Exercice 9.4: Variations sur l'intégrale de Gauss

(1) $G_1 = \displaystyle\int_0^{+\infty} e^{-\frac{x^2}{2}}\ \mathrm{d}x = \frac{\sqrt{2\pi}}{2}.$

$G_2 = \dfrac{1}{\sqrt{2\pi}} \cdot \displaystyle\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\ \mathrm{d}x$

Soit $g(x) = e^{-\frac{x^2}{2}}$.

$\forall x \in \mathbb{R},\ g(-x) = gx)$ donc g est une fonction paire.

Ainsi:

$$\int_0^{+\infty} g(x)\ \mathrm{d}x = \int_{-\infty}^{0} g(x)\ \mathrm{d}x = \frac{\sqrt{2\pi}}{2}$$

Donc:

$$\int_{-\infty}^{+\infty} g(x)\ \mathrm{d}x = \int_{-\infty}^{0} g(x)\ \mathrm{d}x + \int_0^{+\infty} g(x)\ \mathrm{d}x = \frac{\sqrt{2\pi}}{2} + \frac{\sqrt{2\pi}}{2} = \sqrt{2\pi}$$

Donc

$$G_2 = 1$$

(2) Soit

$$\forall x \in \mathbb{R},\ \Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\ \mathrm{d}t$$

(a) Φ est une primitive de g donc Φ est dérivable et :

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} \cdot g(x).$$

(b)

$$\begin{aligned} \Phi(0) = &\ \frac{1}{\sqrt{2\pi}} \cdot \textstyle\int_{-\infty}^{0} e^{-\frac{t^2}{2}}\ \mathrm{d}t \\ = &\ \frac{1}{\sqrt{2\pi}} \cdot \textstyle\int_{-\infty}^{0} g(t)\ \mathrm{d}t \\ \Rightarrow \Phi(0) = &\ \frac{1}{2} \end{aligned}$$

(c)

$$\begin{aligned}
\Phi(-x) = &\ \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-x} e^{-\frac{t^2}{2}}\ \mathrm{d}t \\
u = -t \Rightarrow &\ u : +\infty \to x \\
&\ \mathrm{d}u = -\mathrm{d}t \\
&\ e^{-\frac{t^2}{2}}\ \mathrm{d}t = -e^{-\frac{u^2}{2}}\ \mathrm{d}u \\
\Rightarrow \Phi(-x) = &\ -\frac{1}{\sqrt{2\pi}} \cdot \int_{+\infty}^{+x} e^{-\frac{t^2}{2}}\ \mathrm{d}t \\
= &\ \frac{1}{\sqrt{2\pi}} \cdot \int_{x}^{+\infty} e^{-\frac{t^2}{2}}\ \mathrm{d}t
\end{aligned}$$

Ainsi:

$$\begin{aligned}
\Phi(x) + \Phi(-x) = &\ \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\ \mathrm{d}t + \frac{1}{\sqrt{2\pi}} \cdot \int_{x}^{+\infty} e^{-\frac{t^2}{2}}\ \mathrm{d}t \\
= &\ \frac{1}{\sqrt{2\pi}} \cdot \left(\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\ \mathrm{d}t + \int_{x}^{+\infty} e^{-\frac{t^2}{2}}\ \mathrm{d}t \right) \\
= &\ \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}}\ \mathrm{d}t
\end{aligned}$$

Donc:

$$\Phi(x) + \Phi(-x) = G_2 = 1$$