Fuzzy set estimation using interval tools; Application to localization

L. Jaulin, J. Tillet, F. Le Bars and R. Boukezzoula

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Characteristic functions for set computation

One robot at position (x_1, x_2) measures a distance $[d]$ to the landmark $\mathbf{m}$.
The corresponding *granule* is

$$\mathbb{Z} = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \sqrt{(x_1 - m_1)^2 + (x_2 - m_2)^2} \in [d] \right\}$$

Its *characteristic function* is $\zeta(\mathbf{x})$.

With more landmarks, we have more granules

$$\mathbb{Z}_j = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \sqrt{(x_1 - m_1(j))^2 + (x_2 - m_2(j))^2} \in [d_j] \right\}$$

j	1	2	3	4
$[d_j]$	$[2,4]$	$[4,6]$	$[7,9]$	$[4,6]$
$\mathbf{m}(j)$	$(-1,3)$	$(5,2)$	$(8,-1)$	$(1,-5)$

and more characteristic function $\zeta^j(\mathbf{x})$.

We define a *score function* $\sigma : \{0,1\}^m \mapsto [0,1]$ as

$$\sigma(0,0,\dots,0) = 0$$
$$\sigma(1,1,\dots,1) = 1$$
$$\forall j, a_j \leq b_j \Rightarrow \sigma(a_1,\dots,a_m) \leq \sigma(b_1,\dots,b_m)$$

The *membership function* associated with the granules $\mathbb{Z}_j$ and the score function σ is

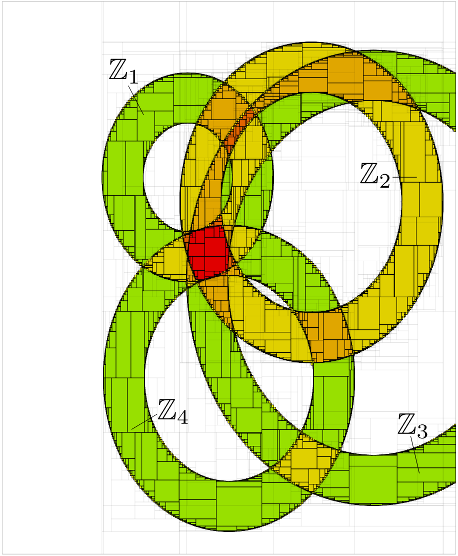
$$\mu(\mathsf{x}) = \sigma\left(\zeta^1(\mathsf{x}), \ldots, \zeta^m(\mathsf{x})\right)$$

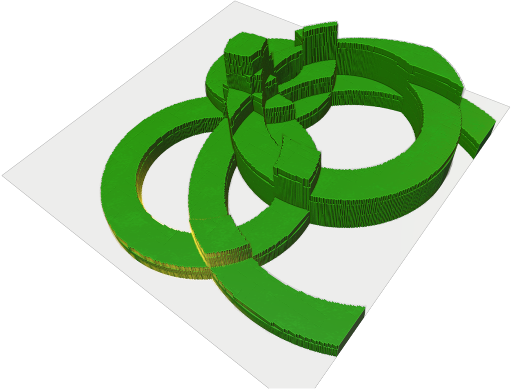
For instance, if

$$\sigma(b_1, b_2, b_3, b_4) = \frac{b_1 + 2b_2 + b_3 + b_4}{5}$$

we get

$$\mu(\mathsf{x}) = \frac{\zeta^1(\mathsf{x}) + 2\zeta^2(\mathsf{x}) + \zeta^3(\mathsf{x}) + \zeta^4(\mathsf{x})}{5}$$


$\mathbb{Z}_1$
$\mathbb{Z}_2$
$\mathbb{Z}_3$
$\mathbb{Z}_4$



https://replit.com/@aulin/Alpha-cut-characterization

Set algebra v.s. score function

The set associated with the granules $\mathbb{Z}_j$, the score function σ, the degree $\alpha \in [0,1]$ is

$$\mathbb{X} = \left\{ \mathsf{x} \,|\, \sigma\left(\zeta^1(\mathsf{x}), \ldots, \zeta^m(\mathsf{x})\right) \geq \alpha \right\}.$$

