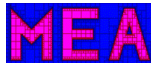


Chain of set inversion problems; Application to reachability analysis

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Backward reach set

Given $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$ and \mathbb{Y} . Set

$$\varphi^k = \underbrace{\mathbf{f} \circ \mathbf{f} \circ \dots \circ \mathbf{f}}_{k \text{ times}},$$

For a given \bar{k} , compute

$$\mathbb{X}_0 = (\varphi^{\bar{k}})^{-1}(\mathbb{Y}).$$

Equivalently, solve the chain

$$\mathbb{X}_0 = \mathbf{f}^{-1}(\mathbb{X}_1), \mathbb{X}_1 = \mathbf{f}^{-1}(\mathbb{X}_2), \dots, \mathbb{X}_{\bar{k}} = \mathbb{Y}$$

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Thick set inversion

Links and chains

Test-cases for the backward reach set

Robust case

With an input vector $\mathbf{u}(k) \in [\mathbf{u}]$:

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)).$$

The initial feasible set \mathbb{X}_0 becomes also uncertain: it depends on $\mathbf{u}(k)$.

Define φ^k as:

$$\begin{aligned}\varphi^1(\mathbf{x}(0), \mathbf{u}(0)) &= \mathbf{f}(\mathbf{x}(0), \mathbf{u}(0)) \\ \varphi^{k+1}(\mathbf{x}(0), \mathbf{u}(0:k)) &= \mathbf{f}(\varphi^k(\mathbf{x}(0), \mathbf{u}(0:k-1)), \mathbf{u}(k))\end{aligned}$$

Define

$$\mathbb{X}_0^C = \{\mathbf{x}_0 | \forall \mathbf{u}(0 : \bar{k} - 1) \in [\mathbf{u}]^{\bar{k}}, \varphi^{\bar{k}}(\mathbf{x}(0), \mathbf{u}(0 : \bar{k} - 1)) \in \mathbb{Y}\}$$

$$\mathbb{X}_0^D = \{\mathbf{x}_0 | \exists \mathbf{u}(0 : \bar{k} - 1) \in [\mathbf{u}]^{\bar{k}}, \varphi^{\bar{k}}(\mathbf{x}(0), \mathbf{u}(0 : \bar{k} - 1)) \in \mathbb{Y}\},$$

we have

$$\mathbb{X}_0^C \subset \mathbb{X}_0 \subset \mathbb{X}_0^D.$$

The sets \mathbb{X}_0^C and \mathbb{X}_0^D are the *minimal* and the *maximal backward reach set*.

Thick set inversion

Set inversion problem

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

If $\mathbf{f}_{\mathbf{p}}$ depends on a parameter vector $\mathbf{p} \in \mathbb{R}^q$.

$$\mathbb{X}^{\subset} \subset \mathbb{X}(\mathbf{p}) \subset \mathbb{X}^{\supset},$$

where

$$\mathbb{X}^{\subset} = \bigcap_{\mathbf{p} \in [\mathbf{p}]} \mathbf{f}_{\mathbf{p}}^{-1}(\mathbb{Y}) = \{\mathbf{x} | \forall \mathbf{p} \in [\mathbf{p}], \mathbf{f}_{\mathbf{p}}(\mathbf{x}) \in \mathbb{Y}\}$$

$$\mathbb{X}^{\supset} = \bigcup_{\mathbf{p} \in [\mathbf{p}]} \mathbf{f}_{\mathbf{p}}^{-1}(\mathbb{Y}) = \{\mathbf{x} | \exists \mathbf{p} \in [\mathbf{p}], \mathbf{f}_{\mathbf{p}}(\mathbf{x}) \in \mathbb{Y}\}.$$

The pair $[[X]] = [X^C, X^D]$ is a *thick set*. It partitions \mathbb{R}^n into three zones: the clear zone X^C , the penumbra $X^D = X^D \setminus X^C$ and the dark zone $\mathbb{R}^n \setminus X^D$.

