An introduction to Mobile Robotics Seen through the eyes of distributed computing

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June 15, 2022

Part I: An introduction to Distributed Task solvability

(slightly biased towards geometric methods though)

Based on slides from Jérémy Ledent

Introduction

The distributed computing setting

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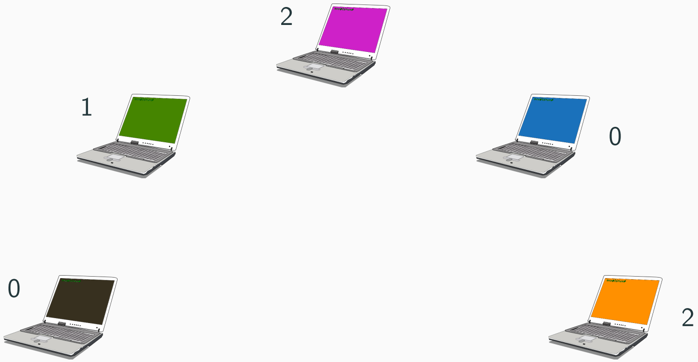

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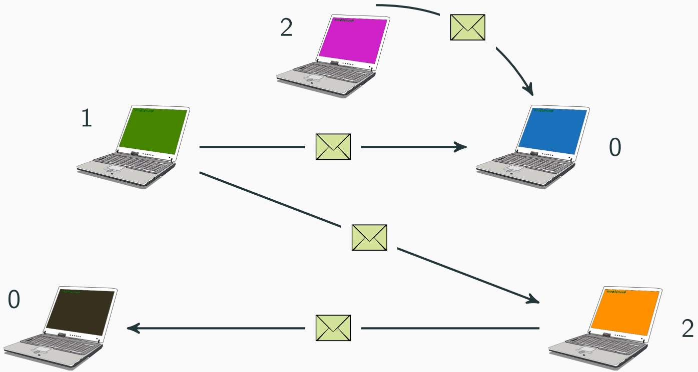

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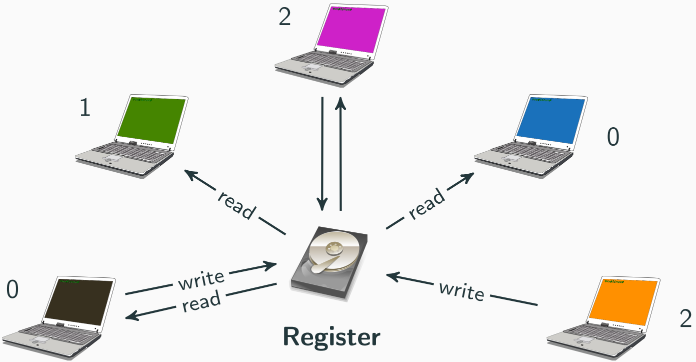

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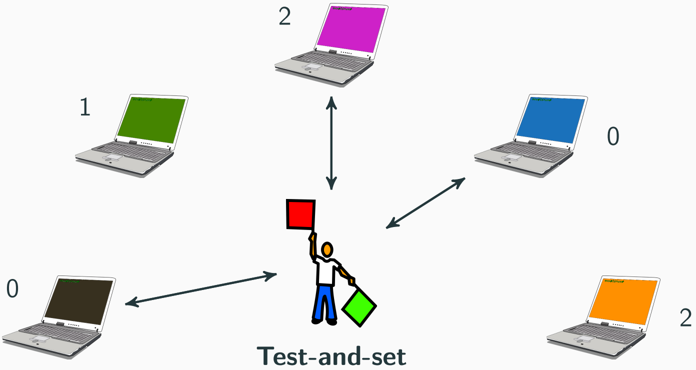

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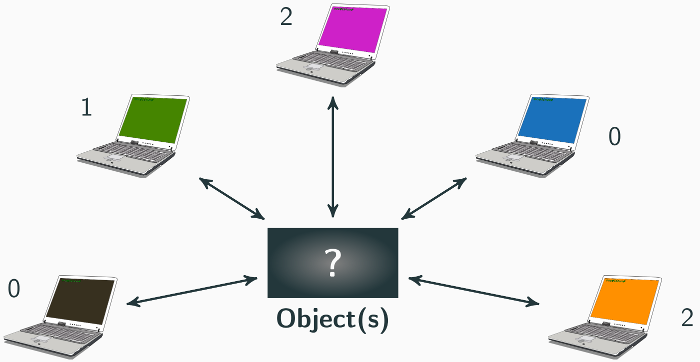