Example 1. We have

$$\mathbb{Z}_1 \cap \mathbb{Z}_2 = \left\{ \mathbf{x} \,|\, \frac{\zeta^1(\mathbf{x}) + \zeta^2(\mathbf{x})}{2} \geq 1 \right\}$$

Thus

$$\cap \leftrightarrow \left\{ \begin{array}{c} \sigma(b_1, b_2) = \frac{b_1 + b_2}{2} \\ \alpha = 1 \end{array} \right.$$

Equivalently

$$\mathbb{Z}_1 \cap \mathbb{Z}_2 = \left\{ \mathsf{x} \,|\, \min(\zeta^1(\mathsf{x}), \zeta^2(\mathsf{x})) \geq 1 \right\}$$

Thus

$$\cap \leftrightarrow \left\{ \begin{array}{c} \sigma(b_1, b_2) = \min(b_1, b_2) \\ \alpha = 1 \end{array} \right.$$

Example 2. We have

$$\mathbb{Z}_1 \cup \mathbb{Z}_2 = \left\{ \mathbf{x} \,\middle|\, \frac{\zeta^1(\mathbf{x}) + \zeta^2(\mathbf{x})}{2} \geq \frac{1}{2} \right\}$$

Thus

$$\cup \leftrightarrow \left\{ \begin{array}{c} \sigma(b_1, b_2) = \frac{b_1 + b_2}{2} \\ \alpha = \frac{1}{2} \end{array} \right.$$

Equivalently

$$\mathbb{Z}_1 \cup \mathbb{Z}_2 = \left\{ \mathbf{x} \,|\, \max(\zeta^1(\mathbf{x}), \zeta^2(\mathbf{x})) \geq 1 \right\}$$

Thus

$$\cup \leftrightarrow \begin{cases} \sigma(b_1, b_2) = \max(b_1, b_2) \\ \alpha = 1 \end{cases}$$

Example 3. We have

$$(\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4) = \left\{ \mathbf{x} \mid \frac{\zeta^1(\mathbf{x}) + 2\zeta^2(\mathbf{x}) + \zeta^3(\mathbf{x}) + \zeta^4(\mathbf{x})}{5} \geq \frac{1}{2} \right\}$$

Thus

$$\begin{cases} \sigma(b_1, \ldots, b_4) = \frac{b_1 + 2b_2 + b_3 + b_4}{5} \\ \alpha = \frac{1}{2} \end{cases}$$

Example 4. We have

$$\bigcap^{\{q\}} \mathbb{Z}_j = \left\{ \mathsf{x} \,|\, \frac{1}{m}\sum_j \zeta^j(\mathsf{x}) \geq 1 - \frac{q}{m} \right\}$$

Thus

$$\begin{cases} \sigma(b_1, \ldots, b_m) = \frac{1}{m}\sum_j b_j \\ \alpha = 1 - \frac{q}{m} \end{cases}$$

Example 5. We have

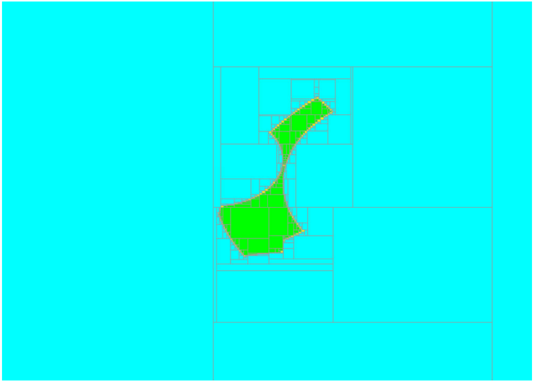
$$\mathbb{Z}_1 \cap \bigcap^{\{q\}} \mathbb{Z}_j. = \left\{ \mathbf{x} \,|\, \min\left(\zeta^1(\mathbf{x}), \frac{1}{m} \sum_j \zeta^j(\mathbf{x}) \right) \geq 1 - \frac{q}{m} \right\}$$

