i \neq j$) are assumed to be statistically independent)

Optimal filter equations (Bayesian filtering)



(a) Prediction

Convolution of the **prior conditional density** with the state transition density. The transition density accounts for the deterministic dynamics f_k and its uncertainty (process noise $\mathbf{w}_k$).

(b) Predicted density, new measurement

Step (a) results in the **predicted conditional density**, whose support is usually larger than the prior density.
A measurement $\mathbf{y}_k$ is now available. It will introduce information in the estimation.

(c) Correction

The predicted density is multiplied with the measurement density, leading to the **posterior conditional density**.
Its support is usually smaller than the predicted density. It yields a refined estimate $\hat{\mathbf{x}}_k$ and covariance $\hat{\mathbf{P}}_k$.
Then, one can iterate back to step (a) for further time-steps.

Linear-Gaussian models

Linear-Gaussian state evolution (dynamical model):

$$\begin{aligned} &\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{G}_k \mathbf{u}_k + \mathbf{w}_k \\ &\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k) \qquad (\mathbf{w}_k \in \mathbb{R}^d, \mathbf{Q}_k \in \mathbb{R}^{d \times d}) \end{aligned} \tag{5}$$

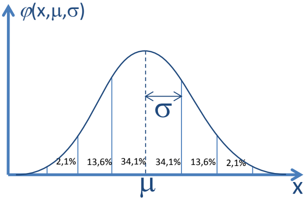
Linear-Gaussian observation equation (sensor model):

$$\begin{aligned} &\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \\ &\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k) \qquad (\mathbf{v}_k \in \mathbb{R}^{d_m}, \mathbf{R}_k \in \mathbb{R}^{d_m \times d_m}) \end{aligned} \tag{6}$$

Gaussian initial state uncertainty:

$$\mathbf{x}_0 \sim \mathcal{N}(\mathbf{s}_0, \mathbf{P}_0) \qquad (\mathbf{s}_0 \in \mathbb{R}^d, \mathbf{P}_0 \in \mathbb{R}^{d \times d})$$

Gaussian density (recall)



Only two parameters:

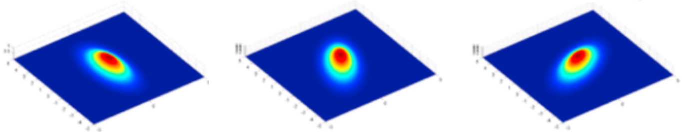
- Expectancy (theoretical mean)
- Covariance (dispersion and correlations)

(the covariance P is the square of the standard deviation σ)

Gaussian pdf:

$$\varphi(\mathbf{x};\boldsymbol{\mu},\mathbf{P}) = \frac{1}{\sqrt{(2\pi)^d|\mathbf{P}|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{P}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

Illustration in 2D: $\mathbf{P} = \begin{bmatrix} a & c \\ c & b \end{bmatrix} > 0$



Gaussian Bayesian filter

State density propagation (dynamics, Chapman-Kolmogorov equation):

$$
\begin{array}{ccccc}
p(\mathbf{x}_k|\mathbf{Y}_{k-1}) & = \int & p(\mathbf{x}_k|\mathbf{x}_{k-1}) & p(\mathbf{x}_{k-1}|\mathbf{Y}_{k-1}) & \mathrm{d}\mathbf{x}_{k-1} \\
\mathcal{N}(\mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1}) & \leftarrow & \mathcal{N}(\mathbf{x}_k, \mathbf{Q}_k) & \mathcal{N}(\mathbf{x}_{k-1}, \mathbf{P}_{k-1}) &
\end{array} \tag{7}
$$

State density correction/update (measurements, Bayes rule):

$$
\begin{array}{cccc}
p(\mathbf{x}_k|\mathbf{Y}_k) & \propto & p(\mathbf{y}_k|\mathbf{x}_k) & p(\mathbf{x}_k|\mathbf{Y}_{k-1}) \\
\mathcal{N}(\mathbf{x}_k, \mathbf{P}_k) & \leftarrow & \mathcal{N}(\mathbf{y}_k, \mathbf{R}_k) & \mathcal{N}(\mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})
\end{array} \tag{8}
$$

$\Rightarrow$ Gaussian models yield Gaussian conditional state densities

Kalman Filter equations

Prediction step:

$$\begin{aligned}\widehat{\mathbf{x}}_{k|k-1} &= \mathbf{F}_k \widehat{\mathbf{x}}_{k-1|k-1} + \mathbf{G}_k \mathbf{u}_k \\ \widehat{\mathbf{P}}_{k|k-1} &= \mathbf{F}_k \widehat{\mathbf{P}}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k\end{aligned} \tag{9}$$

