Efficient resolution of logical models

ENSTA-IA303

Alexandre Chapoutot and Sergio Mover

ENSTA Paris

2020-2021

About the "second part" of IA303

Contact information:

- via email: sergio.mover <at> lix.polytechnique.fr
- "on-demand" office hours: Friday from 2pm to 3pm
- Get in touch also if you are interested in research in:
  - Formal methods (formal verification, decision procedures, . . .)
  - Artificial intelligence (mainly related to the use logics, symbolic AI, planning, . . .)
  - Cyber-Physical systems (e.g., hybrid systems, . . .)

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In the next 4 classes:

- Introduction to Satisfiability Modulo Theories (SMT) (week 4, today)
- How some theory solvers works: EUF (week 5) and LRA/LIA (week 6)
- Some applications of SMT in formal verification (week 7)

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Main take-away points:

- SMT is a powerful and mature tool, with wide applications
- How to formalize a problem in SMT and use an SMT solver
- How an SMT solver works and why is it efficient (despite the problem complexity)

Lecture 4: Satisfiability Modulo Theories and DPLL($\mathcal{T}$)

Main goals for today

In class[1]:

- Why and where do we use Satisfiability Modulo Theories (SMT)?
- What is SMT precisely?
- How can we efficiently decide the SMT problem?

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[1] Main references:

- The Calculus of Computation [Bradley and Manna, 2007], Chapter 2 (First-Order Logic) and Chapter 3 (First-Order Theories)
- Satisfiability Modulo Theories [Barrett et al., 2009]
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In class[1]:

- Why and where do we use Satisfiability Modulo Theories (SMT)?
- What is SMT precisely?
- How can we efficiently decide the SMT problem?

In the tutorial:

- How do you use an SMT solver?
- How do you formalize a problem (i.e., encode) as an SMT problem?
- How can you write a program that uses an SMT solver?
- How can you automate the encoding generation for a class of problems?

---

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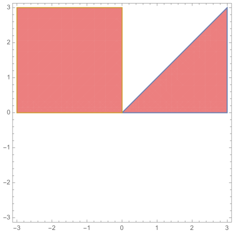
SMT in a nutshell - informal intuition

- Extend of the propositional satisfiability problem to represent specific domains. For example, real numbers:

$$(x - y \geq 0 \vee x < 0) \wedge y > 0$$



- There are more theories: bitvectors, rational and integers linear arithmetic , uninterpreted functions, arrays, strings, separation logics, floating point arithmetic, ...
- SMT is an expressive language.

What problems can we solve with SMT?

We use SMT to express different have constraints satisfaction problems for several applications:

- Formal Verification for software, hardware, cyber-physical systems, protocols, neural networks (e.g., [Henzinger et al., 2004, McMillan and Padon, 2020, Dutertre et al., 2018, Cimatti et al., 2016, Cimatti et al., 2015])

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- Planning (e.g., [Wolfman and Weld, 1999, Cashmore et al., 2020, Cimatti et al., 2018])
- ...

Active area of research!
With applications in formal methods, programming languages, software engineering, AI, ...