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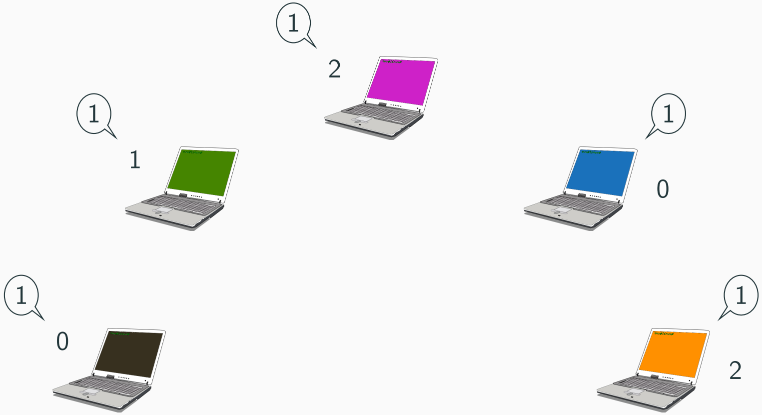

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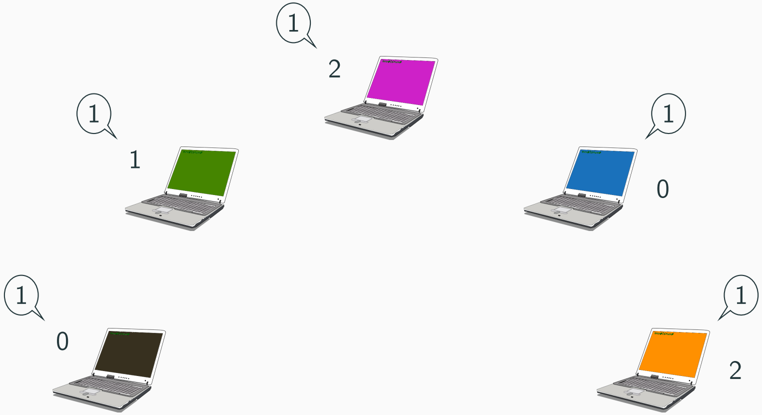

Task specification: $(0,1,2,0,2) \rightarrow (1,1,1,1,1)$ ✓ or ✗ ?

Question: Can we solve the task T using the objects $X_1,\ldots,X_n$?

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Main focus today: Proving impossibility results.

⚠ Usually *not* a matter of computing power:
Dealing with incomplete or unreliable information.

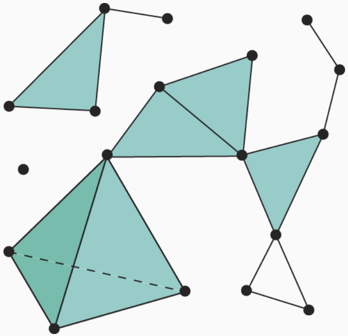
Combinatorial Topology

Simplicial complexes

Definition

A **simplicial complex** is a pair $\langle V, S \rangle$ where

- V is a set of vertices
- S is a downward-closed family of subsets of V called simplices. (i.e., $X \in S$ and $Y \subseteq X$ implies $Y \in S$)



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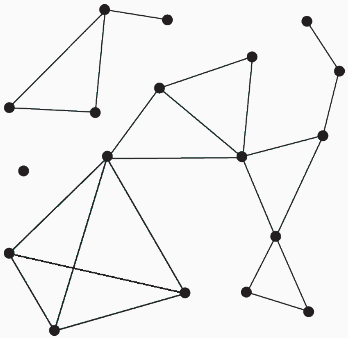

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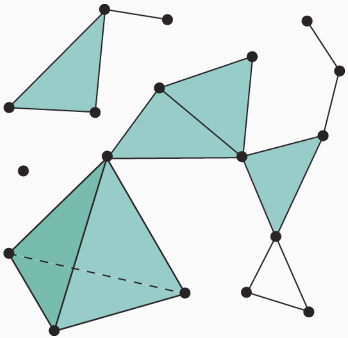

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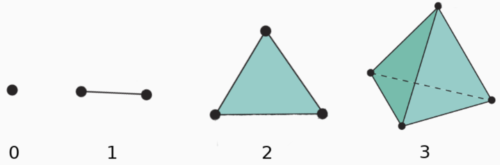

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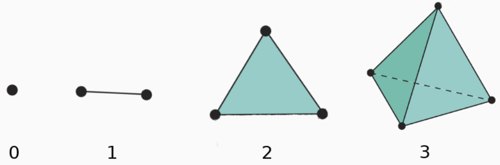

- A simplicial complex is pure if all maximal simplices are of the same dimension.