Thus

$$\begin{cases} \sigma(b_1, \ldots, b_m) = \min\left(b_1, \frac{1}{m} \sum_j b_j\right) \\ \alpha = 1 - \frac{q}{m} \end{cases}$$

σ	α	$\mathbb{X}$
$\frac{b_1+b_2}{2}$	1	$\mathbb{Z}_1 \cap \mathbb{Z}_2$
$\frac{b_1+b_2}{2}$	0.5	$\mathbb{Z}_1 \cup \mathbb{Z}_2$
$\frac{b_1+2b_2+b_3+b_4}{5}$	0.5	$(\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$
$\frac{1}{m}\sum_j b_j$	$1 - \frac{q}{m}$	$\bigcap^{\{q\}} \mathbb{Z}_j$
$\min\left(b_1, \frac{1}{m}\sum_j b_j\right)$	$1 - \frac{q}{m}$	$\mathbb{Z}_1 \ \cap \ \bigcap^{\{q\}} \mathbb{Z}_j.$

```
from codac import *
from MuSolve import SepMu
from vibes import vibes
M = [(-1,3),(5,2),(8,-1),(1,-5)]
D=[Interval(2,4),Interval(4,6),Interval(7,9),Interval(4,6)]
def σ(s):
   b1,b2,b3,b4 = s
   return (b1+2*b2+b3+b4)/5
S = []
for i,m in enumerate(M):
   f=Function("x","y",f"({m[0]}-x)^2 + ({m[1]}-y)^2")
   S.append(SepFwdBwd(f,sqr(D[i])))
   SIVIA([[-10,12],[-12,10]],SepMu(S,σ,0.7))
```



https://replit.com/@aulin/Alpha-cut-one

	Characteristic functions	Set algebra
Operators	$\max, \min, +, -, \cdot$	$\cup, \cap, \neg$
Graduality	Yes	No

Transformation

Take

$$\frac{\zeta^1(x) + 2\zeta^2(x) + \zeta^3(x) + \zeta^4(x)}{5} \geq \frac{1}{2}$$

$\mathbb{Z}_3\mathbb{Z}_4$ \ $\mathbb{Z}_1\mathbb{Z}_2$	00	01	11	10
00	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
01	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{2}{5}$
11	$\frac{2}{5}$	$\frac{4}{5}$	1	$\frac{3}{5}$
10	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{2}{5}$

$\mu(\mathbf{x}) = \frac{\zeta^1(\mathbf{x})+2\zeta^2(\mathbf{x})+\zeta^3(\mathbf{x})+\zeta^4(\mathbf{x})}{5}$

$\mathbb{Z}_3\mathbb{Z}_4$ \ $\mathbb{Z}_1\mathbb{Z}_2$	00	01	11	10
00	0	0	1	0
01	0	1	1	0
11	0	1	1	1
10	0	1	1	0

$\mu(\mathbf{x}) \geq 0.5$

$$\frac{\zeta^1(\mathbf{x})+2\zeta^2(\mathbf{x})+\zeta^3(\mathbf{x})+\zeta^4(\mathbf{x})}{5} \geq \frac{1}{2}$$

$$\rightarrow \quad (\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$$

Reciprocally

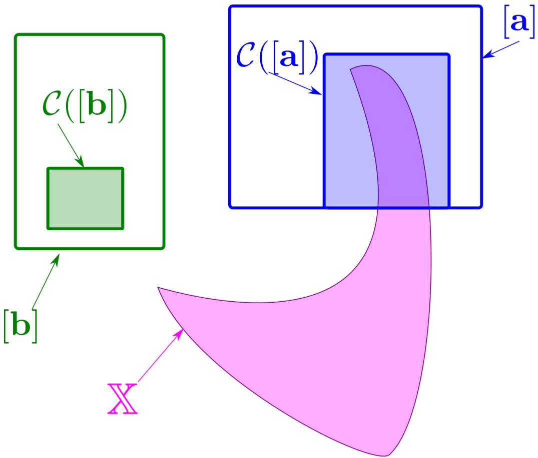
$$(\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$$

