# Reachability Analysis for Separable ODE with respect to Time Varying Bounded Uncertainties

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# Already known result

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**Definition**
We say a system of differential equations is equivalent to a differential inclusion if they have the same solutions.

**Theorem**[1]
When $\mathcal{U}$ is compact and $f(x, \mathcal{U}) = F(x)$ is convex for all $x \in \mathbb{R}^p$, then $\dot{x}(t) = f(x(t), u(t))$ for $u(t) \in \mathcal{U}$ is equivalent to $\dot{x}(t) \in F(x(t))$.

[1] J.W. Nieuwenhuis, "Some Remarks on Set-Valued Dynamical Systems", 1980

# Formulation

# The problem

## Initial Value Problem

$$\begin{cases} \dot{x} = f(t, x(t), u(t)) \\ x(t_0) \in \mathcal{X} \end{cases} \tag{1}$$

## Separable with respect to uncertainties

$$f(t, x, u) = h_0(x, t) + \sum_{i=1}^{m} g_i(u) h_i(x, t) \tag{2}$$

or

$$f(t, x, u) = g(u)^{\mathsf{T}} \cdot h(x, t) \tag{3}$$

with $g_0(u) = 1$.

# Problem hypotheses

## Hypotheses

$$\begin{cases} \dot{x} = f(t, x(t), u(t)) = g(u)^\intercal \cdot h(x, t) \\ x(0) \in \mathcal{X} \\ u(t) \in \mathcal{U} \end{cases} \tag{4}$$

with

- $\mathcal{X} \subset \mathbb{R}^n$ bounded
- $\mathcal{U} \subset \mathbb{R}^m$ bounded
- $t \mapsto u(t)$ measurable

Carathéodory, for all $u : [0, T] \to \mathcal{U}$:

- There exists $m : [0, T] \to \mathbb{R}_+$ such that $\forall (t, x),\ \|f(t, x, u(t))\| \leq m(t)$
- There exists $k : [0, T] \to \mathbb{R}_+$ such that $\forall (t, x_1, x_2),\ \|f(t, x_1, u(t)) - f(t, x_2, u(t))\| \leq k(t)\, \|x_1 - x_2\|$

# Difference with other tools

Other tools use at least Riemann-integrable uncertainties:

Flow*: uncertainties are assumed continuous

CORA: uncertainties are assumed Riemann-integrable

# Enclosure of the solution

# Inclusion using Lebesgue-integration

### Lemma

Let a measurable function $u : [0, T] \to \mathcal{U}$ with a bounded set $\mathcal{U} \subset \mathbb{R}^m$ and a continuous function $g_i : \mathbb{R}^m \to \mathbb{R}$. Let a Lesbegue-integrable function $h_i$ with a decomposition in positive functions: $h_i = h_i^+ - h_i^-$.
Let $\mathcal{C}$ the closure of the convex hull of $\{g(\alpha) \mid \alpha \in \mathcal{U}\}$. Then

$$\int_0^T g_i(u(s))h_i(s)\,ds \in \left\{ \alpha \int_0^T h_i^+(s)\,ds - \beta \int_0^T h_i^-(s)\,ds \;\middle|\; \alpha \in \mathcal{C},\ \beta \in \mathcal{C} \right\} \quad (5)$$

# Some limitations

## Need decompositions

Let $h : [0, 2T] \to \mathbb{R}$ and $u : [0, 2T] \to \{-1, 1\} \subset [-1, 1]$ such that

$$\begin{cases} \forall t \leq T, \quad h(t) = -1 \quad \text{and} \quad u(t) = -1 \\ \forall t > T, \quad h(t) = 1 \quad \text{and} \quad u(t) = 1 \end{cases} \tag{6}$$

We have

$$\int_0^{2T} u(t)h(t)\,dt = 2T \tag{7}$$

$$\forall \alpha \in \mathbb{R},\ \alpha \int_0^{2T} h(t)\,dt = 0 \tag{8}$$

So

$$\int_0^{2T} u(s)h(s)\,ds \notin \left\{ \alpha \int_0^{2T} h(s)\,ds \;\middle|\; \alpha \in [-1, 1] \right\} \tag{9}$$

# Some limitations

## Only valid in 1D

Let $h(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, $\forall t \leq T$, $u(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\forall t > T$, $u(t) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

$$\int_0^{2T} u(t)^\intercal h(t)\, dt = \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{4} \end{pmatrix}^\intercal \cdot \begin{pmatrix} 2T^2 \\ 8T^3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{4} \end{pmatrix}^\intercal \cdot \int_0^{2T} h(t)\, dt \quad (10)$$

but

$$\begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{4} \end{pmatrix} \notin \mathrm{Conv}\left( \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \right) \quad (11)$$

# Our operator

## The dynamics

$$\dot{x}(t) = g(u(t))^{\intercal} \cdot h(x(t), t) \tag{12}$$

with $x(0) \in \mathcal{X}_0 \subset \mathbb{R}^n$, $u(t) \in \mathcal{U} \subset \mathbb{R}^k$ and $g(u(t)) \in \mathbb{R}^m$.