Correction step (where $\mathbf{y}_k - \mathbf{H}_k \widehat{\mathbf{x}}_{k|k-1}$ is the *innovation* term and $\mathbf{K}_k$ is the Kalman gain):

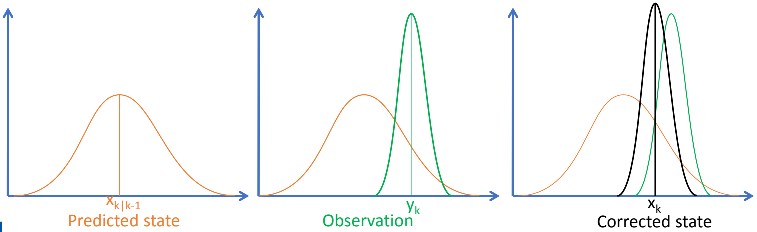
$$\begin{aligned}\widehat{\mathbf{x}}_{k|k} &= \widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \left(\mathbf{y}_k - \mathbf{H}_k \widehat{\mathbf{x}}_{k|k-1} \right) \\ \widehat{\mathbf{P}}_{k|k} &= (\mathbf{I}_d - \mathbf{K}_k \mathbf{H}_k)\, \widehat{\mathbf{P}}_{k|k-1} \\ \mathbf{K}_k &= \widehat{\mathbf{P}}_{k|k-1} \mathbf{H}_k^T \left(\mathbf{R}_k + \mathbf{H}_k \widehat{\mathbf{P}}_{k|k-1} \mathbf{H}_k^T \right)^{-1}\end{aligned} \tag{10}$$

1D illustration of the correction step

Assume $H = 1$, $x_{k|k-1} \in \mathbb{R}$, $y_k \in \mathbb{R}$, then:

$$K_k = \frac{\widehat{P}_{k|k-1}}{R_k + \widehat{P}_{k|k-1}}$$

$$\widehat{x}_{k|k} = \widehat{x}_{k|k-1} + K_k(y_k - \widehat{x}_{k|k-1}) = \frac{\frac{\widehat{x}_{k|k-1}}{\widehat{P}_{k|k-1}} + \frac{y_k}{R_k}}{\frac{1}{\widehat{P}_{k|k-1}} + \frac{1}{R_k}} = \frac{\widehat{x}_{k|k-1} R_k + y_k \widehat{P}_{k|k-1}}{\widehat{P}_{k|k-1} + R_k}$$



2D illustration: $\mathbf{x}_k = [p_k, v_k]^T$ $\quad \mathbf{y}_k = p_k + \eta_k$

- $\mathbf{x}_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{k-1} + w_k \qquad w_k \sim \mathcal{N}\left(0, \begin{bmatrix} 1^2 & 0 \\ 0 & 0.001^2 \end{bmatrix}\right)$
- $y_k = p_k + \eta_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_k + \eta_k \qquad \eta_k \sim \mathcal{N}(0, 1^2)$

Initialization: $\widehat{\mathbf{x}}_{0|0} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \quad \widehat{\mathbf{P}}_{0|0} = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \quad \left(\mathbf{x}_{0|0} = \begin{bmatrix} 20 \\ 12 \end{bmatrix}\right)$

Prediction step: $(\Delta t = 1 \text{ s})$:

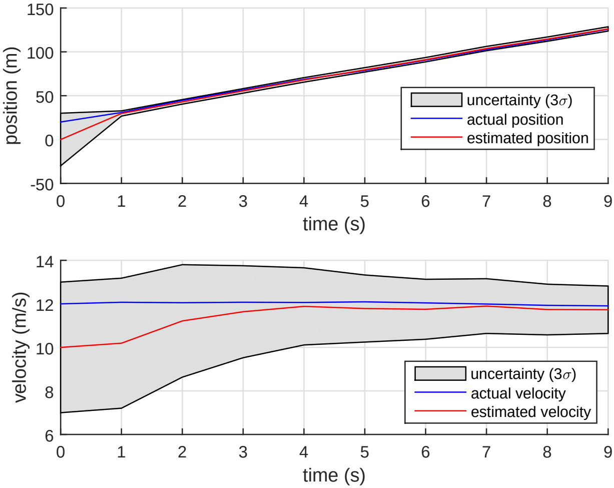
$$\widehat{\mathbf{x}}_{1|0} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad \widehat{\mathbf{P}}_{1|0} = \begin{bmatrix} 102 & 1 \\ 1 & 1.001 \end{bmatrix} \qquad \left(\mathbf{x}_1 = \begin{bmatrix} 30.83 \\ 12.07 \end{bmatrix}\right)$$