$$\rightarrow \qquad \max \begin{pmatrix} \min(\zeta^1(\mathbf{x}), \zeta^2(\mathbf{x})) \\ \min(\zeta^2(\mathbf{x}), \zeta^4(\mathbf{x})) \\ \min(\zeta^2(\mathbf{x}), \zeta^3(\mathbf{x})) \\ \min(\zeta^1(\mathbf{x}), \zeta^3(\mathbf{x}), \zeta^4(\mathbf{x})) \end{pmatrix} \geq 1$$

μ-factory

A *contractor* $\mathscr{C}$ [1] $\mathbb{X}$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that

$$
\begin{array}{ll}
\mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\
\mathscr{C}([\mathbf{x}]) \cap \mathbb{X} \subset [\mathbf{x}] \cap \mathbb{X} & \text{(consistency)} \\
[\mathbf{x}] \subset [\mathbf{y}] \ \Rightarrow \ \mathscr{C}([\mathbf{x}]) \subset \mathscr{C}([\mathbf{y}]) & \text{(monotonicity)}
\end{array}
$$


$\mathcal{C}([\mathbf{b}])$
$[\mathbf{b}]$
$\mathcal{C}([\mathbf{a}])$
$[\mathbf{a}]$
$\mathbb{X}$

A contractor for

$$\mathbb{X} = (\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$$

is

$$\mathscr{C}_{\mathbb{X}} = (\mathscr{C}_{\mathbb{Z}_1} \cap \mathscr{C}_{\mathbb{Z}_2}) \cup (\mathscr{C}_{\mathbb{Z}_2} \cap \mathscr{C}_{\mathbb{Z}_4}) \cup (\mathscr{C}_{\mathbb{Z}_2} \cap \mathscr{C}_{\mathbb{Z}_3}) \cup (\mathscr{C}_{\mathbb{Z}_1} \cap \mathscr{C}_{\mathbb{Z}_3} \cap \mathscr{C}_{\mathbb{Z}_4})$$

Reification

The constraint

$$\mathbf{x} \in (\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$$

can be rewritten as

$$b_i = (\mathbf{x} \in \mathbb{Z}_i)$$
$$1 = (b_1 \wedge b_2) \vee (b_2 \wedge b_4) \vee (b_2 \wedge b_3) \vee (b_1 \wedge b_3 \wedge b_4)$$

The constraint

$$\mathbf{x} \in (\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$$

can be rewritten as

$$\mathbf{z}(i) \in \mathbb{Z}_i$$
$$b_i = (\mathbf{x} = \mathbf{z}(i))$$
$$1 = (b_1 \wedge b_2) \vee (b_2 \wedge b_4) \vee (b_2 \wedge b_3) \vee (b_1 \wedge b_3 \wedge b_4)$$

The constraint

$$\mathbf{x} \in (\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$$

can be rewritten as

$$\mathbf{z}(i) \in \mathbb{Z}_i$$
$$b_i = (\mathbf{x} = \mathbf{z}(i))$$
$$\sigma(b_1, \ldots, b_4) \geq 1$$

The constraint

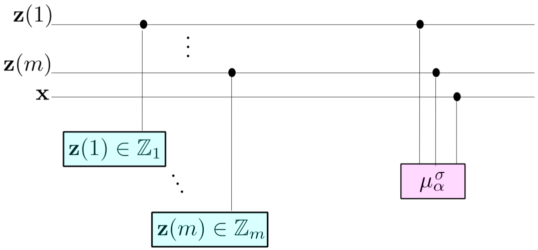
$$\mathbf{x} \in (\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$$

can be rewritten [3] as

$$\mathbf{z}(i) \in \mathbb{Z}_i$$
$$\sigma((\mathbf{x} = \mathbf{z}(1)), \ldots, (\mathbf{x} = \mathbf{z}(m))) \geq 1$$