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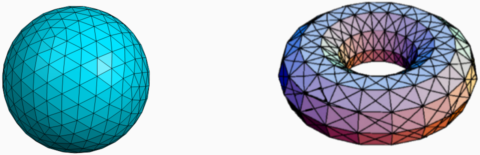

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Simplicial maps	$\simeq$	Continuous functions

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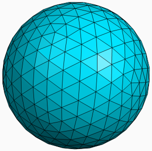

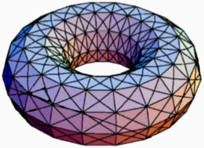

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Simplicial complexes are a good setting for algebraic topology:

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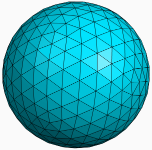

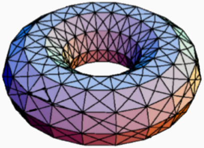

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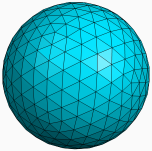

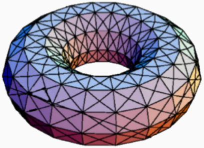

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- Homology

Asynchronous Computability via Combinatorial Topology

Chromatic simplicial complexes

We fix a *finite* set P of colors/processes.

Definition

A **chromatic simplicial complex** is given by $\langle V, S, \chi \rangle$ where

- $\langle V, S \rangle$ is a simplicial complex
- $\chi : V \to P$ assigns colors to vertices, such that every simplex $X \in S$ has vertices of different colors $(\forall u, v \in X.\, \chi(u) \neq \chi(v))$

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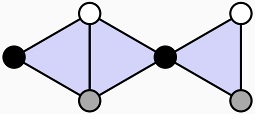
Example: a pure chromatic simplicial complex of dimension 2:



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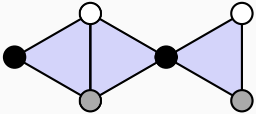

[1] All the pictures will have 3 processes in order to remain 2-dimensional; this is of course not a technical requirement.

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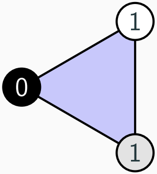

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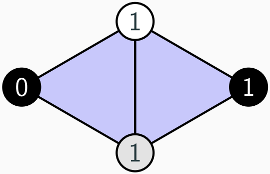

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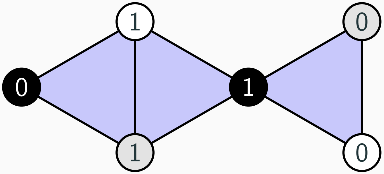

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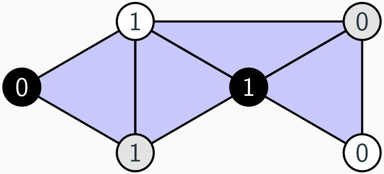

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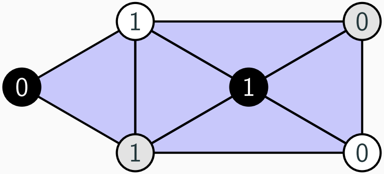

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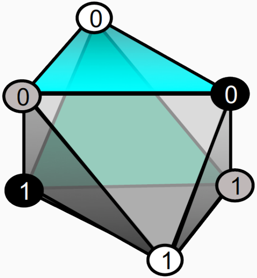

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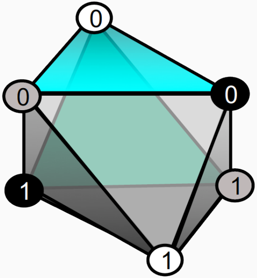

Remark: for $n+1$ processes, we get a combinatorial n-sphere.

The immediate snapshot object

```
immediate_snapshot : int -> int array
```

Fix a number n of processes.
We suppose given a shared array A of size n.
Only process P_i can write in $A[i]$, but everyone can read it.

