Construction of isolating blocks for dynamical systems

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Dynamical system

We consider the autonomous nonlinear system:

$$\dot{x} = f(x),$$

with states $x \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$.
Assuming f Lipschitz continuous, for any initial condition x_0 there exists a solution curve $\varphi(t, x_0)$, with $\varphi(0, x_0) = x_0$ and $\dot{\varphi}(t, x_0) = f(\varphi(t, x_0))\ \forall t$.

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Goal

Try to describe the dynamic of the system without explicit solving.

E.g.: invariants computations, Lyapunov functions, isolating blocks, . . .

Definitions (1)

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Given a compact subset $\mathcal{B}$ of the state space, we denote by $\mathrm{Inv}(\mathcal{B}, \varphi)$ the (possibly empty) invariant contained in $\mathcal{B}$, i.e.

$$\mathrm{Inv}(\mathcal{B}, \varphi) := \{x \in \mathcal{B} : \varphi(\mathbb{R}, x) \subset B\}.$$

It is an isolated invariant if $\mathrm{Inv}(\mathcal{B}, \varphi) \subseteq \mathrm{int}\mathcal{B}$ (robustness to small perturbations of the system).

Definitions (2)

Isolating blocks

$\mathcal{B}$ is an isolating block if:

1. $\mathcal{B}^{-} = \{x \in \mathcal{B} : \varphi([0, T], x) \not\subset \mathcal{B}, \quad \forall T > 0\}$ is closed
2. $\mathrm{Inv}_T(\mathcal{B}, \varphi) = \{x \in \mathcal{B} : \varphi([-T, T], x) \subset B\} \subset \mathrm{int}\mathcal{B}, \forall T > 0$

Every isolated invariant is associated to an isolating block.

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Wazewski property

Given $\mathcal{B}$ an isolating block, if $\mathcal{B}^-$ is not a deformation retract of $\mathcal{B}$ then $\mathrm{int}\mathcal{B}$ contains a non-empty isolated invariant.

We can have more information about the isolated invariant given the topology of $\mathcal{B}^-$ (see e.g. Conley index as used in Stephens and Wanner, 2014).

Questions

Main concerns are:

- how to "define" a candidate isolating block;
- how to compute a candidate isolating block;
- how to prove isolating block properties.

Every steps can be connected to the other.

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In this presentation

Generation of templated isolating blocks

Checking whether a template set is an isolating block

Given $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ some functions, we consider templates of the form:

$$\mathcal{B} = \{x : g_j(x) \geqslant 0, \ \forall j = 1, \ldots, m\}$$

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Sufficient condition

$\mathcal{B}$ (compact) is an isolating block if for each $i = 1, \dots, m$:

$$g_i = 0 \wedge \left(\bigwedge_{j \neq i} g_j \geqslant 0 \right) \wedge \left(\bigwedge_{j=1}^{k} \mathcal{L}_f^{(j)}(g_i) = 0 \right) \implies \mathcal{L}_f^{(k+1)}(g_i) \geqslant 0$$

for all $k = 1, \dots, N_i - 2$, where N_i is the order of the differential radical ideal of g_i, and where $\mathcal{L}_f^{(j)}(g_i)$ is the j-th order Lie derivative of g_i w.r.t. f.

Practical test

Check up to a given order k:

$$g_i = 0 \wedge \left(\bigwedge_{j \neq i} g_j \geqslant 0 \right) \wedge \left(\bigwedge_{j=1}^{k} \mathcal{L}_f^{(j)}(g_i) = 0 \right) \implies \mathcal{L}_f^{(k+1)}(g_i) > 0$$

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For examples:

- Order 1 condition for isolating blocks:

$$g_i = 0 \wedge g_j \geqslant 0 \; (j \neq i) \implies \mathcal{L}_f(g_i) \neq 0$$