We define

$$\mu_\alpha^\sigma(\mathsf{x},\mathsf{z}(1),\ldots,\mathsf{z}(m)) \Leftrightarrow \sigma((\mathsf{x}=\mathsf{z}(1)),\ldots,(\mathsf{x}=\mathsf{z}(m))) \geq \alpha$$



Constraint network [2]

$$\mu_\alpha^\sigma \supset \bigcap^{\{q]}$$

We mean that we propose a generalization of the relaxed intersection algorithm

Take

$$\sigma((\mathsf{x} = \mathsf{z}(1)), \ldots, (\mathsf{x} = \mathsf{z}(m))) \geq \alpha$$
$$\mathsf{z}(i) \in [\mathsf{z}](i)$$
$$\mathsf{x} \in [\mathsf{x}]$$

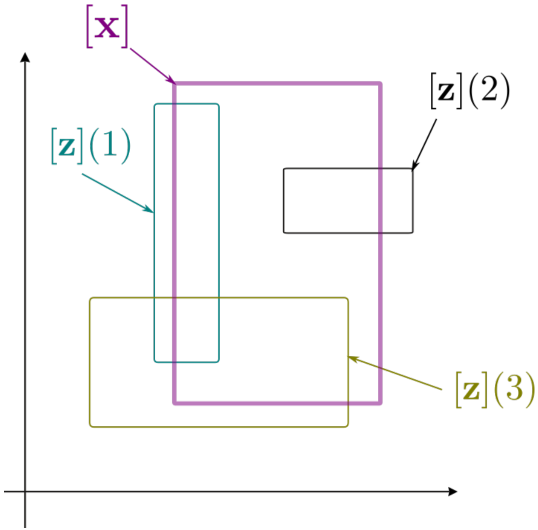
Contract $[\mathsf{z}](i)$ and $[\mathsf{x}]$.

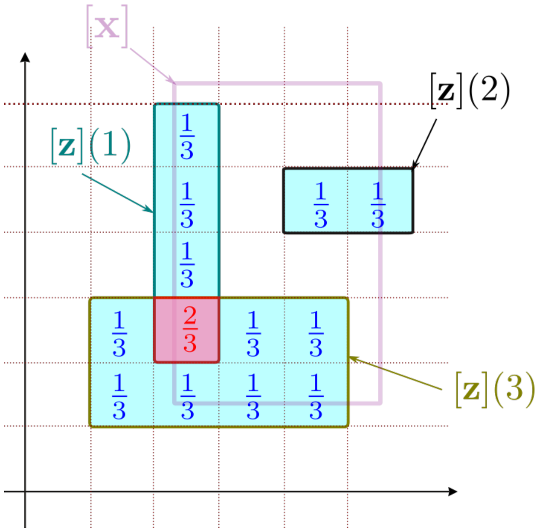
Example

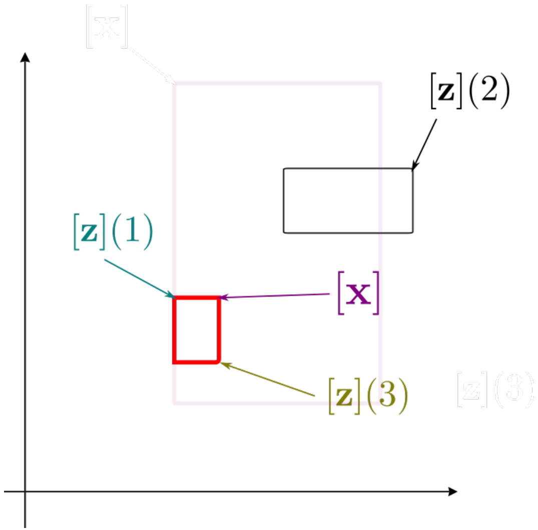
$$\sigma\left((\mathsf{x} = \mathsf{z}(1)), \ldots, (\mathsf{x} = \mathsf{z}(m))\right) \geq \frac{2}{3}$$

where

$$\sigma(b_1, b_2, b_3) = \frac{b_1 + b_2 + b_3}{3}$$


[x]
[z](2)
[z](1)
[z](3)