Software verification

```c
float weighted_sum(unsigned int x,
                   unsigned int y) {
  unsigned int i;
  float sum;

  if (y > x) {    // swap x and y
    x = x^y; y = y^x; x = x^y;
  }

  sum = 0;
  for (i = 0; i <= (x-y)-1; ++i) {
    float tmp;
    tmp = ((i + 1)) / (x - y);
    sum = sum + tmp;
  }

  return sum;
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Compute $\sum_{i=1}^{|x-y|} \left(\frac{i}{|x-y|}\right)$

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Software verification uses SMT to faithfully model the program semantic

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First-Order Logic

Extends Propositional Logic with *predicates*,*functions*, and *quantifiers* to reason about infinite domains.

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Syntax

A *term* t is either:

- A *constant* $a, b, c, \ldots, 0, 1, \ldots$ (or a 0-ary function)
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A FOL formula ϕ is either:

- an atom a
- $\neg\psi$, with ψ a FOL formula
- $\psi_1 \wedge \psi_2$, with ψ_1 and ψ_2 FOL formulas.
  Other operators: $\psi_1 \vee \psi_2 := \neg(\neg\psi_1 \wedge \neg\psi_2)$, $\psi_1 \rightarrow \psi_2 := \neg\psi_1 \vee \psi_2$
- $\exists x.\psi$, with x a variable and ψ a FOL formula
- $\forall x.\phi$, with x a variable and ψ a FOL formula

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- $\forall x.\phi$, with x a variable and ψ a FOL formula
  We will "almost always" avoid quantifiers

First-Order Logic - Some examples of syntax

- Terms:
  - $1, a, b$ are constants
  - x, y, z are variables
  - $f(x)$, $g(x, z)$, and $g(f(x), y)$ are functions

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  - $f(x)$, $g(x,z)$, and $g(f(x), y)$ are functions
- Predicates
  - $p(a,b)$, $p(a, f(x))$, $q(x,y)$
- FOL formulas
  - $p(a,b)$
  - $\neg p(a,b)$
  - $(p(a,b) \wedge q(x,y))$
  - $\forall x. x = y \wedge f(x,y)$
  - $\forall x. \exists y. (x = y \rightarrow f(x) = f(y))$

First-Order Logic - Semantic

While Propositional Logic evaluates over true and false, in FOL we have domains and assignments .

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- $\mathcal{D}_{\mathcal{I}}$ is the *domain* of $\mathcal{I}$: it's a non-empty set of elements (e.g., values, objects, ...)
- $\alpha_{\mathcal{I}}$ is an *assignment* that maps constants, functions, and predicates to elements, functions, and predicates of $\mathcal{D}_{\mathcal{I}}$:

$$\alpha_{\mathcal{I}}(x) := x_{\mathcal{I}} \qquad x_{\mathcal{I}} \in \mathcal{D}_{\mathcal{I}}$$

$$\alpha_{\mathcal{I}}(f) := f_{\mathcal{I}} \qquad f_{\mathcal{I}} : \mathcal{D}_{\mathcal{I}}^{n} \to \mathcal{D}_{\mathcal{I}}$$

Note that constants are 0-ary functions!$\alpha_{\mathcal{I}}(p) \quad := p_{\mathcal{I}} \qquad p_{\mathcal{I}} : \mathcal{D}_{\mathcal{I}}^{n} \to \{tru$

First-Order Logic - an example of interpretation

- Consider the FOL formula [2]: $x + y > z \to y > z - x$
- A possible intepretation over the interger numbers $\mathbb{Z}$
  - $\mathcal{D}_{\mathcal{I}} = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  - the function $+$, $-$ are assigned to the the plus and minus function in $\mathbb{Z}$ (i.e., $+_{\mathbb{Z}}, -_{\mathbb{Z}}$)
  - the predicates $>$ is assigned to $>_{\mathbb{Z}}$
  - x, y, z are 0-ary functions of integer type
  - $\alpha_{\mathcal{I}} := \{x \mapsto 13, y \mapsto 2, z \mapsto 4, > \mapsto >_{\mathbb{Z}}, + \mapsto +_{\mathbb{Z}}, - \mapsto -_{\mathbb{Z}}, \ldots\}$

[2] Example 2.7 from [Bradley and Manna, 2007]

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When does an interpretation $\mathcal{I} = (\mathcal{D}_\mathcal{I}, \alpha_\mathcal{I})$ satisfies a FOL formula ϕ, $\mathcal{I} \models \phi$?