When P_i calls `immediate_snapshot(x)`:

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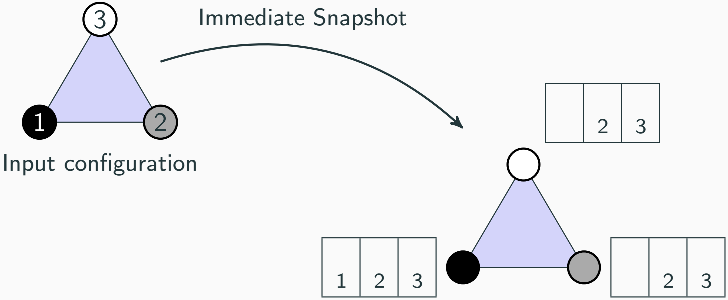
Protocol complex for immediate snapshot



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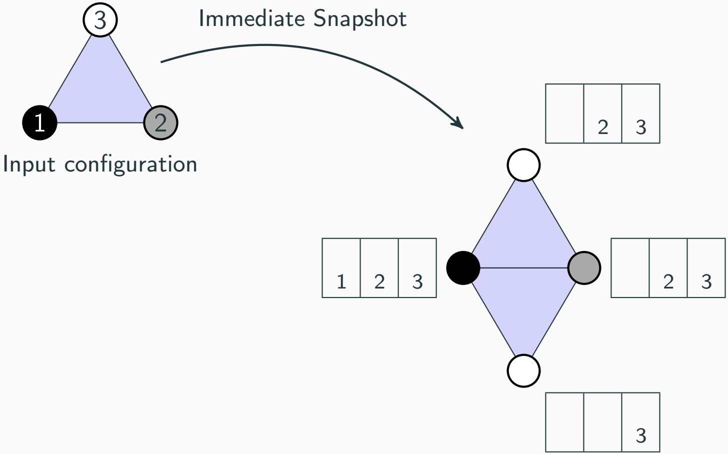

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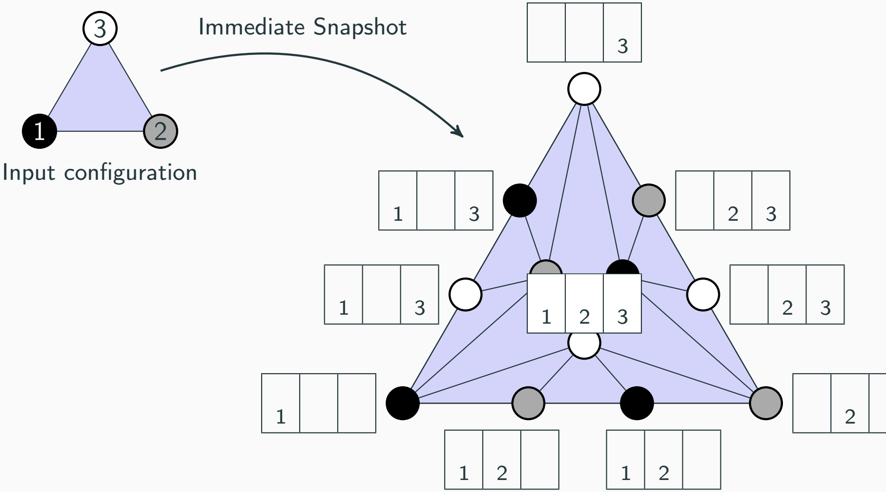

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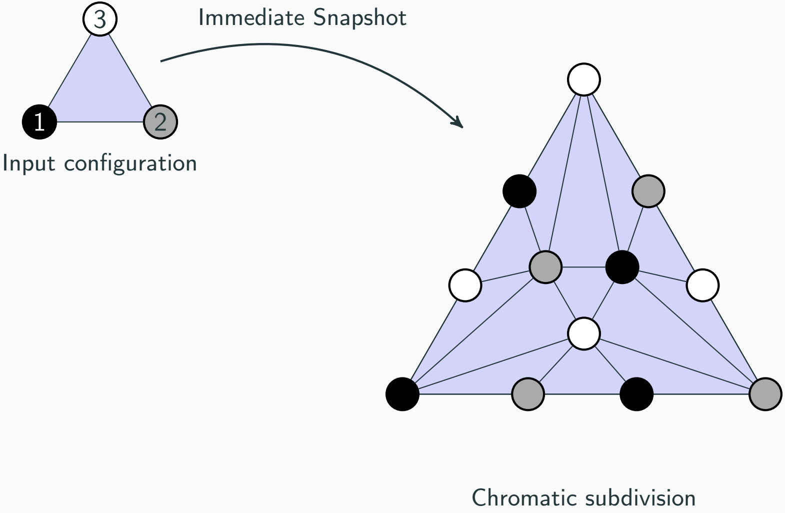
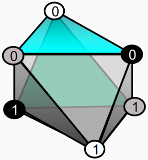
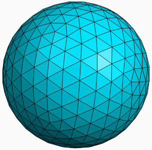




Input complex

Immediate snapshot



Protocol complex

Key property: the topology is preserved.