- Order 2 condition for isolating blocks:

$$g_i = 0 \wedge g_j \geqslant 0 \; (j \neq i) \wedge \mathcal{L}_f(g_i) = 0 \implies \mathcal{L}_f^{(2)}(g_i) > 0$$

How it is done (1)

Fribourg et al., 2015

Given a polynomial system and polynomial template $\mathcal{B}$: relaxations by Sum of Squares using Putinar positivstellensatz.
Find polynomials α_j ($j = 0, 1, \ldots k$) and Sum Of Square polynomials β_l ($l = 0, 1, \ldots, m, l \neq i$) such that:

$$\mathcal{L}_f^{k+1}(g_i) = \beta_0 + \sum_{l \neq i} \beta_l g_l + \alpha_0 g_i + \sum_{j=1}^{k} \alpha_j \mathcal{L}_f^{(j)}(g_i).$$

- Requires to solve Linear Matrix Inequality (LMI) problem via a Semi-definite Program (SDP).
- Fribourg et al. 2016: adaptation to differential inclusion.

How it is done (2)

Stephens and Wanner, 2014

Considering a general compact set $\mathcal{B}$ and its boundary $\mathcal{M} = \partial\mathcal{B}$ is a smooth orientable manifold. Conditions of isolating block tested via an inclusion test between the nodal domains of two functions (defined w.r.t first and second lie derivative on $\mathcal{M}$).
Use of interval arithmetic for a rigorous test.

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Strongly related to checking the previous condition at order 2 via interval arithmetic.

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Problem

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- Need to consider parametric template g_i;
- Find parameters such that $\mathcal{B}$ is compact and an isolating block;
- + other conditions (containment / exclusion of equilibrium points, criterion to optimize, etc).

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- Find parameters such that $\mathcal{B}$ is compact and an isolating block;
- $+$ other conditions (containment / exclusion of equilibrium points, criterion to optimize, etc).

Case study:

- $\mathcal{B}(p) = \{x : g(p, x) \geqslant 0\}$ with g being a quadratic function of x (ellipsoidal template) and p some parameters (center, radius, orientation, etc ...).
- Only consider the Order 2 condition, i.e. find p s.t.:

$$g_i(p, x) = 0 \wedge \mathcal{L}_f(g_i)(p, x) = 0 \implies \mathcal{L}_f^{(2)}(g_i)(p, x) > 0,\ \forall x \in \mathcal{B}.$$

How to compute (1)

Using Putinar and SOS relaxation: Find polynomials α_0,α_1 and SOS polynomial β_0 such that $\forall x$:

$$\mathcal{L}_f^2(g_i)(p,x) = \beta_0(x) + \alpha_0(x)g(p,x) + \alpha_1(x)\mathcal{L}_f(g_i)(p,x).$$

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We do not consider this approach right now

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Interval Branch & Prune framework with quantified constraint contractor.

Branch & Prune framework

Constraint equivalent to:

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Ineffective contractors on $g_i(p,x) \neq 0$ or $\mathcal{L}_f(g_i)(p,x) \neq 0$. Since a union of these constraints: ineffective contractor on the QCSP.

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However, the negation of the QCSP is:

$$g_i(p,x) = 0 \wedge \mathcal{L}_f(g_i)(p,x) = 0 \wedge \mathcal{L}_f^{(2)}(g_i)(p,x) \leqslant 0,\ \exists x \in \mathcal{B},$$

which can be directly treated by the method from Ishii et al.,2012.

Ishii et al. 2012: B & P projection method

Considers NCSP problems of the form:

$$c(p,x) \equiv h(p,x) = 0 \wedge g(p,x) \leqslant 0,$$

with $p \in P$ (P a k-dimensional box), $x \in X$ (X a n-dimensional box), $h : \mathbb{R}^{k+n} \rightarrow q$ (with $q \leqslant n$) and $g : \mathbb{R}^{k+n} \rightarrow \mathbb{R}^l$.