Notation. The *thick set inversion problem* is denoted by

$$\llbracket X \rrbracket = \mathbf{f}_{[p]}^{-1}(Y).$$

One box in the penumbra. Consider the thick set inversion problem $\llbracket \mathbb{X} \rrbracket = \mathbf{f}_{[\mathbf{p}]}^{-1}(\llbracket \mathbf{y} \rrbracket)$ where

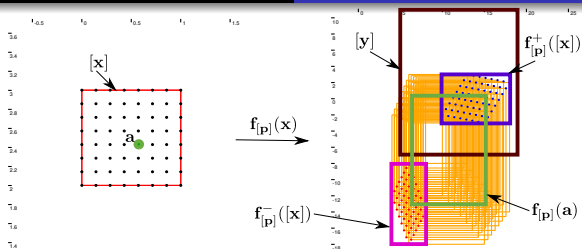
$$\mathbf{f}_{\mathbf{p}}(\mathbf{x}) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Take $[\mathbf{p}] = [2, 3] \times [2, 5] \times [4, 5] \times [-6, -1]$, $[\mathbf{y}] = [5, 19] \times [-7, 11]$
and $\mathbf{x} \in [\mathbf{x}] = [0, 1] \times [2, 3]$.

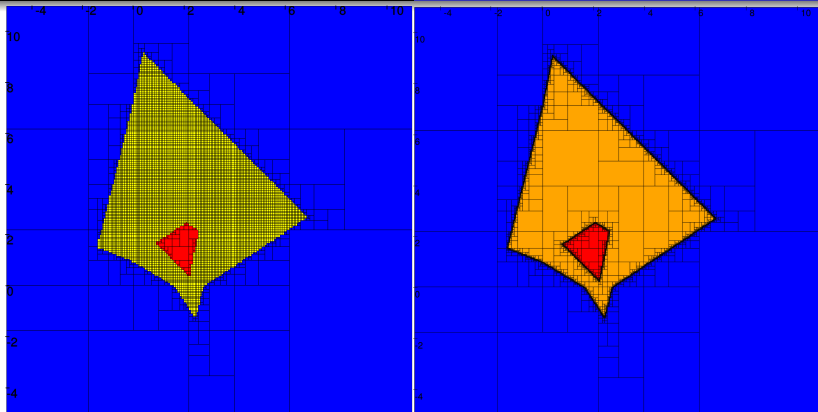
$$\mathbf{f}^-(\mathbf{x}, [\mathbf{p}]) \in \left[\mathbf{f}_{[\mathbf{p}]}^- \right] ([\mathbf{x}]) = \begin{pmatrix} 2 \cdot [0, 1] + 2 \cdot [2, 3] \\ 4 \cdot [0, 1] - 6 \cdot [2, 3] \end{pmatrix} = \begin{pmatrix} [4, 8] \\ [-18, -8] \end{pmatrix}$$

and

$$\mathbf{f}^+(\mathbf{x}, [\mathbf{p}]) \in \left[\mathbf{f}_{[\mathbf{p}]}^+ \right] ([\mathbf{x}]) = \begin{pmatrix} 3 \cdot [0, 1] + 5 \cdot [2, 3] \\ 5 \cdot [0, 1] - 1 \cdot [2, 3] \end{pmatrix} = \begin{pmatrix} [10, 18] \\ [-3, 3] \end{pmatrix}.$$



We conclude that $[x]$ is inside the penumbra



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Links

If, for a box $[p] \subset \mathbb{R}^q$, the set

$$f_{[p]}(x) = f(x, [p]) = \{y \in \mathbb{R}^p \mid \exists p \in [p], y = f_p(x)\}$$

is a box, then f is said to be a *link*.

Links will be composed later to build a *chain*.

Due to their specific box-shaped structure, link can be inverted with respect to x without bisections on the p -space.

Example. The function

$$f(\mathbf{x}, p) = 20e^{-x_1 p} - 8e^{-x_2 p}.$$

is a scalar link function.