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- $\mathcal{I} \models \top$ and $\mathcal{I} \not\models \bot$
- we evaluate a term t with $\alpha_{\mathcal{I}}(t)$, recursively:
  - for a variable x $\alpha_{\mathcal{I}}(x)$, $\alpha_{\mathcal{I}}(a)$
  - $\alpha_{\mathcal{I}}(f(t_1, \ldots, t_n)) = \alpha_{\mathcal{I}}(f)(\alpha_{\mathcal{I}}(t_1), \ldots, \alpha_{\mathcal{I}}(t_n))$
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- $\mathcal{I} \models \phi_1 \wedge \phi_2$ iff $\mathcal{I} \models \phi_1$ and $\mathcal{I} \models \phi_2$
- $\mathcal{I} \models \exists x.\psi$, if there is some $a \in \mathcal{D}_\mathcal{I}$, $(\mathcal{D}_\mathcal{I}, \alpha_\mathcal{I}[x \mapsto a]) \models \psi$
- $\mathcal{I} \models \forall x.\phi$, if for all $a \in \mathcal{D}_\mathcal{I}$, $(\mathcal{D}_\mathcal{I}, \alpha_\mathcal{I}[x \mapsto a]) \models \psi$.

First-Order Logic - Semantic (Example)

- formula [3] $\phi := x + y > z \to y > z - x$
- interpretation $\mathcal{I} = (\mathbb{Z}, \alpha_{\mathcal{I}})$:

$$\alpha_{\mathcal{I}} := \{x \mapsto 13, y \mapsto 2, z \mapsto 4, > \mapsto >_{\mathbb{Z}}, + \mapsto +_{\mathbb{Z}}, - \mapsto -_{\mathbb{Z}}, \ldots\}$$

- the truth value of ϕ under $\mathcal{I}$ is:
  1. $\mathcal{I} \models x + y > 0$ — since $\mathcal{I}[x + y > 0] = 13_{\mathbb{Z}} +_{\mathbb{Z}} 42_{\mathbb{Z}} >_{\mathbb{Z}} 1_{\mathbb{Z}}$
  2. $\mathcal{I} \models y > z - x$ — since $\mathcal{I}[y > z - x] = 42_{\mathbb{Z}} >_{\mathbb{Z}} 1_{\mathbb{Z}} -_{\mathbb{Z}} 13_{\mathbb{Z}}$
  3. $\mathcal{I} \models \phi$ — by 1,2, and the semantic of $\to$

[3] Example 2.8 from [Bradley and Manna, 2007]

First-Order Logic - Satisfiability and Validity

A FOL formula ϕ is:

- *satisfiable* iff there exists and interpretation $\mathcal{I}$ such that $\mathcal{I} \models \phi$
- *valid* iff for all interpretations $\mathcal{I}$, $\mathcal{I} \models \phi$

Decidability results (see [Bradley and Manna, 2007]):

- validity is semi-decidable: if ϕ is valid, then there exists a procedure that eventually terminates and says yes
- satisfiability is undecidable.

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So, what can we do?

Restricting the domains and interpretations of FOL

In a lot of cases we know the domains and operations appearing in the formulas. For example:

- planning with resources: integer or real numbers
- numerical programs manipulating memory: arrays and integer numbers
- microcode (of CPUs): bounded-length bit vectors
- html web sanitizers: strings
- ...

If we restrict FOL to such domains and operations the satisfiability problem becomes decidable.

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If we restrict FOL to such ~~domains and operations~~ first-order theories the satisfiability problem for quantifier-free formulas becomes decidable.

First-Order Theories - Definition

Theory $\mathcal{T}$

A theory $\mathcal{T}$ is defined with:

- a signature Σ: set of constants, functions, predicates
- a set of axioms $\mathcal{A}$: set of closed FOL formulas (i.e., no free variables) containing constants, functions, predicates from Σ

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Validity and Satisfiability

A Σ-formula ϕ is valid in the theory $\mathcal{T}$ ($\mathcal{T}$-*valid*, written as $\models_{\mathcal{T}} \phi$), if:

- **for all** the interpretations $\mathcal{I}$ such that $\mathcal{I}$ satisfies all the axioms of $\mathcal{T}$ (i.e., $\mathcal{I} \models_{\mathcal{T}} A$, for every axiom $A \in \mathcal{A}$ - this is called a $\mathcal{T}$-*interpretation*)
- $\mathcal{I}$ also satisfy ϕ ($\mathcal{I} \models_{\mathcal{T}} \phi$)

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An example please!

