Numerical methods for dynamical systems

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Part VIII

Numerical methods for IVP-DDE

Initial Value Problem of Ordinary Differential Equations

Consider an IVP for ODE, over the time interval $[0, t_{\text{end}}]$

$$\dot{\mathbf{y}} = f(t, \mathbf{y}) \quad \text{with} \quad \mathbf{y}(0) = \mathbf{y}_0$$

IVP has a unique solution $\mathbf{y}(t; \mathbf{y}_0)$ if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz in $\mathbf{y}$

$$\forall t, \forall \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n, \exists L > 0, \quad \| f(t, \mathbf{y}_1) - f(t, \mathbf{y}_2) \| \leq L \| \mathbf{y}_1 - \mathbf{y}_2 \| \quad .$$

Goal of numerical integration

- Compute a sequence of time instants: $t_0 = 0 < t_1 < \cdots < t_n = t_{\text{end}}$
- Compute a sequence of values: $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n$ such that

$$\forall \ell \in [0, n], \quad \mathbf{y}_\ell \approx \mathbf{y}(t_\ell; \mathbf{y}_0) \ .$$

- s.t. $\mathbf{y}_{\ell+1} \approx \mathbf{y}(t_\ell + h; \mathbf{y}_\ell)$ with an error $\mathcal{O}(h^{p+1})$ where
  - h is the integration **step-size**
  - p is the **order** of the method

New problems to solve

System of Delay Differential Equations (DDE)

$$\begin{aligned} \dot{\mathbf{y}}(t) &= f(t, \mathbf{y}(t), \mathbf{y}(t-\tau)) & t_0 \leqslant t \leqslant t_{end} \\ \mathbf{y}(t) &= \phi(t) & t \leqslant t_0 \end{aligned}$$

System of Neutral Delay Differential Equations (NDDE)

$$\begin{aligned} \dot{\mathbf{y}}(t) &= f(t, \mathbf{y}(t), \mathbf{y}(t-\tau), \dot{\mathbf{y}}(t-\sigma)) & t_0 \leqslant t \leqslant t_{end} \\ \mathbf{y}(t) &= \phi(t) & t \leqslant t_0 \end{aligned}$$

Remark

For $t \geqslant t_0$ it can be that $t - \tau < t_0$ so an *initial function* $\phi(t)$ (history) is needed

We focus on DDE in this lecture.

Constant delay τ and σ are non-negative values

Variable or time dependant delay $\tau(t)$ and $\sigma(t)$

State dependant delay $\tau(t, y(t))$ and $\sigma(t, y(t))$

Remark: constant and time dependant delay are well studied in the literature. State variable is still an open problem.

Remark: We will focus on constant delays

It is related to Lipschitz property, in particular for

$$\begin{aligned} \dot{\mathbf{y}}(t) &= f(t, \mathbf{y}(t), \mathbf{y}(t-\tau(t,\mathbf{y}(t))), \dot{\mathbf{y}}(t-\sigma(t,\mathbf{y}(t))) & t_0 \leqslant t \leqslant t_{\text{end}} \\ \mathbf{y}(t) &= \phi(t) & t \leqslant t_0 \end{aligned}$$

if

$$\inf_{[t_0,t_f]\times\mathbb{R}^d} \tau(t,x) = \tau_0 > 0 \qquad \inf_{[t_0,t_f]\times\mathbb{R}^d} \sigma(t,x) = \sigma_0 > 0$$

then problem reduces to IVP-ODE, on interval $[t_0, t_0 + H]$ with $H = \min(\tau_0, \sigma_0)$, such that

$$\begin{aligned} \dot{\mathbf{y}}(t) &= f(t, \mathbf{y}(t), \phi(t-\tau(t,\mathbf{y}(t))), \dot{\phi}(t-\sigma(t,\mathbf{y}(t))) \leqslant t_{\text{end}} \\ \mathbf{y}(t) &= \phi(t_0) \end{aligned}$$

This is named *method of steps* but it does not always work in particular when delays τ or σ vanishes around t^*

DDE is not an ODE

Consider the system in 1D

$$\begin{cases} \dot{y}(t) = -y(t-1), & t \geqslant 0 \\ y(t) = 1, & t \leqslant 0 \end{cases}$$

As $\dot{y}(0)^- = 0$ and $\dot{y}(0)^+ = -y(t-1) = -1$ the derivative function has a jump at $t = 0$.

And the second derivative $\ddot{y}(t)$ is given by

$$\ddot{y} = -\dot{y}(t-1)$$

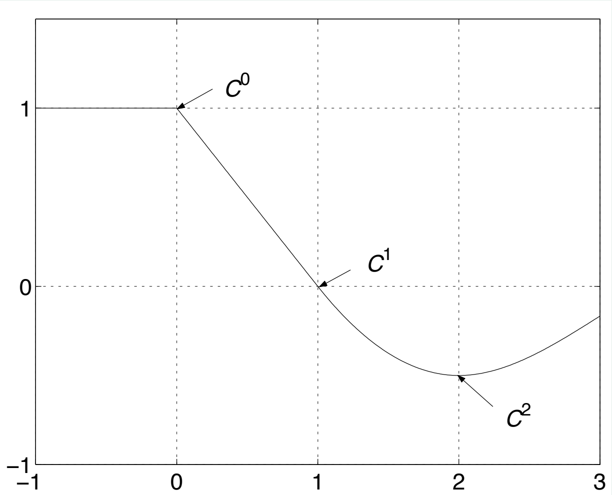
and so it has a jump at $t = 1$.