[x]
[z](1)
[z](2)
[z](3)
1/3
2/3


[x]
[z](2)
[z](1)
[x]
[z](3)
[z](3)

Moreover, we get the correspondences

$$\mathsf{z}(1) = \mathsf{z}(3) = \mathsf{x} \neq \mathsf{z}(2)$$

or equivalently

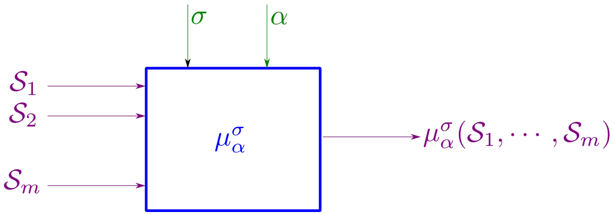
$$b_1 = b_3 = 1 \text{ and } b_2 = 0$$

Proposition (De Morgan rule). The complementary set of the $\mathbb{X} = \left\{ \mathsf{x} \mid \sigma\left(\zeta^1(\mathsf{x}), \ldots, \zeta^m(\mathsf{x})\right) \geq \alpha \right\}$ is

$$\overline{\mathbb{X}} = \left\{ \mathsf{x} \mid \bar{\sigma}\left(\bar{\zeta}^1(\mathsf{x}), \ldots, \bar{\zeta}^m(\mathsf{x})\right) > \bar{\alpha} \right\}$$

where

$$\begin{aligned} \bar{\zeta}^j &= 1 - \zeta^j \\ \bar{\sigma}\left(b_1, \ldots, b_m\right) &= 1 - \sigma\left(1 - b_1, \ldots, 1 - b_m\right) \\ \bar{\alpha} &= 1 - \alpha. \end{aligned}$$


σ
α
$\mathcal{S}_1$
$\mathcal{S}_2$
$\mathcal{S}_m$
μ_α^σ
$\mu_\alpha^\sigma(\mathcal{S}_1, \cdots, \mathcal{S}_m)$

```
from codac import *
from MuSolve import SepMu
from vibes import vibes
M = [(-1,3),(5,2),(8,-1),(1,-5)]
D=[Interval(2,4),Interval(4,6),Interval(7,9),Interval(4,6)]
def σ(s):
   b1,b2,b3,b4 = s
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S = []
for i,m in enumerate(M):
   f=Function("x","y",f"({m[0]}-x)^2 + ({m[1]}-y)^2")
   S.append(SepFwdBwd(f,sqr(D[i])))
   SIVIA([[-10,12],[-12,10]], SepMu(S,σ,0.7) )
```

Fuzzy sets

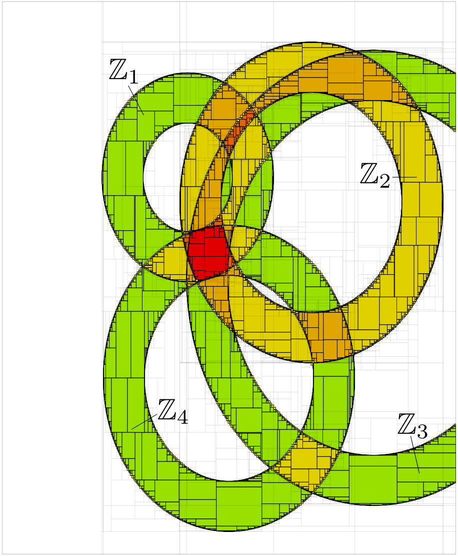
Consider m granules $\mathbb{Z}_1,\ldots,\mathbb{Z}_m$ of $\mathbb{R}^n$, and a score function σ. The corresponding fuzzy set $\mathbb{X}$ is