Equalities and Uninterpreted functions

The Theory of Equalities and Uninterpreted functions $\mathcal{T}_E$ is defined as:

- the signature $\Sigma_E := \{=, a, b, c, \ldots, f, g, h, \ldots, p, q, r, \ldots\}$
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- the set of axioms $\mathcal{A}$:
  1. $\forall x. x = x$ [reflexivity]
  2. $\forall x, y. x = y \rightarrow y = x$ [symmetry]
  3. $\forall x, y, z.((x = y \wedge y = z) \rightarrow x = z)$ [transitivity]

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  4. Function and predicate congruence
     - For each $n \in \mathbb{N}$ and n-ary function symbol f:

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Some concrete examples

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Is sat? Is valid?

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Hint: treat $*$ and $+$ as an uninterpreted function.

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Some examples are from [Bradley and Manna, 2007] and [Barrett et al., 2009]

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Is the satisfiability of EUF-formulas decidable?

Some examples are from [Bradley and Manna, 2007] and [Barrett et al., 2009]

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Is the satisfiability of EUF-formulas decidable? More next week!

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Some theories of interest I

- Linear rational and Integer Arithmetic (LRA and LIA) - (week 3)

$$(x + y < 3 \wedge y > 2) \rightarrow x < 1$$

Used to model arithmetic (note that constraints are linear!)

- Difference logic

$$(a - b \leq 3 \wedge c - a \leq 2) \vee b - c \leq 10$$

Used to model arithmetic (note the restrictions, only difference of 2 constants, no strict inequalities)

- Reals (i.e., polynomial inequalities over the reals)

$$a^2 + 3ab + c \leq 3 \vee a - b \leq 2$$

Model geometric problems, problems in engineering, . . .

Some theories of interest II

- Arrays

$$\neg((write(a, i, v_1) \wedge j = i + 1) \rightarrow (read(a, i) < read(a, j, v)))$$

Model unbounded memory in programs

- Bit-Vectors - a bit-vector $x_{[n]}$ is a vector of bits of length n

$$x_{32}[15:0] = y_{[16]}[7:0] :: y_{[16]}[15:8]$$

Model hardware operations and low-level software

- Strings

$$y = \text{“}a'' \cdot x \wedge x = z \cdot \text{“}b'') \rightarrow y = \text{“}a'' \cdot w \cdot \text{“}b''$$

Model string constraints, for example for testing or security

Satisfiability Modulo Theory Problem

SMT Problem

The problem of deciding the satisfiability of quantifier-free formulas expressed in some *decidable* first order theory $\mathcal{T}$

Some remarks:

- Usually quantifier-free formulas, but SMT solver can deal with quantifiers (semi-decidable or focus on decidable subsets, like Effectively Propositional Logic)
- Also consider formulas obtained combining multiple theories, e.g., $\mathcal{T}_1 \cup T_2 \cup \ldots \cup T_n$

How can we decide the SMT problem?

Assumption from here onwards:

- ϕ is a quantifier-free formula
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- We can enumerate all the μ^b Propositional models of $(P_1 \vee P_2) \wedge (P_3 \vee P_4)$

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- For each model μ^b, we can check if the conjunction is *consistent* in the theory $\mathcal{T}$:

$$x > 3 \wedge \neg x + y = 0 \wedge y < 0 \wedge \neg x < 3$$

It's satisfiable: $\mu := \{x \mapsto 4, y \mapsto 0\}$

This is the lazy approach to SMT (e.g., see [Sebastiani, 2007])

Offline lazy approach to SMT solving

procedure $\mathcal{T}$-DPLL-offline(ϕ)
  $\phi^b := TO_{\mathbb{B}}(\phi)$
  while true **do**
    $res, \mu^b := DPLL(\phi^b)$
    if $res = true$ **then**
      $\mu := TO_{\mathcal{T}}(\mu^b)$