## The operator

$$\begin{aligned}\mathbb{P}_{x_0,(\alpha_i),(\beta_i)}(\varphi) = t \mapsto x_0 + \int_0^t h_0(\varphi(s), s)\, ds \\ + \sum_{i=1}^{m} \alpha_i \int_0^T h_i^+(\varphi(s), s)\, ds \\ - \sum_{i=1}^{m} \beta_i \int_0^T h_i^-(\varphi(s), s)\, ds \end{aligned} \tag{13}$$

with $x_0 \in \mathcal{X}_0$ and for all $i \in [\![1, m]\!]$, $\alpha_i$ and $\beta_i$ belong to the closed convex hull of $\{g_i(v) \mid v \in \mathcal{U}\}$.

# Fixed-point theorem

## Theorem

For all $x_0$, $(\alpha_i)$ and $(\beta_i)$, let $\varphi_{x_0,(\alpha_i),(\beta_i)}$ such that

$$\forall t \in [0, T],\ \mathbb{P}_{x_0,(\alpha_i),(\beta_i)}\left(\varphi_{x_0,(\alpha_i),(\beta_i)}\right)(t) \subset \varphi_{x_0,(\alpha_i),(\beta_i)}(t) \qquad (14)$$

and

$$\varphi = \bigcup_{\substack{x_0 \in \mathcal{X}_0 \\ \forall i \in [\![1,m]\!],\ \alpha_i \in \mathrm{Conv}(\{g_i(v)\ \mid\ v \in \mathcal{U}\}) \\ \forall i \in [\![1,m]\!],\ \beta_i \in \mathrm{Conv}(\{g_i(v)\ \mid\ v \in \mathcal{U}\})}} \varphi_{x_0,(\alpha_i),(\beta_i)} \qquad (15)$$

Then $\varphi$ is an over-approximation of the reachable set.

# Application

# Algorithm to compute an over-approximation

## Algorithm

We use Taylor Models as sets representations.

1. Compute a raw enclosure of the solution
2. Decompose the functions $h_{i \geq 1}(x(t), t)$ as difference of positive functions using the raw enclosure
3. Compute the polynomial expansion up to the expected order
4. Find a valid remainder

# Decomposition

### Affine decomposition

Assume for all $t \in [0, T]$, $h_i(x(t), t) \in [a, b]$ and $a < 0 < b$. We define

$$\begin{cases} h_i^+(x(t), t) = \frac{b}{b-a} h_i(x(t), t) - \frac{ab}{b-a} \\ h_i^-(x(t), t) = \frac{a}{b-a} h_i(x(t), t) - \frac{ab}{b-a} \end{cases} \tag{16}$$

and we have $h_i(x(t), t) = h_i^+(x(t), t) - h_i^-(x(t), t)$.

### Optimality

This decomposition minimizes $\left\| h_i^+ \right\|_1 + \left\| h_i^- \right\|_1$.

# Examples

# Example 1

## Simple dynamics

$$\dot{x}(t) = (0.1 - t)u(t) \tag{17}$$

with $x(0) = 0$ and $\forall t \in [0, 0.2],\ u(t) \in [-1, 1]$

**Case: $u$ constant**

If $u$ is constant ($u(t) = u(0) \in [-1, 1]$), then $x(0.2) = 0$ and $x(t) \in [-0.005, 0.005]$.

**Exact reachable set**

The exact reachable set at time $t = 0.2$ is $x(0.2) \in [-0.01, 0.01]$.

# Example 1: Decomposition

## Decomposition

For all $t \in [0, 0.2]$, $h_1(x(t), t) = (0.1 - t) \in [-0.1, 0.1]$.
We deduce $h_1(x, t) = h_1^+(x, t) - h_1^-(x, t)$ with

$$\begin{cases} h_1^+(x, t) = 0.1 - 0.5t \\ h_1^-(x, t) = 0.5t \end{cases} \tag{18}$$

## Equivalent dynamics

The dynamics becomes

$$\dot{x}(t) = (0.1 - 0.5t)\, u(t) - (0.5t)\, u(t) \tag{19}$$

# Example 1: Over-approximation

We replace the occurrences of $u(t)$ by $\mathrm{TM}(\alpha,\ [0])$ and $\mathrm{TM}(\beta,\ [0])$ with $\alpha \in [-1,1]$ and $\beta \in [-1,1]$.
We start with an expansion to the order 0 in time:
$\varphi_0(x_0,t) = \mathrm{TM}(x_0,\ [0])$.
Then, to expected an higher order expansion, we iterate $\varphi_{n+1} = \mathbb{P}(\varphi_n)$:

$$\begin{aligned}
\varphi_1(x_0,t) &= \mathrm{TM}(x_0,\ [0]) + \mathrm{TM}(\alpha,\ [0]) \int_0^t h_1^+\left(\varphi_0(x_0,s),s\right) ds \\
&\qquad - \mathrm{TM}(\beta,\ [0]) \int_0^t h_1^-\left(\varphi_0(x_0,s),s\right) ds \\
&= \mathrm{TM}(x_0,\ [0]) + \mathrm{TM}(\alpha,\ [0]) \cdot \mathrm{TM}\left(0.1t - 0.25t^2,\ [0]\right) \\
&\qquad - \mathrm{TM}(\beta,\ [0]) \cdot \mathrm{TM}\left(0.25t^2,\ [0]\right) \\
&= \mathrm{TM}\left(x_0 + 0.1\alpha t - 0.25(\alpha+\beta)t^2,\ [0]\right)
\end{aligned}$$

with $x_0 = 0$, $\alpha \in [-1,1]$ and $\beta \in [-1,1]$.