Correction step:

$$\mathbf{K}_1 = \begin{bmatrix} 0.9903 \\ 0.0097 \end{bmatrix} \qquad y_k - \mathbf{H}_k \mathbf{x}_{k|k-1} = 29.91 - 10 = 19.91$$

$$\widehat{\mathbf{x}}_{1|1} = \begin{bmatrix} 29.7 \\ 10.2 \end{bmatrix} \qquad \widehat{\mathbf{P}}_{1|1} = \begin{bmatrix} 0.9903 & 0.0097 \\ 0.0097 & 1.0903 \end{bmatrix}$$

2D illustration: $\mathbf{x}_k = [p_k, v_k]^T$ $\quad$ $\mathbf{y}_k = p_k + \eta_k$



Extended Kalman Filter for nonlinear models

Prediction step:

$$\begin{aligned} \widehat{\mathbf{x}}_{k|k-1} &= f(\widehat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_k) \\ \widehat{\mathbf{P}}_{k|k-1} &= \mathbf{F}_{k-1}\widehat{\mathbf{P}}_{k-1|k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_k \\ \mathbf{F}_{k-1} &= \frac{\partial f}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\widehat{\mathbf{x}}_{k-1}} \in \mathbb{R}^{d\times d} \end{aligned} \tag{11}$$

Correction step (where $\mathbf{y}_k - h(\widehat{\mathbf{x}}_{k|k-1})$ is the *innovation* term and $\mathbf{K}_k$ is the Kalman gain):

$$\begin{aligned} \widehat{\mathbf{x}}_{k|k} &= \widehat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k\left(\mathbf{y}_k - h(\widehat{\mathbf{x}}_{k|k-1})\right) \\ \widehat{\mathbf{P}}_{k|k} &= (\mathbf{I}_d - \mathbf{K}_k\mathbf{H}_k)\,\widehat{\mathbf{P}}_{k|k-1} \\ \mathbf{K}_k &= \widehat{\mathbf{P}}_{k|k-1}\mathbf{H}_k^T\left(\mathbf{R}_k + \mathbf{H}_k\widehat{\mathbf{P}}_{k|k-1}\mathbf{H}_k^T\right)^{-1} \\ \mathbf{H}_k &= \frac{\partial h}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\widehat{\mathbf{x}}_{k|k-1}} \in \mathbb{R}^{d_m\times d} \end{aligned} \tag{12}$$

Limitations and potential solutions

Extended Kalman Filter limitations:

- Local linearization of f and h: instability for severe non-linearities or large initial state uncertainty
- Gaussian approximations: no longer valid for non-linear models: introduce some bias

Advanced non-linear Gaussian filters:

- Unscented Kalman Filter (UKF): avoids linarization (Unscented Transform).
- Ensemble Kalman Filter (EnKF): the covariance matrix is replaced by the sample covariance.

Advanced non-linear non-Gaussian filter:

- Particle Filter (PF): Monte-Carlo approximation of the Optimal Filter

Inertial Measurement Unit

Relativity principle:
Imagine a box without access to the outside world.

What can you measure inside the box?

Inertial Measurement Unit

Relativity principle:
Imagine a box without access to the outside world.

What can you measure inside the box?

- Specific accelerations (i.e., non gravitational)
- Angular velocities

What **cannot** you measure?

Inertial Measurement Unit

Relativity principle:
Imagine a box without access to the outside world.

What can you measure inside the box?

- Specific accelerations (i.e., non gravitational)
- Angular velocities

What **cannot** you measure?

- Gravity field

Inertial Measurement Unit

Accelerometer measurement:

$$\gamma_m = b + (1 + f)\gamma + w_\gamma \tag{13}$$

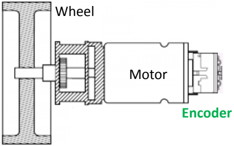
γ	:	actual specific acceleration
γ_m	:	measured specific acceleration
b	:	bias (constant)
f	:	scale factor
w_γ	:	sensor noise

Gyrometer measurement:

$$\omega_m = d + (1 + \tau)\omega + w_\omega \tag{14}$$

ω	:	actual absolute angular velocity
ω_m	:	measured angular velocity
d	:	angular drift (constant)
τ	:	scale factor
w_ω	:	sensor noise