Proposition. Consider ℓ scalar link functions, $f_i(\mathbf{x}, \mathbf{p}_i)$, where vectors $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_\ell\}$ are vector components of the vector \mathbf{p} . The function

$$\mathbf{f}(\mathbf{x}, \mathbf{p}) = \begin{pmatrix} f_1(\mathbf{x}, \mathbf{p}_1) \\ \vdots \\ f_\ell(\mathbf{x}, \mathbf{p}_\ell) \end{pmatrix}$$

is a link.

Example. The function

$$f(\mathbf{x}, \mathbf{p}) = \begin{pmatrix} p_1 x_1 x_2 + p_1 p_2 \sin(x_2) \\ p_3 x_1 + p_3 p_4 \cos(x_1) x_2 \end{pmatrix}$$

is a link.

Consequence.

If $f_p(x)$ is a link, for all x , the set $f_{[p]}(x)$ is a box.

If $x \in [x]$, the two bounds of the box $f_{[p]}(x)$ are included inside the boxes $\left[f_{[p]}^- \right] ([x])$ and $\left[f_{[p]}^+ \right] ([x])$.

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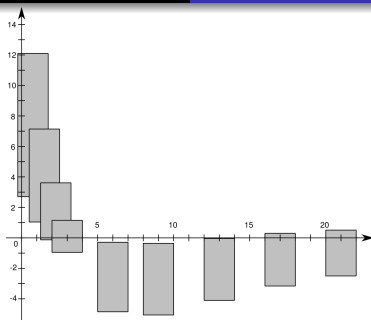
Parameter estimation with uncertain time measurement

Example. Let us estimate the parameters q_1 and q_2 of the model

$$y(\mathbf{q}, t) = 20e^{-q_1 t} - 8e^{-q_2 t}.$$

We assume that 10 measurements y_i have been collected at time t_i .

i	$[t_i]$	$[y_i]$
1	[0.25, 1.25]	[2.7, 12.1]
2	[1, 2]	[1.04, 7.14]
3	[1.75, 2.75]	[-0.13, 3.61]
4	[2.5, 3.5]	[-0.95, 1.15]
5	[5.5, 6.5]	[-4.85, -0.29]
6	[8.5, 9.5]	[-5.06, -0.36]
7	[12.5, 13.5]	[-4.1, -0.04]
8	[16.5, 17.5]	[-3.16, 0.3]
9	[20.5, 21.5]	[-2.5, 0.51]
10	[24.5, 25.5]	[-2, 0.6]



Data $([t_i], [y_i])$

We are interested by the thick set

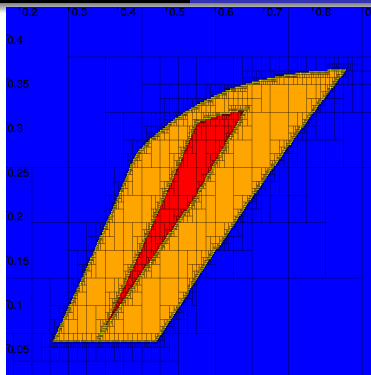
$$[[Q]] = \{\mathbf{q} \in \mathbb{R}^2 \mid \forall i, y(\mathbf{q}, t_i) \in [y_i]\}.$$

If we set

$$\begin{aligned} \mathbf{f}_{[t]}(\mathbf{q}) &= \begin{pmatrix} y(\mathbf{q}, [t_1]) \\ \vdots \\ y(\mathbf{q}, [t_{10}]) \end{pmatrix} \\ [y] &= [y_{10}] \times \cdots \times [y_{10}] \\ [t] &= [t_1] \times \cdots \times [t_{10}] \end{aligned}$$

then, we have

$$[[Q]] = \mathbf{f}_{[t]}^{-1}([y]).$$



Thick set $[[Q]]$

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Chain

Chain. The function $\varphi_{\mathbf{p}}(\mathbf{x})$ is a *chain* if it can be written as

$$\varphi_{\mathbf{p}}(\mathbf{x}) = \mathbf{f}_{\mathbf{p}_\ell}^\ell \circ \dots \circ \mathbf{f}_{\mathbf{p}_2}^2 \circ \mathbf{f}_{\mathbf{p}_1}^1(\mathbf{x})$$

where $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_\ell\}$ are vector components of \mathbf{p} and the $\mathbf{f}_{\mathbf{p}_k}^k$ are links.