The (binary) consensus task

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Each process P_i has a binary input $\mathrm{in}_i \in \{0,1\}$.
After communicating, it decides an output $\mathrm{d}_i \in \{0,1\}$.

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Specification:

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- **Validity:** $\mathrm{d}_i \in \{\mathrm{in}_i \mid 1 \leq i \leq n\}$ for all i.

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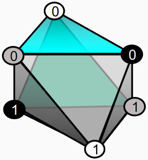
Examples: for 3 processes,

- if the inputs are $(0,0,0)$, the outputs must be $(0,0,0)$.
- if the inputs are $(1,0,1)$, the outputs can be $(0,0,0)$ or $(1,1,1)$.

Topological characterization of task solvability



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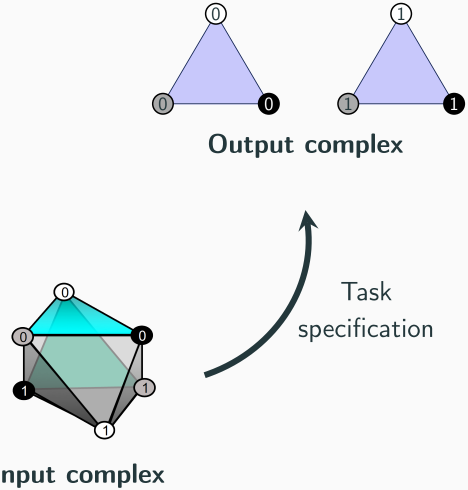
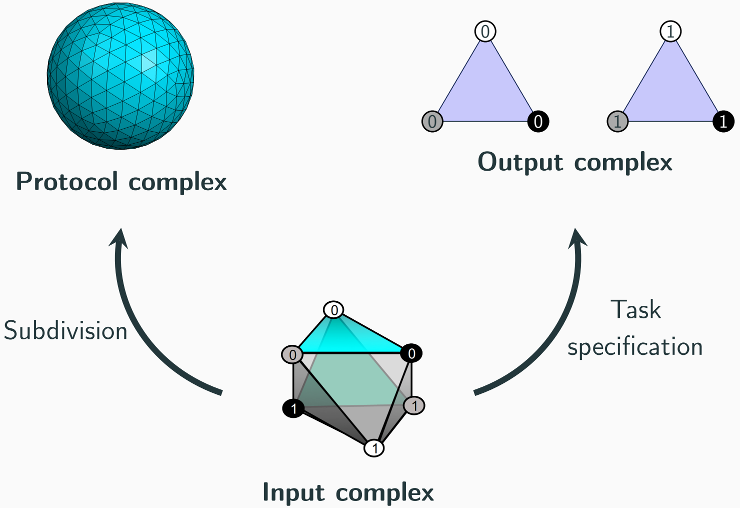
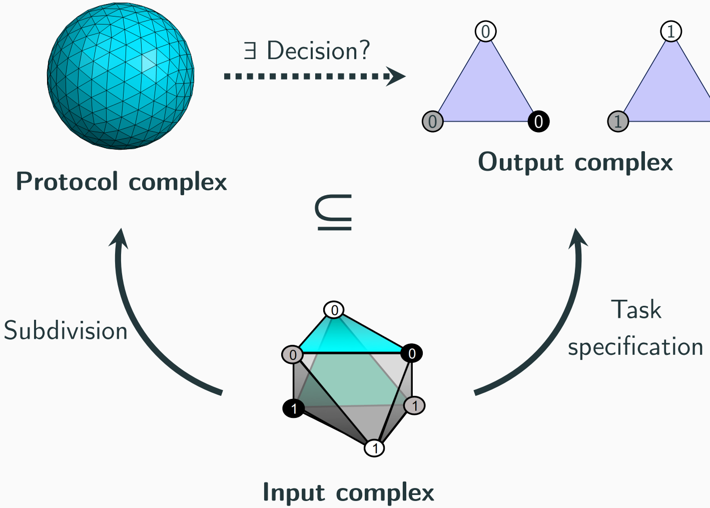


Topological characterization of task solvability



Topological characterization of task solvability



Asynchronous Computability Theorem (ACT)

Theorem (Herlihy and Shavit, 1999)

A task is solvable by a wait-free protocol using read/write registers if and only if there exists a simplicial map from the protocol complex into the output complex that satisfies the task specification.