$$\mu_{\mathbb{X}}:\left\{\begin{array}{ccc}\mathbb{R}^n & \rightarrow & [0,1] \\ \mathbf{x} & \rightarrow & \sigma\left(\zeta^1(\mathbf{x}),\ldots,\zeta^m(\mathbf{x})\right)\end{array}\right.$$

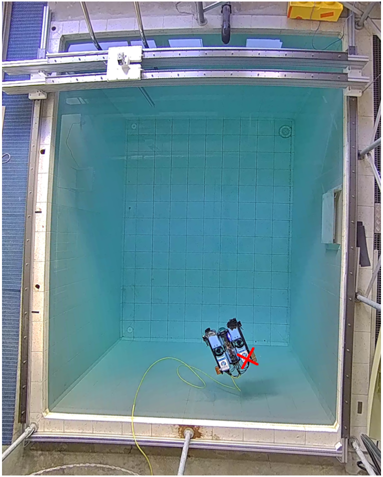
An α-cut of a fuzzy set $\mathbb{X}$ is the *crisp* set:

$$\mathbb{X}_\alpha = \{\mathsf{x} \mid \mu_{\mathbb{X}}(\mathsf{x}) \geq \alpha\} = \left\{\mathsf{x} \mid \sigma\left(\zeta^1(\mathsf{x}), \ldots, \zeta^m(\mathsf{x})\right) \geq \alpha\right\}$$

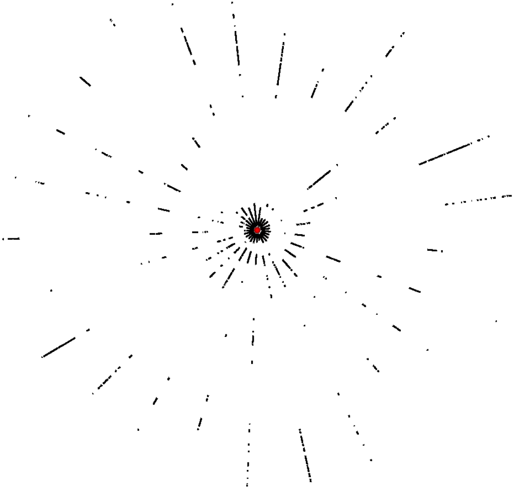
where α is *membership degree*.

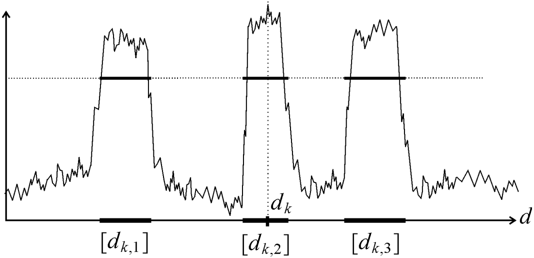

$\mathbb{Z}_1$
$\mathbb{Z}_2$
$\mathbb{Z}_3$
$\mathbb{Z}_4$

Application to localization



Underwater robot in a pool, equipped with a sonar, a compass and a manometer.



