ENSTA Paris
2020-2021

Numerical methods for dynamical systems

Homework n° 4

Goal(s)

- ★ Implementation of one-step methods for discontinuous dynamocal systems
- ★ Simulation of a few number of systems

For the next exercises, we will consider different dynamical systems to test the method. In particular, we will consider

— F4

$$\dot{y} = \begin{cases} -\frac{2}{21} - \frac{120(t-5)}{1+4(t-5)^2} & \text{if} \quad t \leqslant 10 \\ -2y & \text{if} \quad t > 10 \end{cases}$$

with $y(0) = 1.0$ and the simulation end time is 20 seconds.

— Boucing ball

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -9.81 \end{aligned} \quad \text{if} \quad (y_1 \leqslant 0 \wedge y_2 < 0) \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.0 \\ -0.8y_2 \end{pmatrix}$$

with $y_1(0) = 10$ and $y_2(0) = 15.0$. **Warning :** in this system after detecting the event it is necessary to apply a reset function which will change the state vectors. Final simulation time is 20 (or less)

— F1

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = \begin{cases} 2ay_2 - (\pi^2 + a^2)y_1 + 1 & \text{if} \quad \lfloor t \rfloor \quad \text{is even} \\ 2ay_2 - (\pi^2 + a^2)y_1 - 1 & \text{if} \quad \lfloor t \rfloor \quad \text{is odd} \end{cases}$$

with $a = 0.1$, $y_1(0) = 0$ and $y_2(0) = 0$, the final simulation time is 20 seconds.

We recall in Algorithm 1

Data : f_1 the dynamic, f_2 the dynamic, g the zero-crossing function, $\mathbf{y}_0$ initial condition, t_0 starting time, t_{end} end time, h integration step-size, tol

```
t ← t0;
y ← y0;
f ← f1;
while t < t_end do
    Print(t, y);
    y1 ← Euler(f,t,y,h);
    y2 ← Heun(f,t,y,h);
    if ComputeError(y1,y2) is smaller than tol then
        if g(y) · g(y1) < 0 then
            Compute p(t) from y, f(y), y1 and f(y1);
            [t−,t+] = FindZero (g(p(t)));
            Print (t + t−, p(t−));
            f ← f2;
            y ← p(t+);
            t ← t + t+;
        end
        y ← y1;
        t ← t + h;
        h ← ComputeNewH (h,y1,y2);
    end
    h ← h/2
end
```

Algorithme 1 : Pseudo code of simulation engine with one-step variable step-size method with discontinuous dynamical systems

Exercise 1 – Implementation

Question 1

We consider a second adaptive Runge-Kutta method which is the Bogacki-Shampine method. Its Butcher tableau is

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{3}{4} & 0 & \frac{3}{4} & & \\
1 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \\
\hline
 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \\
 & \frac{7}{24} & \frac{1}{4} & \frac{1}{3} & \frac{1}{8}
\end{array}$$

The integration methods is defined by

$$\begin{aligned}
\mathbf{k}_1 &= f(t_n, \mathbf{y}_n) \\
\mathbf{k}_2 &= f(t_n + \frac{1}{2}h_n, \mathbf{y}_n + \frac{1}{2}h_n\mathbf{k}_1) \\
\mathbf{k}_3 &= f(t_n + \frac{3}{4}h_n, \mathbf{y}_n + \frac{3}{4}h_n\mathbf{k}_2) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + h_n\left(\frac{2}{9}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{4}{9}\mathbf{k}_3\right) \\
\mathbf{k}_4 &= f(t_n + h_n, \mathbf{y}_{n+1}) \\
\mathbf{z}_{n+1} &= \mathbf{y}_n + h_n\left(\frac{7}{24}\mathbf{k}_1 + \frac{1}{4}k_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{8}\mathbf{k}_4\right)
\end{aligned}$$

Implement this method (with the computeNewH method)

Question 2

Implement of method to compute a polynomial interpolation of the solution $y(t)$ by using Hermite's Cubic Splines

`https://en.wikipedia.org/wiki/Cubic_Hermite_spline`

Question 3

Implement a method to search for zeros of univariate functions. In particular, we will use Secant method defined by

$$x_{i+1} = x_i + \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} f(x_i) \quad .$$

It requieres two initial values x_0 and x_1 which shall enclose the solution.

Question 4

Solve the problems with your simulation engine.

TO SUBMIT

— A small report should be sent summarize the answers to the questions.
— This report should be associated to the source code.

Send the archive containing the report and the source codes in a mail which title is

[numerical methods for dynamical systems] FIRSTNAME LASTNAME

to `alexandre.chapoutot@ensta-paris.fr`
before the next lecture, Friday October 16, 2020.