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Traditional B & P builds a paving in the (p,x) space of the solutions, but here only interested in the solutions projected on p, i.e.

$$\Sigma_p = \{p \in P : \exists x \in X, c(p,x)\}.$$

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- decompose the (p,x) space but avoid decomposing more in x when existentially QCSP is satisfied. To do so, use of neighboring information that prunes the projected space and specific bisection strategy;
- use of parametric interval Newton operator to solve the equations.

Experimental results

Implementation of the method from Ishii et al. 2012 in Ibex, with default tune.

- Hahn example (2 dimensions, template with up to 3 parameters, $\epsilon = 0.01$)

$$\begin{cases} \dot{x} = -x + 2x^2 y \\ \dot{y} = -y \end{cases}$$

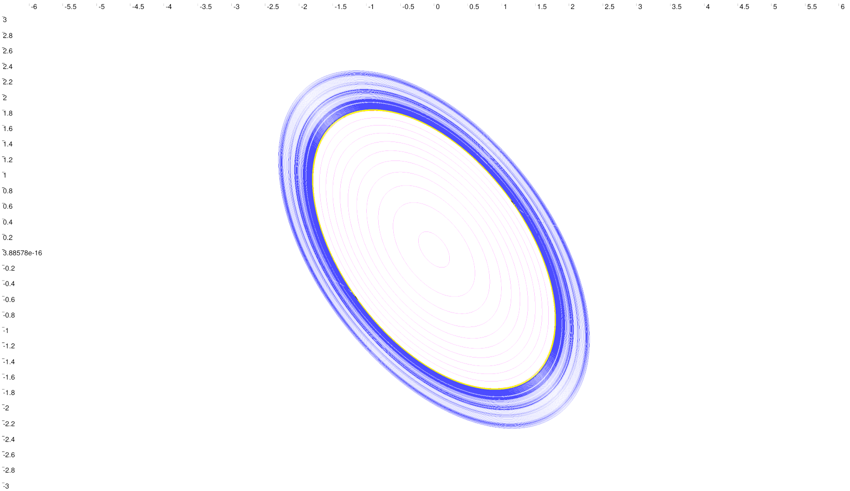
- Stephens and Wanner example (3 dimensions, 1 parameter template, $\epsilon = 0.025$)

$$\begin{cases} \dot{x} = 2x(z - y) \\ \dot{y} = 1 + z - x^2 \\ \dot{z} = -1 + y + x^2 \end{cases}$$

Hahn example

$g(p,x)$	CPU	# reject
$x^2 + y^2 + xy - p$	0.17 s	40
$p_2 x^2 + y^2 + xy - p_1$	43.7 s	5628
$p_1 x^2 + p_2 y^2 + p_3 xy - 1$	MI	MI

Hahn example



1-Parameter template

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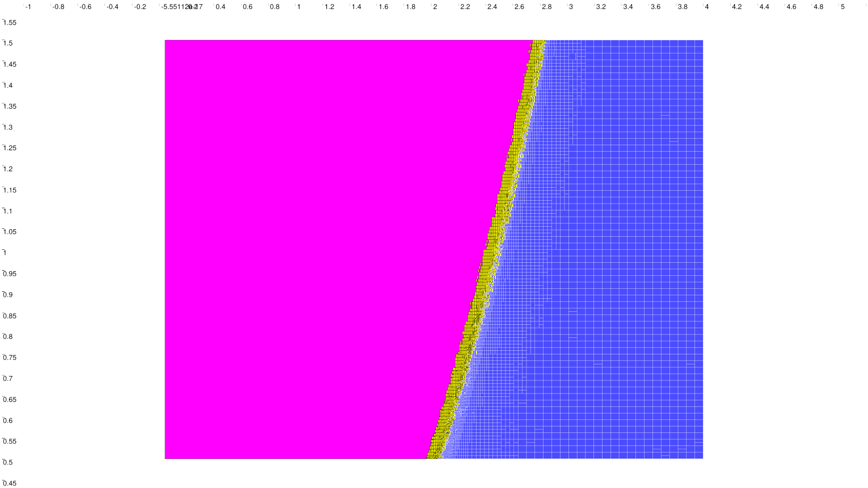
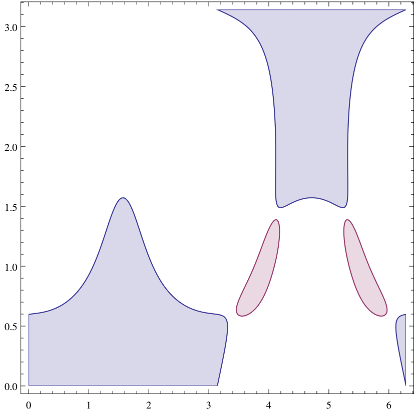


