Numerical methods for dynamical systems

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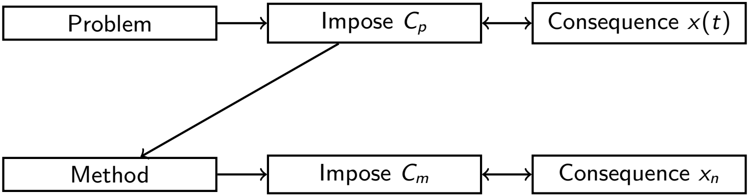
2020-2021

Part V

Stability analysis

Part 5. Section 1
Introduction to stability of numerical methods

Note: there are several kinds of stability.



From a generic point of view we have:

- Impose a certain conditions C_p on IVP which force the exact solution $x(t)$ to exhibit a certain stability
- Apply a numerical method on IVP
- **Question:** what conditions must be imposed on the method such that the approximate solution $(x_n)_{n\in\mathbb{N}}$ has the same stability property?

Consider, a perturbed IVP

$$\dot{\mathbf{y}} = f(t, \mathbf{y}) + \delta(t) \qquad \text{with} \quad \mathbf{y}(0) = \mathbf{y}_0 + \delta_0 \quad \text{and} \quad t \in [0, b]$$

$(\delta(t), \delta_0)$ denotes the perturbations

Definition: totally stable IVP

From

- $(\delta(t), \delta_0)$ and $(\delta^*(t), \delta_0^*)$ two perturbations
- $\mathbf{y}(t)$ and $\mathbf{y}^*(t)$ the associated solutions

if

$$\forall t \in [0, b], \forall \varepsilon > 0, \exists K > 0,$$
$$\| \delta(t) - \delta^*(t) \| \leq \varepsilon \wedge \| \delta_0 - \delta_0^* \| \leq \varepsilon \Rightarrow \| \mathbf{y}(t) - \mathbf{y}^*(t) \| \leq K\varepsilon$$

then IVP is **totally stable**.

We consider the application of numerical method on a perturbed IVP so we have a perturbed numerical scheme

Definition: zero-stability

From

- δ_n and δ_n^* two discrete-time perturbation
- $\mathbf{y}_n$ and $\mathbf{y}_n^*$ the associated numerical solution

if

$$\forall n \in [0, N], \forall \varepsilon > 0, \exists K > 0, \forall h \in (0, h_0] \quad \| \delta_n - \delta_n^* \| \leq \varepsilon \Rightarrow \| \mathbf{y}_n - \mathbf{y}_n^* \| \leq K\varepsilon$$

then the method is **zero-stable**

In a different point of view, we want to solve $\dot{y} = 0$ with $y(0) = y_0$ and so numerical method should produce as a solution $y(t) = y_0$. (It is obvious for RK methods)

Zero stability for multi-step methods

First and second characteristic polynomials for linear multi-step methods are

$$\rho(z) = \sum_{i=0}^{k} \alpha_i z^i \qquad \text{and} \qquad \sigma(z) = \sum_{i=0}^{k} \beta_i z^i$$

Root condition

A linear multi-step method satisfies the **root condition** is the roots of the first characteristic polynomial ρ have modulus less than or equal to one and those of modulus one are simple.

Theorem

A multi-step method is zero stable is it satisfies the root condition.

Theorem

No zero-stable linear k-step method can have order exceeding $k+1$

Consistency of numerical methods

We denote by $\Phi_f(t_n, \mathbf{y}_n; h)$ a Runge-Kutta method such that

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\Phi_f(t_n, \mathbf{y}_n; h)$$

If Φ_f is such that

$$\lim_{h \to 0} \Phi_f(t_n, \mathbf{y}_n; h) = f(t_n, \mathbf{y}_n) \ .$$

then the Runge-Kutta method is **consistent** to the IVP.

As a consequence, the truncation error is such that:

$$\lim_{h \to 0} \mathbf{y}(t_{n+1}) - \mathbf{y}_n - h\Phi_f(t_n, \mathbf{y}_n; h) = 0$$