Consequence: No dependency effect and no wrapping effect.

Example. The function

$$\varphi_{\mathbf{p}}(\mathbf{x}) = \begin{pmatrix} \sin(p_3 x_1^2) + p_4 \\ \frac{p_5(p_1 e^{x_2} + p_1 p_2 x_1 x_2)}{p_3 x_1^2} \end{pmatrix}$$

is not a link. Now, if we define

$$\mathbf{f}_{\mathbf{p}_1}^1(\mathbf{x}) = \begin{pmatrix} p_1 e^{x_2} + p_1 p_2 x_1 x_2 \\ p_3 x_1^2 \end{pmatrix}$$

$$\mathbf{f}_{\mathbf{p}_2}^2(\mathbf{z}) = \begin{pmatrix} \sin(z_2) + p_4 \\ \frac{p_5 z_1}{z_2} \end{pmatrix}$$

with $\mathbf{p}_1 = (p_1, p_2, p_3)$ and $\mathbf{p}_2 = (p_4, p_5)$ then, we have

$$\varphi_{\mathbf{p}}(\mathbf{x}) = \mathbf{f}_{\mathbf{p}_2}^2 \circ \mathbf{f}_{\mathbf{p}_1}^1(\mathbf{x}).$$

Since both $\mathbf{f}_{\mathbf{p}_2}^2, \mathbf{f}_{\mathbf{p}_1}^1$ are links, $\varphi_{\mathbf{p}}(\mathbf{x})$ is a chain.

Test-cases for the backward reach set

Test-case 1: linear system

Consider the linear system

$$\mathbf{x}(k+1) = \mathbf{A} \cdot \mathbf{x}(k)$$

where

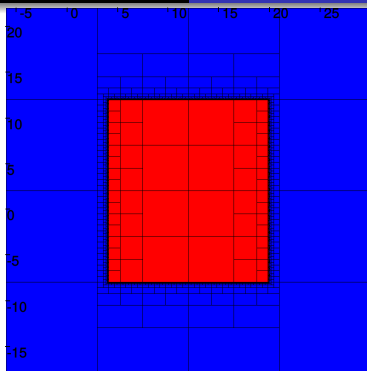
$$\mathbf{A}(k) \in \left(\begin{array}{cc} [2.5, 3] & [2, 3] \\ [4, 4.5] & [-3, -2] \end{array} \right)$$

plays the role of \mathbf{p} .

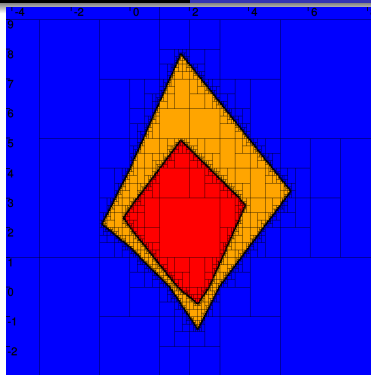
We want to reach the target

$$\mathbb{Y} = [4, 20] \times [-8, 12]$$

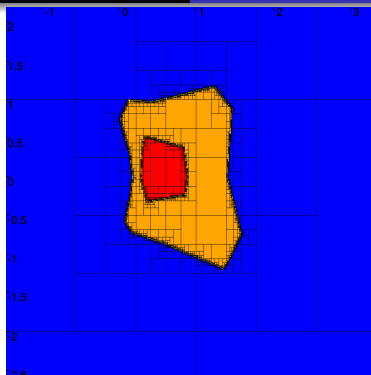
at time $\bar{k} = 3$.