We have reduced a computational question ("Is the task solvable?") to a topological one ("Is there a simplicial map?").

Algebraic topology excels at answering such questions!

- Simplicial maps preserve k-connectedness.
- Compute algebraic invariants of spaces.

Some results in the field

Asynchronous Computability Theorem (ACT)

Theorem (Herlihy and Shavit, 1999)

A task is solvable by a **wait-free** protocol using **read/write registers** if and only if there is a decision map from the protocol complex into the output complex that satisfies the task specification.

For instance: Update/scan wait-free protocols are:

- $(n-1)$-connected (no hole in any dimension)
- whatever number of communication rounds

Applications

- k-set agreement: generalisation of consensus; processes must terminate with at most k different values, taken from the initial values
- we cannot even solve k-consensus ($k \geq 1$) on such a machine!
- Approximate agreement: end up with "close enough" decisions: Possible!

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What if:

- we replace "wait-free" by "t-resilient"?
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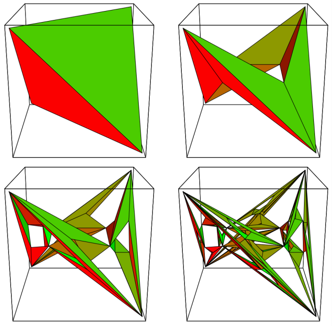
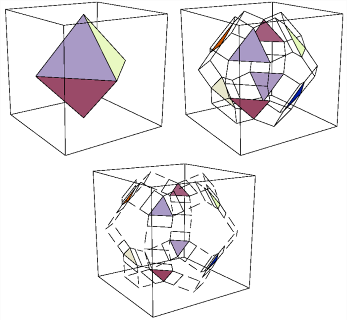
- we use other objects instead of read/write registers?
  ⟶ Many results with atomic operations (test&set, fetch&add etc.), (semi-) synchronous broadcast etc.

Protocol complexes for other objects

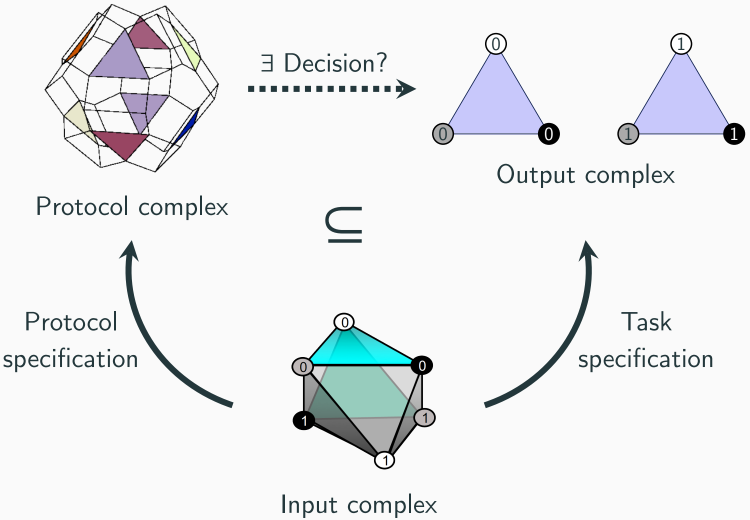


For test-and-set protocols
Herlihy, Rajsbaum, PODC'94

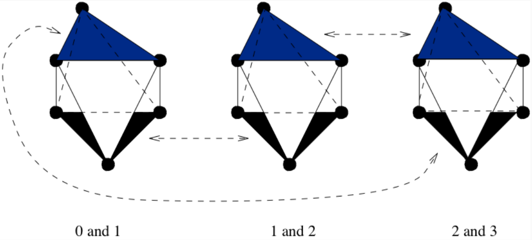


For synchronous message-passing
Herlihy, Rajsbaum, Tuttle, 2001

Example: task solvability for 3 procs, one round, 1-resilient synchronous broadcast



Ex: output complex, 3 processes, for the 2-set agreement



3 sphere, glued together, minus the simplex formed of the 3 values: connected but not 1-connected! (i.e. simply-connected) - compare with the output complex for consensus: 2 disconnected triangles!