Wheel odometry



Model 1:

$$\Delta p_m = (1+f)\Delta p + w \tag{15}$$

Δp	:	actual wheel position variation since last measureme
Δp_m	:	measured wheel position variation
f	:	scale factor
$w \sim \mathcal{N}(0, \sigma^2)$	:	sensor noise

Model 2:

$$\Delta p_m = \Delta p + w_{\Delta p} \tag{16}$$

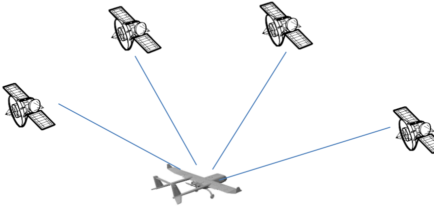
$w_{\Delta p} \sim \mathcal{N}(0, (\Delta p \sigma)^2)$: sensor noise

Global Navigation System by Satellite (GNSS)

Pseudo-measurements for satellite i:

$$\begin{aligned} PR_i &= \|\mathbf{p} - \mathbf{p}_i\| + cB_h + w_{pr} \\ PV_i &= \left\langle \mathbf{v} - \mathbf{v}_i, \frac{\mathbf{p}-\mathbf{p}_i}{\|\mathbf{p}-\mathbf{p}_i\|} \right\rangle + c\dot{B}_h + w_{pv} \end{aligned} \tag{17}$$

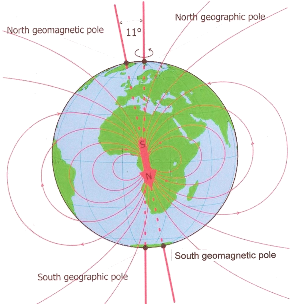
- $\mathbf{p}$, $\mathbf{v}$: receptor's position and velocity
- $\mathbf{p}_i$, $\mathbf{v}_i$: satellite's position and velocity
- c : speed of light
- B_h : receptor's clock bias
- w_{pr}, w_{pv} : measurement noises.



From a minimum of 4 satellites, instantaneous navigation solution:

$$\begin{aligned} \mathbf{p}_m &= \mathbf{p} + \mathbf{w}_p \\ \mathbf{v}_m &= \mathbf{v} + \mathbf{w}_v \end{aligned} \tag{18}$$

Magnetometer

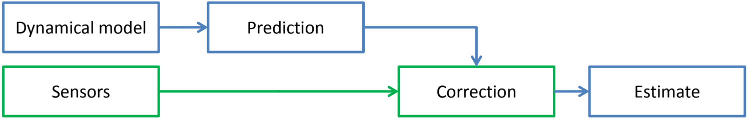


$$\mathbf{b}_m = \mathbf{R}_{earth \to body} \mathbf{b} + \mathbf{w} \tag{19}$$

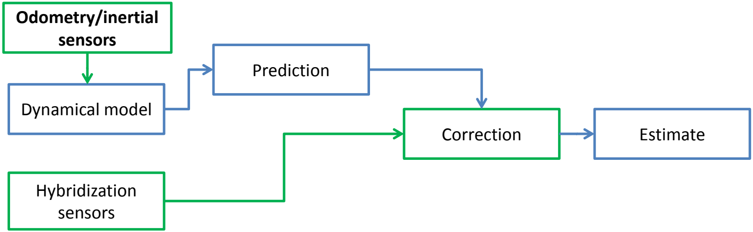
$\mathbf{b}$	:	actual local magnetic field
$\mathbf{b}_m$	:	measured magnetic field
$\mathbf{R}_{earth \to body}$	:	sensor attitude matrix in Earth frame
$\mathbf{w}$	:	sensor noise

Two possible implementations of a Kalman filter

- Implementation with a theoretical dynamical model

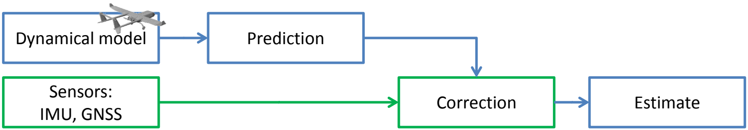


- Navigation by odometry/inertial Hybridization

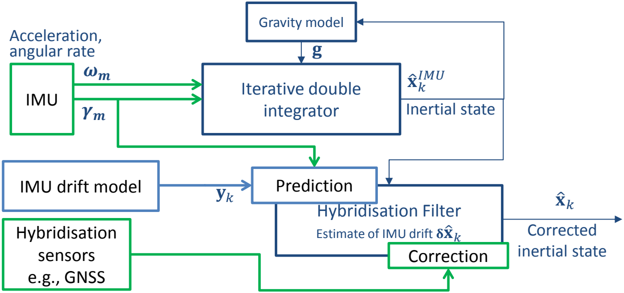


Example: IMU hybridization

- Implementation with a specific aircraft dynamical model



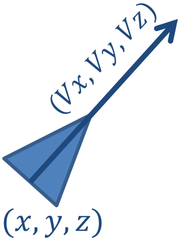
- Navigation by odometry/inertial Hybridization