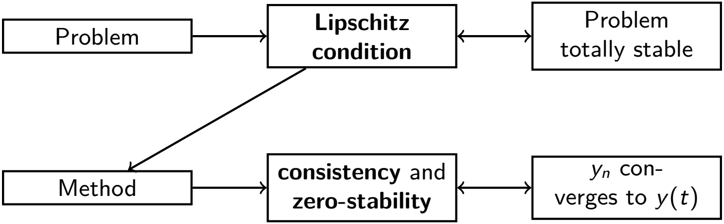
Consistency for s-stage RK methods

A necessary and sufficient condition is that

$$\sum_{i=1}^{s} b_i = 1$$

A Runge-Kutta method is said **convergent** if

$$\lim_{h \to 0} \mathbf{y}_n = \mathbf{y}(t_n)$$



Part 5. Section 2

Linear stability analysis for one-step methods

We consider the IVP:

$$\dot{y} = \lambda y \quad \text{with} \quad \lambda \in \mathbb{C}, \Re(\lambda) < 0$$

Applying a RK method, we get

$$y_{n+1} = R(\hat{h}) y_n \quad \text{with} \quad \hat{h} = \lambda h$$

$R(\hat{h})$ is called the *stability function* of the method.

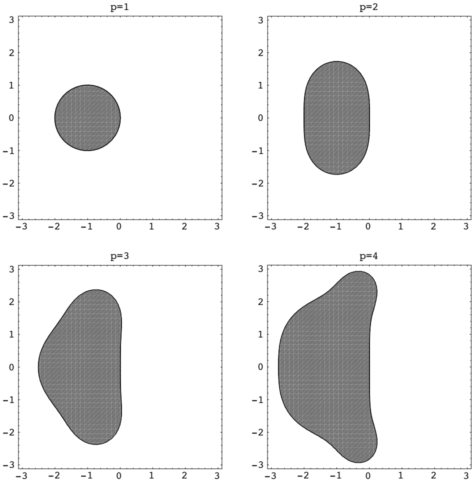
Stability function of RK methods

$$R(\hat{h}) = \frac{\det\left(I - \hat{h}A + \hat{h}\mathbb{1}b^t\right)}{\det\left(I - \hat{h}A\right)}$$

So, $\lim_{n\to\infty} x_n = 0$ when $|R(\hat{h})| < 1$

The stability function for s-stage ($s = 1, 2, 3, 4 \Rightarrow p = s$) ERK is reduced to a polynomial function:

$$R(\hat{h}) = 1 + \hat{h} + \frac{1}{2!}\hat{h}^2 + \cdots + \frac{1}{s!}\hat{h}^s$$



The stability function for s-stage ($s > 4 \Rightarrow p < s$) ERK is reduced to a polynomial function:

$$R(\hat{h}) = 1 + \hat{h} + \frac{1}{2!}\hat{h}^2 + \cdots + \frac{1}{p!}\hat{h}^p + \sum_{q=p+1}^{s} \gamma_q \hat{h}^q$$

with γ_q depending only on the coefficients of the ERK methods.

For example,

- for RKF45 ($s = 5$ and $p = 4$)

$$R(\hat{h}) = 1 + \hat{h} + \frac{1}{2!}\hat{h}^2 + \frac{1}{6}\hat{h}^3 + \frac{1}{24}\hat{h}^4 + \frac{1}{104}\hat{h}^5$$

- DOPIR54 ($s = 6$ and $p = 5$)

$$R(\hat{h}) = 1 + \hat{h} + \frac{1}{2!}\hat{h}^2 + \frac{1}{6}\hat{h}^3 + \frac{1}{24}\hat{h}^4 + \frac{1}{120}\hat{h}^5 + \frac{1}{600}\hat{h}^6$$

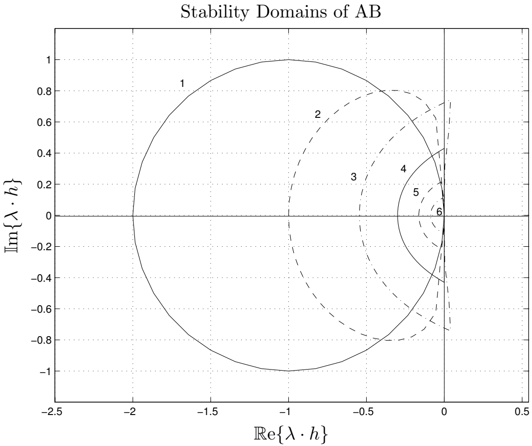
Part 5. Section 3
Linear stability analysis for multi-step methods

We consider the scalar linear IVP

$$\dot{y} = \lambda y \quad \text{with} \quad \lambda \in \mathbb{C}, \Re(\lambda) < 0$$

For linear problem, the **stability polynomial** of a multi-step method is

$$\pi(r, \hat{h}) = \rho(r) - \hat{h}\sigma(r) \quad \text{with} \quad \hat{h} = \lambda h$$