      $res := \mathcal{T}consistent(\mu)$
      if $res = true$ **then**
        return `SAT`
      else
        $\phi^b := \phi^b \wedge \neg\mu^b$
    else
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- Boolean reasoning: add a blocking clause μ_b to avoid to "visit" the same Boolean model

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- Only blocks a complete model μ^b "weak" pruning of the search space
  - What is the effect on the backjumping of CDCL?
  - What is the effect on learning clauses?
- Check for $\mathcal{T}$-consistency of full models μ could be unsatisfiable "earlier"
  - Can we detect unsatisfiability due to theory "earlier" in the search?
  - Can we generalize Boolean constraint Propagation to the theory $\mathcal{T}$?

Could we have a tighter integration of CDCL and the $\mathcal{T}$-Solver?

The online approach - DPLL($\mathcal{T}$)

Similar architecture to CDCL, but integrates the theory reasoning:

procedure DPLL-$\mathcal{T}(\phi)$
  May pre-process ϕ (e.g., propagation)
  $\mu := \emptyset$
  $\phi^b := TO_{\mathbb{B}}(\phi); \mu^b = TO_{\mathbb{B}}(\mu)$
  while true **do**
    $\mathcal{T} - Decide(\phi^b, \mu^b)$
    while true **do**
      $res := \mathcal{T} - Deduce(\phi^b)$
      if $res = true$ **then**
        $\mu := TO_{\mathcal{T}}(\mu^b)$
        return `SAT`
      else if $res = conflict$ **then**
        $lvl := \mathcal{T} - Analyze(\phi^b, \mu^b)$
        if $lvl = 0$ **then**
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Similar architecture to CDCL, but integrates the theory reasoning:

- *decision*: choose an unassigned literal l from ϕ^b (similar to DPLL)
- *deduce*: iteratively deduces a literal l^b s.t. $\phi^b \wedge \mu^b \models l^b$
  - In case, add l to μ and check the consistency of μ (in the theory)
  - Optimized with $\mathcal{T}$*-propagation* and *early pruning*.

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while true **do**
$\boxed{\mathcal{T} - Decide(\phi^b, \mu^b)}$
while true **do**
$\boxed{res := \mathcal{T} - Deduce(\phi^b)}$
if $res = true$ **then**
$\mu := TO_{\mathcal{T}}(\mu^b)$
return `SAT`
else if $res = conflict$ **then**
$\boxed{lvl := \mathcal{T} - Analyze(\phi^b, \mu^b)}$
if $lvl = 0$ **then**
return `UNSAT`
else
$\mathcal{T} - Backtrack(lvl, \phi^b, \mu^b)$

Similar architecture to CDCL, but integrates the theory reasoning:

- *decision*: choose an unassigned literal l from ϕ^b (similar to DPLL)
- *deduce*: iteratively deduces a literal l^b s.t. $\phi^b \wedge \mu^b \models l^b$
  - In case, add l to μ and check the consistency of μ (in the theory)
  - Optimized with $\mathcal{T}$*-propagation* and *early pruning*.
- *analyze*: detect the conflict clauses and determines the decision level to backtrack to.
  - Produces also a *theory conflicts*

The online approach - DPLL($\mathcal{T}$)

procedure DPLL-$\mathcal{T}(\phi)$
    May pre-process ϕ (e.g., propagation)
    $\mu := \emptyset$
    $\phi^b := TO_{\mathbb{B}}(\phi); \mu^b = TO_{\mathbb{B}}(\mu)$
    while true **do**
        $\boxed{\mathcal{T} - Decide(\phi^b, \mu^b)}$
        while true **do**
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            if $res = true$ **then**
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- *backtrack*: block the conflict clause and bactracks to the level lvl (similar to DPLL)
    - $\mathcal{T}$-backjumping and $\mathcal{T}$-learning

$\mathcal{T}$-backjumping and $\mathcal{T}$-learning

When we invoke the $\mathcal{T}$-solver on an assignment μ, and μ is not consistent:

- we would like to infer a small subset $\nu \subseteq \mu$ such that ν is not consistent (i.e., ν is a *conflict set*)

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- we can use $\neg\nu^b$ to guide the conflict analysis of CDCL