Consequence

The 1-round protocol complex is connected, not simply connected:
$\Longrightarrow$ Impossible to solve consensus in 1 round, in a 1-resilient manner, with synchronous broadcast
$\Longrightarrow$ Possible to solve 2-set agreement in 1 round, in a 1-resilient manner, with synchronous broadcast

More generally speaking

In every synchronous broadcast protocol

- With $(n-2)$ steps (for n processes, synchronous message passing, one fault at most, i.e. 1-resilient), the sub-complex with all values equal to zero, and the one with all values equal to one, are connected
- Corollary : there is no algorithm, for this architecture, to solve consensus in (less than) $n-2$ rounds of communication (for at most one round)

Easy...

More generally

For r rounds of communication, and at most k faults in the synchronous model (message passing)

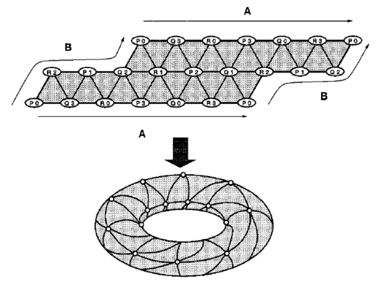
- The (sub-) protocol complex corresponding to an input, homeomorphic to the sphere in dimension $n-1$ (binary input values) $P(S^{n-1})$ is $(n-rk-2)$-connected
- This implies in particular that we have a lower bound of $n-1$ rounds for consensus, with $k=1$ (at most one crash)

Is it that simple?

Renaming

It is known to be implementable on an asynchronous system with message passing, in the presence of faults:

The $(n+1, K)$-renaming starts with $n+1$ processes which all have a name in $0, \ldots, N$. They must terminate with a name in in $0, \ldots, K$ with $n \leq K < N$.



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Renaming

It is known to be implementable on an asynchronous system with message passing, in the presence of faults:

- (Attiya et al. JACM 1990) : wait-free solution for $K \geq 2n+1$, and none when $K \leq n+2$
- By using entirely geometric techniques: it was shown that there is there is no renaming when $K \leq 2n$ (Herlihy and Shavit STOC 1993)

Is it that simple?

Renaming

It is known to be implementable on an asynchronous system with message passing, in the presence of faults:

A mistake in the proof has been found in 2008 (Rajsbaum and Castaneda, PoDC). In fact, this is still computable when $K = 2n$ and $n+1$ is not a power of a prime number!
Example : computable for $(K,n) = (10,5),(14,7)\dots$ but not for $(K,n) = (4,2),(6,3),(8,4)\dots$

Is it that simple? (2)

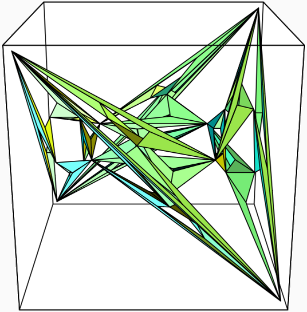
Real multiprocessors use much more refined synchronisation primitives

- test&set
- fetch&add
- compare&swap
- queues...

Exemple : Test&Set

Wait-free protocols with Test&Set

- are all $(n-3)$-connecterd
- are more expressive than scan/update protocols (for instance, we can solve the consensus with 2 processes)
- but we still cannot solve the consensus problem, in the presence of faults, for 3 processes or more



Some references

Other primitives, other models (asynchronous, synchronous, semi-synchronous etc.)...other results

- "Distributed Algorithms", N. Lynch
- "The art of multiprocessor programming", M. Herlihy, N. Shavit
- "Distributed Computing through Combinatorial Algebraic Topology", M. Herlihy, D. Kozlov, S. Rajsbaum

And also "Directed Topology and Concurrency", L. Fajstrup, E. Haucourt, E. Goubault, S. Mimram, M. Raussen

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A deep connection between topology and distributed computing.

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- Applies to a large range of computational models.

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What I did not talk about:

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- Connection with epistemic logic (very good for proving algorithms correct, next time!).
- Connection with swarm robotics: Bernardo, now!.