# Example 1: Over-approximation

### Result

For all $t \in [0, 0.2]$, we have
$x(t) \in \mathrm{TM}\left(x_0 + 0.1\alpha t - 0.25(\alpha + \beta)t^2,\ [0]\right) = [-0.1t, 0.1t]$.
So $x(0.2) \in [-0.02, 0.02]$.

### (Remind) Exact reachable set

The exact reachable set at time $t = 0.2$ is $x(0.2) \in [-0.01, 0.01]$.

# Decreasing exponential

## Dynamics

$$\dot{x}(t) = -u(t)x(t) \tag{20}$$

with $x(0) \in [1, 1.1]$ and $\forall t,\ u(t) \in [1, 2]$.

# Decreasing exponential

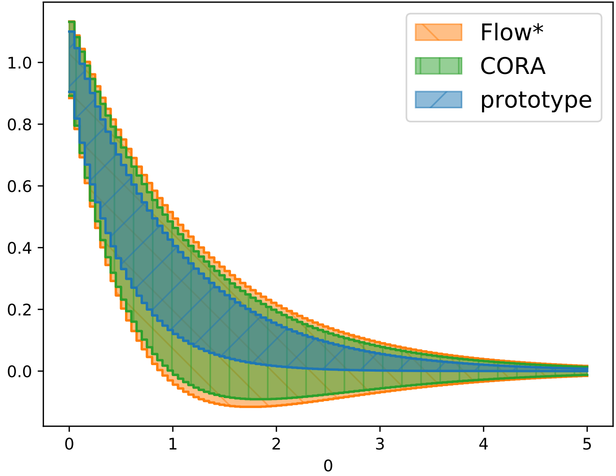


Figure: Over-approximations with fixed time-step equals to 0.05

# Nonlinear perturbation

## Dynamics

$$\begin{cases} \dot{x}(t) = -x(t) + x(t)y(t)u(t) \\ \dot{y}(t) = -y(t) \end{cases} \tag{21}$$

with $x(0) = 1$, $y(0) = 2$ and $u(t) \in [-1, 1]$.

# Nonlinear perturbation

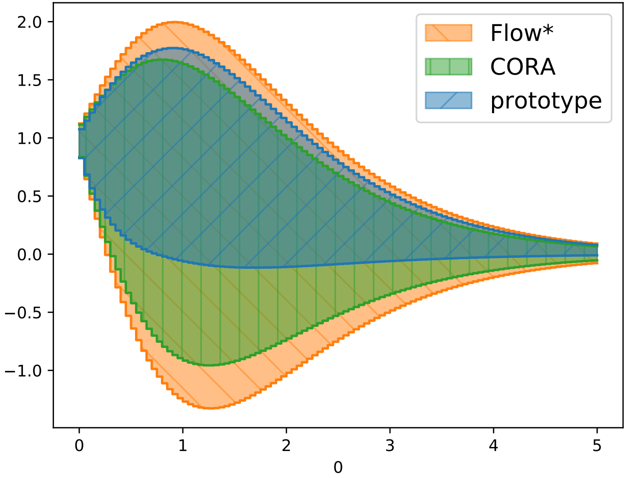


Figure: Over-approximations with fixed time-step equals to 0.05 with order 4 (31 *s*)

# Nonlinear perturbation

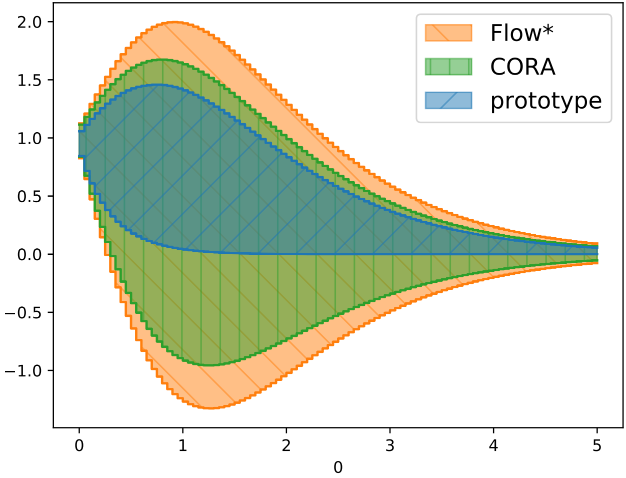


Figure: Over-approximations with fixed time-step equals to 0.05 with order 5 (2 $m$ 20 $s$)

# Conclusion

## Summary

- able to handle measurable bounded uncertainties
- promising results on simple examples

## Futur work

- try different sets representations
- optimize the prototype

Thank you for your attention