2-Parameter template

Stephens and Wanner example

Template: $g(p,x) = x^2 + (y-1)^2 + (z+1)^2 - p$. 13 rejected parameter boxes, 64.3 s for solving. Valid templates for p within $[0.04, 2.60781]$ and $[3.22656, 4]$. Using spherical coordinates:



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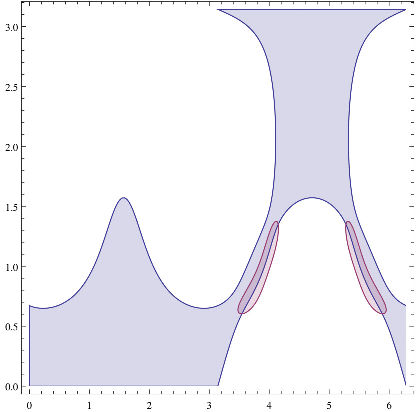

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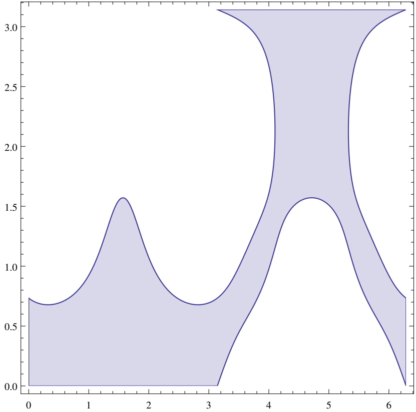

$p = 3.23$

Limitations of the approach

Quite efficient for eliminating parameters not leading to isolating blocks:

- but not directly tackling the original problem (in development . . .)
- not able to contract: how about introducing extra constraints on the template ? (Djaballah et al., 2015)

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Some tuning and improvement may be necessary in order to reach an efficient (memory and time) implementation when considering higher number of parameters and dimensions (critical for more general parametric template).

- computing a specific template (that still needs to be found) ?
- E.g. a template maximizing the volume of $\mathcal{B}$.

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- E.g. a template maximizing the volume of $\mathcal{B}$.

Other interesting templates: "polyfacial" polytopes (edges either exit or entry) using order 1 condition.

Relations to other problems (1)

Effectively handling this problem of isolating blocks would allow to efficiently handling some specials cases.
E.g Barrier functions that need to satisfy:

$$g(p,x) = 0 \implies \mathcal{L}_f(g)(p,x) \leqslant 0,$$

which leads to invariance of the set defined by $g(p,x) \leqslant 0$ (see e.g. the interval method in Djaballah et al., 2015)

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Valid templates on Hahn are actually such kind of functions (flows enter the template on the boundary)

Relations to other problems (2)

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$$\dot{x} = f(x, u)$$

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Recall:

- Problem: Given some initial conditions, is there a control that allows the states to avoid reaching a specific region ?
- Proposed idea:
  - pre-generates template trajectories for simple control laws which can be defined by a functions of the form $g(x_0, x) = 0$ with x_0 the initial condition and x the states;
  - solve the QCSP on the initial conditions consisting of:

$$g(x_0, x) = 0 \implies V(x) \leqslant 0, \forall x$$

where $V(x) > 0$ represents some obstacles.

Discussion