  In practice, we can consider $\mathcal{T}$-propagations (see later) as unit-propagation in the implication graph.
- $\neg\nu^b$ can be learned as a conflict clause by the sat solver

Ideally, the $\mathcal{T}$-solver should search for a minimal conflict set $\nu \subseteq \mu$

- in practice, finding a minimal set ν is expensive
- $\mathcal{T}$-solvers compromise performance and size of the conflict set ν

$\mathcal{T}$-Backjumping and $\mathcal{T}$-Learning

Examples [4] over the theory of Linear Integer Arithmetic:

$\neg(2x_2 - x3 > 2) \vee A_1$	$\neg B_1 \vee A_1$
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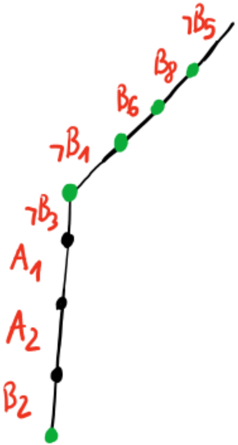

[4] Example 5.2 [Sebastiani, 2007]

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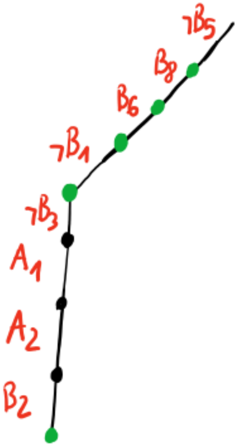

4 Example 5.2 [Sebastiani, 2007]

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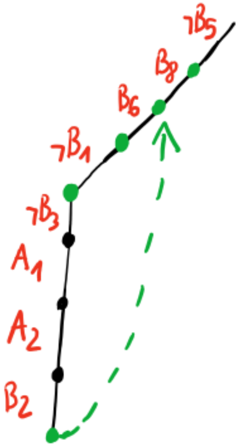

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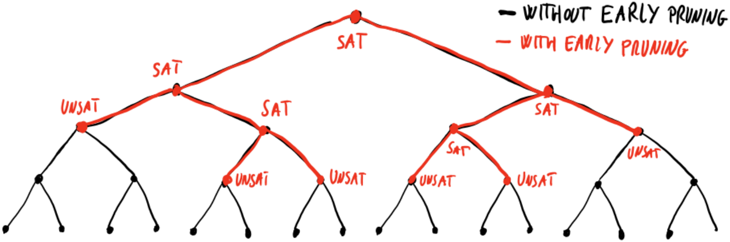
- The solver backtracks removing all literals up to $\{\neg B_5, B_8\}$.

[4] Example 5.2 [Sebastiani, 2007]

Early Pruning

- Check the $\mathcal{T}$-consistency of each partial assignment found by CDCL
- Backtrack immediately if the assignment is not consitent
- Main advantage: prunes the search space



Technical considerations:

- Requires an *incremental* and *backtrackable* $\mathcal{T}$-solver
- Checking consistency for every decision is not cheap!
  Heuristics, use incomplete but cheap consistency checks (e.g., simplex on integer arithmetic)

$\mathcal{T}$-Propagation

- Used in $\mathcal{T} - \textit{decide}$ to deduce the value of unassigned literals:
  - When the current (partial) assignment μ is satisfiable
  - The $\mathcal{T}$-solver can return a set ν of unassigned literals such that $\mu \models_{\mathcal{T}} \nu$
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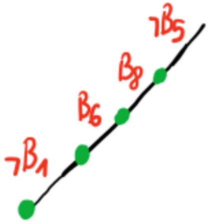

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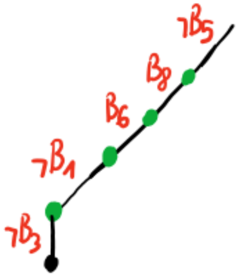

- $\mu^b := \neg B_5, B_8, B_6, \neg B_1$
- $\neg(3x_1 - x_3 \leq 6) \wedge x_3 = 3x_5 + 4 \wedge x_2 - x_4 \leq 6 \wedge (2x_2 - x3 > 2) \models_{\mathcal{T}} \neg(3x_1 - 2x_2 \leq 3)$

Other Approaches to the SMT problem

- The eager approach to SMT: convert the problem to a SAT problem
  Used to decide formulas over the bit-vector theory.

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- The eager approach to SMT: convert the problem to a SAT problem
  Used to decide formulas over the bit-vector theory.
- Abstract DPLL: abstract formulation of DPLL as a transition system
  Allow to reason about the properties of different variants of the algorith (e.g., correctness, completeness, termination)
- Model-Constructing Satisfiability:
  - Assignments (e.g., decisions) to theory variables, not just Propositional
    i.e., no Boolean abstraction anymore, and no enumeration of the models of the Boolean Abstraction.
  - Decisions and explanations can be done for new atoms (obtained from unsatisfiable proofs)
  - Several implementation, efficient for Non-Linear Real Arithmetic

To sum up

What did we see today:

- SMT is a fundamental tool in several area (e.g., verification, program analysis, planning, . . .)
- Satisfiability of (full) First Order logic is undecidable - so what can we do?
- Theories allow us to have decision procedures - motivation to look at the SMT problem
- The lazy approach to SMT: best of both words (CDCL SAT solver) and efficient theory solvers

Next week: how to decide consistency for the theory of Equalities and Uninterpreted Functions

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