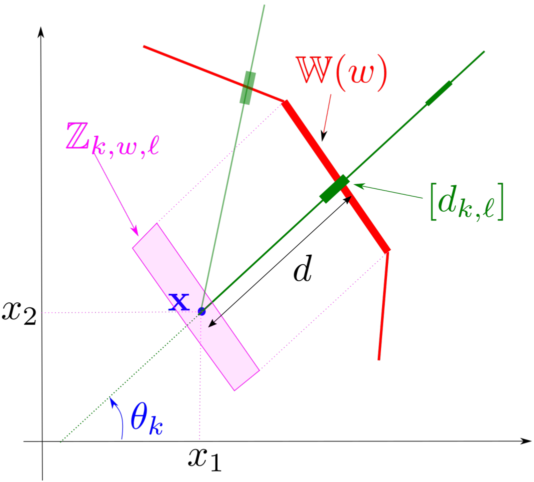


Echo collected after the kth ping

Walls: $\bar{w}$ segments $\mathbb{W}(w)$, $w \in \{1, \ldots, \bar{w}\}$.
Bearings $\theta_1, \ldots, \theta_{\bar{k}}$.
Distance interval $[d_{k,\ell}]$
Location $\mathbf{x} = (x_1, x_2)^\mathsf{T}$

The granules are

$$\begin{aligned}
\mathbb{Z}_{k,w,\ell} &= \{\mathbf{x} \in \mathbb{R}^2 \mid \quad \exists d \in [d_{k,\ell}], \exists \mathbf{m} = (m_1, m_2) \in \mathbb{W}(w) \\
& \qquad\qquad\qquad\quad m_1 = x_1 + d\cos\theta_k, m_2 = x_2 + d\sin\theta_k \quad \} \\
\mathbb{Z}_0 &= \quad \text{Pool}
\end{aligned}$$

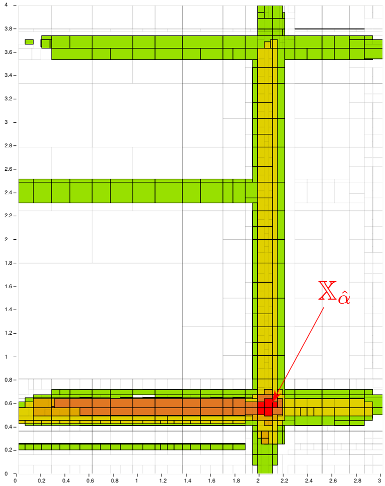

$\mathbb{W}(w)$
$\mathbb{Z}_{k,w,\ell}$
$[d_{k,\ell}]$
d
$\mathbf{x}$
x_2
x_1
θ_k

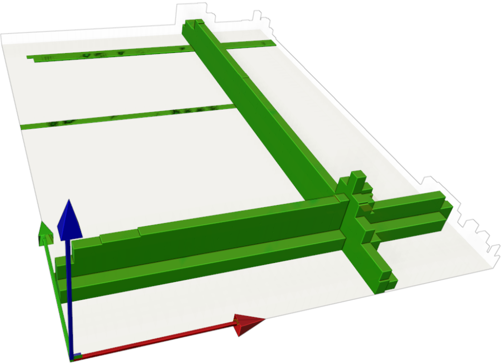
Define the fuzzy set $\mathbb{X}$ as

$$\mu_{\mathbb{X}}(\mathbf{x}) = \frac{1}{\bar{k}} \sum_{k \in \left\{1,\ldots,\bar{k}\right\}} \max_{\ell \in \left\{1,\ldots,\bar{\ell}(k)\right\}} \max_{w \in \{1,\ldots,\bar{w}\}} \min\left(\zeta^{k,w,\ell}(\mathbf{x}), \zeta^{0}(\mathbf{x})\right)$$

Note that

$$\begin{array}{rl} & \max_{\ell \in \left\{1,\ldots,\bar{\ell}(k)\right\}} \max_{w \in \{1,\ldots,\bar{w}\}} \min\left(\zeta^{k,w,\ell}(\mathbf{x}), \zeta^{0}(\mathbf{x})\right) = 1 \\ \Leftrightarrow & \exists \ell \in \left\{1,\ldots,\bar{\ell}(k)\right\}, \exists w \in \{1,\ldots,\bar{w}\}, \mathbf{x} \in \mathbb{Z}_{k,w,\ell} \cap \mathbb{Z}_0, \end{array}$$


$\mathbb{X}_{\hat{\alpha}}$
4
3.8
3.6
3.4
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3



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