The third derivative $y'''(t)$ has a jump at $t = 2$ ans so forth.

$\tau(t)$ and $\sigma(t)$ are assumed continuous

- a jump discontinuity in $\ddot{y}$ is named 1-level primary discontinuity
- a jump discontinuity in y''' is named 2-level primary discontinuity
- ...

Remark that the solution becomes smoother and smoother as the primary discontinuity level increases.



C0
C1
C2
1
0
−1
−1
0
1
2
3

It is important to have a continuous solution of the ODE in order to get values of the solution at time $t - \tau$ for instance.

As previously seen (cf lecture on discontinuous simulation), we can build a polynomial approximation using $\mathbf{y}_n$, $f(\mathbf{y}_n)$, $\mathbf{y}_{n+1}$ and $f(\mathbf{y}_{n+1})$. The accuracy of the approach is interesting with only order 2 RK.

For DDE, we have to find high accurate continuous extension of the solution a.k.a. CERK (Continuous extension Runge-Kutta methods)

Recall that an explicit Runge-Kutta method is defined by:

$$\mathbf{k}_i = f\left(t_n + c_i h_n, \mathbf{y}_n + h\sum_{j=1}^{i-1} a_{ij}\mathbf{k}_j\right) \qquad \mathbf{y}_{n+1} = \mathbf{y}_n + h\sum_{i=1}^{s} b_i\mathbf{k}_i \qquad (1)$$

They are built from order condition which relates the Taylor expansion of the true solution and the Taylor expansion of the numerical solution.

What we want is

$$\eta(t_n + \theta h_n) = y_n + h_n \sum_{i=1}^{s} b_i(\theta) k_i \quad 0 \leqslant \theta \leqslant 1$$

$b_i(\theta)$ are polynomials of a suitable degree such that

$$b_i(0) = 0 \quad \text{and} \quad b_i(1) = b_i \quad i = 1, \ldots, s$$

interpolants of the first class are defined by only using intermediate steps k_i used to define the RK method

interpolants of the second class are defined by adding extra stages

Theorem: interpolants of the first class

Every RK methods (explicit and implicit) of order $p \geqslant 1$ has a continuous extension $\eta(t)$ of order (and degree) $q = 1, \ldots, \lfloor p+1 \rfloor$

Midpoint rule

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

$b_1(\theta) = \theta$

Gauss's method

$$\mathbf{k}_1 = f\left(t_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)h_n, \quad \mathbf{y}_n + h\left(\frac{1}{4}\mathbf{k}_1 + \left(\frac{1}{4} - \frac{\sqrt{3}}{6}\right)\mathbf{k}_2\right)\right)$$

$$\mathbf{k}_2 = f\left(t_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)h_n, \quad \mathbf{y}_n + h\left(\left(\frac{1}{4} + \frac{\sqrt{3}}{6}\right)\mathbf{k}_1 + \frac{1}{4}\mathbf{k}_2\right)\right)$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\left(\frac{1}{2}\mathbf{k}_1 + \frac{1}{2}\mathbf{k}_2\right)$$

$b_1(\theta) = -\frac{\sqrt{3}}{2}\theta\left(\theta - 1 - \frac{\sqrt{3}}{3}\right)$ and

$b_2(\theta) = \frac{\sqrt{3}}{2}\theta\left(\theta - 1 + \frac{\sqrt{3}}{3}\right)$

Interpolants of the second class

Why adding new stages? To reach

$$\max_{0\leqslant\theta\leqslant 1} \mid y_{n+1}(t_n+\theta h_n) - \eta(t_n+\theta h_n) \mid = \mathcal{O}(h_n^{p+1})$$

We will consider CERK methods which will have the FSAL property but adding new stages to reach a given order of the numerical approximation and a given order of the continuous approximation will have some limitations.

order	stages
1	1
2	2
3	4
4	6
5	8
6	11

Example of order 3

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{12}{23} & \frac{12}{23} & & & \\
\frac{4}{5} & -\frac{68}{375} & \frac{368}{375} & & \\
1 & \frac{31}{144} & \frac{529}{1152} & \frac{125}{384} & \\
\hline
 & b_1(\theta) & b_2(\theta) & b_3(\theta) & b_4(\theta)
\end{array}$$

- $b_1(\theta) = \frac{41}{72}\theta^3 - \frac{65}{48}\theta^2 + \theta$
- $b_2(\theta) = -\frac{529}{576}\theta^3 - \frac{529}{344}\theta^2$
- $b_3(\theta) = -\frac{125}{192}\theta^3 - \frac{125}{128}\theta^2$
- $b_4(\theta) = \theta^3 - \theta^2$