Exercice 9.3: Intégrales généralisées
Soit

$$f(x) = \begin{cases} x+1 & -1 \leq x \leq 0 \\ 1-x & 0 < x \leq 1 \\ 0 & x \notin [-1,1] \end{cases}$$

(1) $I = \displaystyle\int_{-\infty}^{+\infty} f(x)\ \mathrm{d}x$.

Soit $c > 1$ et:

$$\begin{aligned} I(c) &= \int_{-c}^{+c} f(x)\ \mathrm{d}x \\ &= \int_{-c}^{-1} f(x)\ \mathrm{d}x + \int_{-1}^{0} f(x)\ \mathrm{d}x + \int_{0}^{1} f(x)\ \mathrm{d}x + \int_{1}^{c} f(x)\ \mathrm{d}x \\ &= 0 + \int_{-1}^{0} (x+1)\ \mathrm{d}x + \int_{0}^{1} (1-x)\ \mathrm{d}x + 0 \\ &= \left[\frac{(x+1)^2}{2}\right]_{-1}^{0} + \left[\frac{-(1-x)^2}{2}\right]_{0}^{1} \\ &= 1 \end{aligned}$$

Donc

$$\lim_{c \to +\infty} I(c) = 1 \in \mathbb{R}$$

Donc I converge et $I = 1$.

(2) $\forall x \in \mathbb{R},\ F(x) = \displaystyle\int_{-\infty}^{x} f(t)\ \mathrm{d}t.$

Donc:

$$
\begin{aligned}
x < -1 \Rightarrow \quad F(x) &= 0 \\
-1 \leq x \leq 0 \Rightarrow \quad F(x) &= \int_{-1}^{x} (t+1)\ \mathrm{d}t \\
&= [\frac{(t+1)^2}{2}]_{-1}^{x} \\
&= \frac{(x+1)^2}{2} \\
0 \leq x \leq 1 \Rightarrow \quad F(x) &= \int_{-1}^{0} (t+1)\ \mathrm{d}t + \int_{0}^{x} (1-t)\ \mathrm{d}t \\
&= [\frac{(t+1)^2}{2}]_{-1}^{0} + [\frac{-(1-t)^2}{2}]_{0}^{x} \\
&= \frac{1}{2} - \frac{(1-x)^2}{2} + \frac{1}{2} \\
&= 1 - \frac{(1-x)^2}{2} \\
x > 1 \Rightarrow \quad F(x) &= I = 1
\end{aligned}
$$