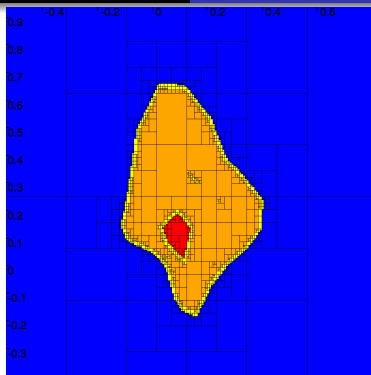
$[[X]](3) = Y$ is thin.



$$\llbracket X \rrbracket(2) = f_{[A]}^{-1}(\llbracket X \rrbracket(3))$$



$$\llbracket X \rrbracket(1) = \mathbf{f}_{[A]}^{-1}(\llbracket X \rrbracket(2))$$



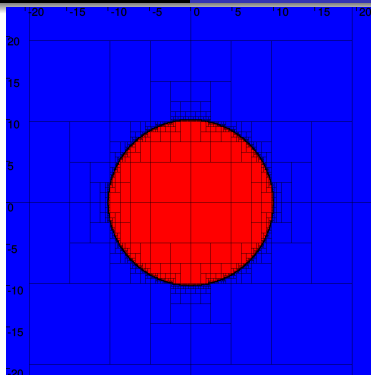
$$\llbracket X \rrbracket(0) = \mathbf{f}_{[A]}^{-1}(\llbracket X \rrbracket(1))$$

Test-case 2: nonlinear system

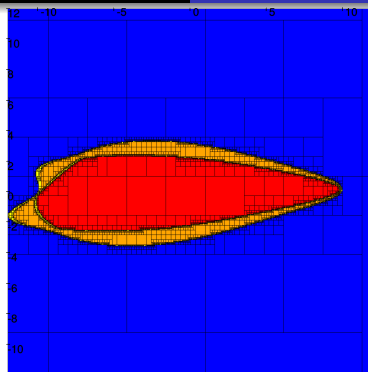
Nonlinear system

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} x_1(k) + x_2^2(k) \cdot u_1(k) \\ \frac{1}{2} \cdot x_1(k) \cdot x_2(k) + u_2(k) \end{pmatrix}$$

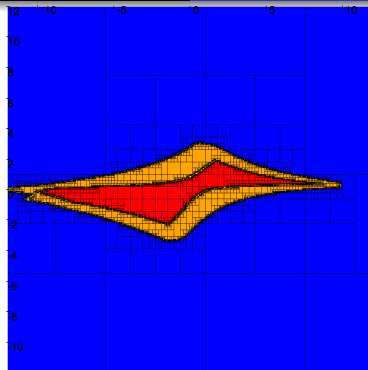
with $u_1 \in [1, 2]$ and $u_2 \in [-2, -1]$. The set \mathbb{Y} is assumed to be a centred disk which has to be reached at time $\bar{k} = 3$.



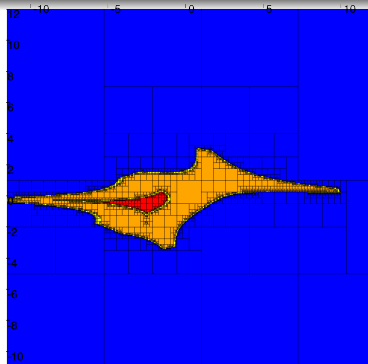
$$[[X]](3) = Y \text{ is thin.}$$



$$\llbracket X \rrbracket(2) = \mathbf{f}_{[A]}^{-1}(\llbracket X \rrbracket(3))$$



$$\llbracket X \rrbracket(1) = \mathbf{f}_{[A]}^{-1}(\llbracket X \rrbracket(2))$$



$$\llbracket X \rrbracket(0) = \mathbf{f}_{[A]}^{-1}(\llbracket X \rrbracket(1))$$