Linear model: **State**: $\mathbf{x}_k = [x_k, y_k, z_k, Vx_k, Vy_k, Vz_k]^T$

Dynamics:

$$\mathbf{F}_k = \begin{bmatrix} 1 & 0 & 0 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta t & 0 \\ 0 & 0 & 1 & 0 & 0 & \Delta t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



$\mathbf{y} = (x, y, z)$

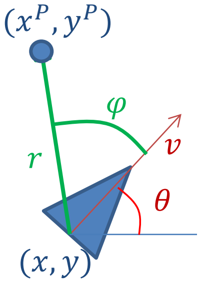
Measurements (e.g., GNSS):

$$\mathbf{H}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Non-linear model: **State**: $\mathbf{x}_k = [x_k, y_k, \theta_k]^T$, **Control**: $\mathbf{u}_k = [v_k, \omega_k]^T$

Dynamics:

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k) = \begin{bmatrix} x_k + v_k \Delta t \cos\theta_k \\ y_k + v_k \Delta t \sin\theta_k \\ \theta_k + \Delta t \omega_k \end{bmatrix}$$



$$\mathbf{F}_k = \begin{bmatrix} 1 & 0 & -v_k \Delta t \sin\theta_k \\ 0 & 1 & v_k \Delta t \cos\theta_k \\ 0 & 0 & 1 \end{bmatrix}$$

Measurements (e.g., radar):

$$\mathbf{y}_k = h(\mathbf{x}_k) = \begin{bmatrix} r_k \\ \varphi_k \end{bmatrix} = \begin{bmatrix} \sqrt{(x_k^P - x_k)^2 + (y_k^P - y_k)^2} \\ \mathrm{atan}\frac{y_k^P - y_k}{x_k^P - x_k} - \theta_k \end{bmatrix}$$

$$\mathbf{H}_k = \begin{bmatrix} -\frac{x_k^P - x_k}{\sqrt{(x_k^P - x_k)^2 + (y_k^P - y_k)^2}} & -\frac{y_k^P - y_k}{\sqrt{(x_k^P - x_k)^2 + (y_k^P - y_k)^2}} & 0 \\ \frac{y_k^P - y_k}{(x_k^P - x_k)^2 + (y_k^P - y_k)^2} & -\frac{x_k^P - x_k}{(x_k^P - x_k)^2 + (y_k^P - y_k)^2} & -1 \end{bmatrix}$$

TP

Partie 1: implémenter un filtre de Kalman pour traiter l'exemple linéaire (slide 25)

1. Prendre en main la structure du code et repérer les différents paramètres.
2. Compléter le code avec les équations du filtre KF (prédiction, correction, slide 12), le modèle dynamique, le modèle de mesure et les matrices jacobiennes et commenter les résultats;
3. (Uniquement pour cette question) Simuler un trou de mesures entre $t = 3$ s et $t = 7$ s en utilisant la variable *is_measurementValid*, qu'observe-t-on ? Expliquer.
4. Faire varier le bruit de dynamique du filtre (matrice **Qf**), qu'observe-t-on ? Expliquer.
5. Faire varier le bruit de mesure du filtre (matrice **Rf**), qu'observe-t-on ? Expliquer.

Partie 2: implémenter un filtre de Kalman Etendu pour traiter l'exemple non-linéaire (slide 26)

1. Compléter le code avec les équations du filtre EKF (slide 16), le modèle dynamique, le modèle de mesure et les matrices jacobiennes et commenter les résultats;
2. Modifier la fréquence des mesures (passer à 1 Hz) en utilisant la variable *dt_mesure*, qu'observe-t-on ? Expliquer.
3. Faire varier le bruit de dynamique du filtre (matrice **Qf**), qu'observe-t-on ? Expliquer.
4. Faire varier le bruit de mesure du filtre (matrice **Rfi**), qu'observe-t-on ? Expliquer.
5. Définir d'autres amers (structure *amers*) et étudier les performances du filtre en fonction du nombre d'amers. Régler le filtre et/ou les positions des amers pour obtenir les meilleures performances possibles et expliquer les résultats.

/!\ Les rapports de TP sont individuels.

Bibliography

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