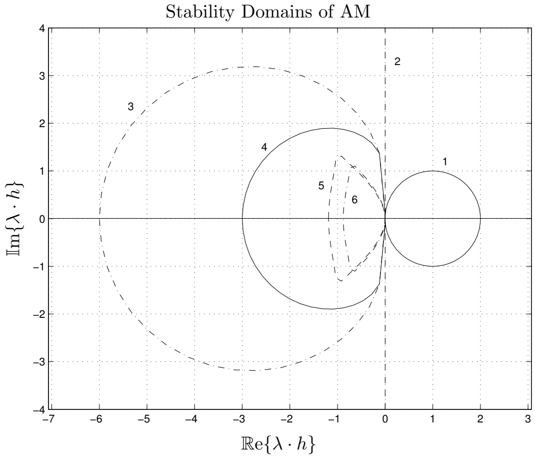
Linear stability of Adams-Moulton methods

We consider the scalar linear IVP

$$\dot{y} = \lambda y \quad \text{with} \quad \lambda \in \mathbb{C}, \Re(\lambda) < 0$$

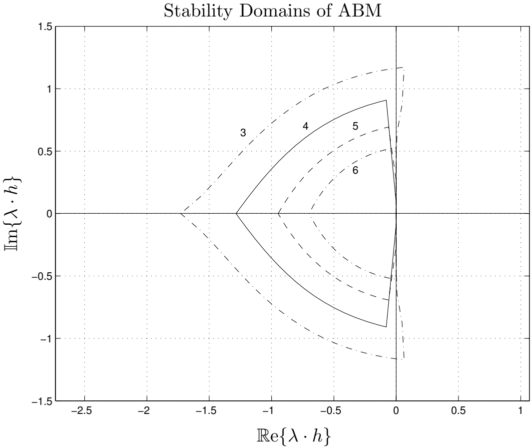
For linear problem, the **stability polynomial** of a multi-step method is

$$\pi(r, \hat{h}) = \rho(r) - \hat{h}\sigma(r) \quad \text{with} \quad \hat{h} = \lambda h$$



We consider the IVP:

$$\dot{x} = \lambda x \quad \text{with} \quad \lambda \in \mathbb{C}, \Re(\lambda) < 0$$

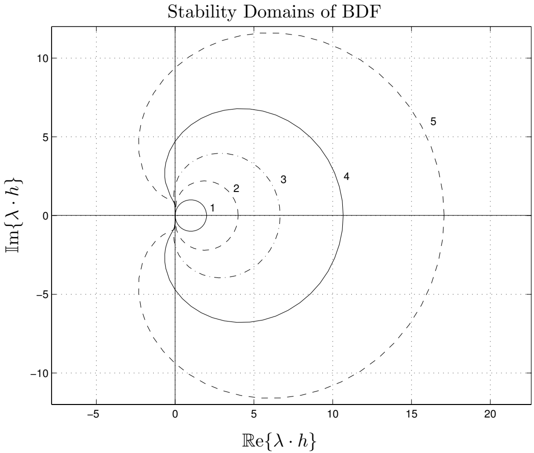


We consider the scalar linear IVP

$$\dot{y} = \lambda y \quad \text{with} \quad \lambda \in \mathbb{C}, \Re(\lambda) < 0$$

For linear problem, the **stability polynomial** of a multi-step method is

$$\pi(r, \hat{h}) = \rho(r) - \hat{h}\sigma(r) \quad \text{with} \quad \hat{h} = \lambda h$$



Part 5. Section 4
Stiffness

Problem 1

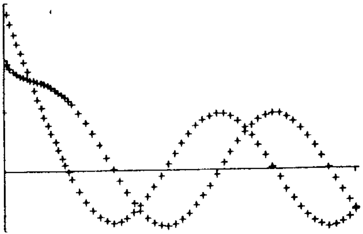
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2\sin(t) \\ 2(\cos(t) - sin(t)) \end{pmatrix}$$

Problem 2

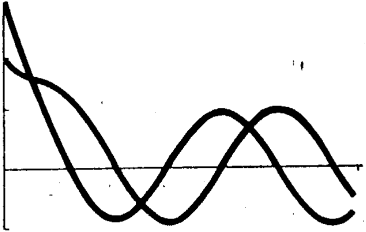
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 998 & -999 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2\sin(t) \\ 999(\cos(t) - sin(t)) \end{pmatrix}$$

Both have the same exact solution:

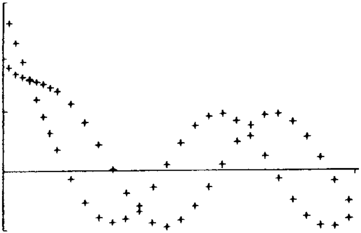
$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = 2\exp(-t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} \quad \text{with initial values} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$



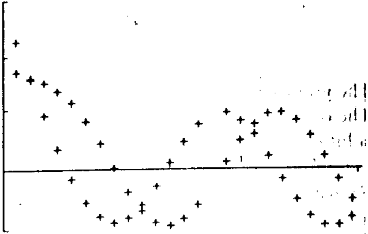
(b) Problem 1, RKF45; $N = 60$.



(c) Problem 2, RKF45; $N = 3373$.



(d) Problem 1, 2-stage Gauss; $N = 29$.



(e) Problem 2, 2-stage Gauss; $N = 24$.

We consider linear constant coefficients IVP of the form:

$$\dot{\mathbf{y}} = A\mathbf{y} + \phi(t)$$

assuming that all eigenvalues λ are such that $\Re(\lambda) < 0$

We denote by

- $\mid \Re(\overline{\lambda}) \mid = \max_{1 \le i \le n} \mid \Re(\lambda_i) \mid$
- $\mid \Re(\underline{\lambda}) \mid = \min_{1 \le i \le n} \mid \Re(\lambda_i) \mid$
- the **stiffness ratio** is defined by $\mid \Re(\overline{\lambda}) \mid / \mid \Re(\underline{\lambda}) \mid$

Stiffness definition - 1 (Lambert)

A linear constant coefficients system is stiff iff all eigenvalues are such that $\Re(\lambda) < 0$ and the stiffness ratio is large.

Others stiffness definitions

Definition 2 (Lambert)

Stiffness occurs when stability requirements, rather than those of accuracy, constrain the step size.

Definition 3 (Lambert)

Stiffness occurs when some components of the solution decay much more quickly than others.

Global definition (Lambert)

If a numerical method with a finite region of absolute stability, applied to a system with any initial values, if forced to use in a certain interval of integration a step size which is excessively small in relation to the smoothness of the exact solution in that interval, then the system is said to be **stiff** in that interval.

A-stability

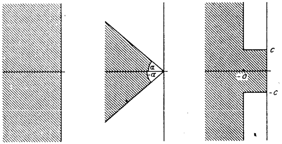
A method is **A-stable** if $\mathcal{R}_s \supseteq \{\hat{h} : \Re(\hat{h}) < 0\}$

$A(\alpha)$-stability

A method is $A(\alpha)$**-stable**, $\alpha \in]0, \pi/2[$, if $\mathcal{R}_s \supseteq \{\hat{h} : -\alpha < \pi - \arg(\hat{h}) < \alpha\}$

Stiffly stability

A method is **stiffly stable** if $\mathcal{R}_S \supseteq \mathcal{R}_1 \cup \mathcal{R}_2$ such that $\mathcal{R}_1 = \{\hat{h} : \Re(\hat{h}) < -a\}$ and $\mathcal{R}_2 = \{\hat{h} : -a \leq \Re(\hat{h}) \leq 0, -c \leq \Im(\hat{h}) \leq c\}$ with a and c two positive real numbers.



Linear stability definition for stiff systems - 2

L-stability

A one step method is ***L*-stable** if

- it is *A*-stable
- and when applied to stable scalar test equations $\dot{y} = \lambda y$ it yields

$$y_{n+1} = \Re(h\lambda)x_n \quad \text{where} \quad |\Re(h\lambda)| \rightarrow 0 \text{ as } \Re(h\lambda) \rightarrow -\infty$$

Relation between the stability definitions

L-stability ⇒ *A*-stability ⇒ stiffly stability ⇒ $A(\alpha)$-stability

Numerical methods for linear stiff problems

Runge-Kutta methods

Method	Order	Linear stability prop.
Gauss	$2s$	A-stability
Radau IA, IIA	$2s-1$	L-stability
Lobatto IIIA, IIIB	$2s-2$	A-stability
Lobatto IIIC	$2s-2$	L-stability

Theorems (Dahlquist barrier)

- Explicit RK cannot be A-stability or stiffly stability or $A(\alpha)$-stability!
- Explicit linear multi-step method cannot be A-stable
- The order of an A-stable linear multi-step method cannot exceed 2
- The second order A stable multi-step method with the smallest error constant (C_3) is the Trapezoidal rule.

For the particular case of BDF

- BF1 and BDF2 are L-stable
- other BDF(3-4-5-6) are $A(\alpha)$-stable
- BF6 has a very narrow stability area, it is not used in practice