Exercice 8.2**: Intégrales g/'enéralisées**

Soit

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{sinon} \end{cases}$$

(1) $I_1 = \displaystyle\int_{-\infty}^{+\infty} f(x)\ \mathrm{d}x$.
Soit $c > \max(|a|, |b|)$ ainsi $-c < a < b < c$ et:

$$\begin{aligned} I_1(c) &= \int_{-c}^{+c} f(x)\ \mathrm{d}x \\ &= \int_{-c}^{a} f(x)\ \mathrm{d}x + \int_{a}^{b} f(x)\ \mathrm{d}x + \int_{b}^{+c} f(x)\ \mathrm{d}x \\ &= 0 + \int_{a}^{b} f(x)\ \mathrm{d}x + 0 \\ &= \int_{a}^{b} \frac{1}{b-a}\ \mathrm{d}x \\ &= \left[\frac{x}{b-a}\right]_a^b \\ &= 1 \end{aligned}$$

Donc

$$\lim_{c \to +\infty} I_1(c) = 1 \in \mathbb{R}$$

Donc I_1 converge et $I_1 = 1$.

(2) $I_2 = \displaystyle\int_{-\infty}^{+\infty} x \cdot f(x)\ \mathrm{d}x$.
Soit $c > \max(|a|, |b|)$ et:

$$\begin{aligned} I_2(c) &= \int_{-c}^{+c} x \cdot f(x)\ \mathrm{d}x \\ &= \int_{-c}^{a} x \cdot f(x)\ \mathrm{d}x + \int_{a}^{b} x \cdot f(x)\ \mathrm{d}x + \int_{b}^{+c} x \cdot f(x)\ \mathrm{d}x \\ &= 0 + \int_{a}^{b} x \cdot f(x)\ \mathrm{d}x + 0 \\ &= \int_{a}^{b} \frac{x}{b-a}\ \mathrm{d}x \\ &= \left[\frac{x^2}{2 \cdot (b-a)}\right]_a^b \\ &= \frac{b+a}{2} \end{aligned}$$

Donc

$$\lim_{c \to +\infty} I_2(c) = \frac{b+a}{2} \in \mathbb{R}$$

Donc I_2 converge et $I_2 = \dfrac{b+a}{2}$.

(3) $I_3 = \displaystyle\int_{-\infty}^{+\infty} x^2 \cdot f(x)\ \mathrm{d}x$.
Soit $c > \max(|a|, |b|)$ et:

$$\begin{aligned} I_3(c) &= \int_{-c}^{+c} x^2 \cdot f(x)\ \mathrm{d}x \\ &= \int_{-c}^{a} x^2 \cdot f(x)\ \mathrm{d}x + \int_{a}^{b} x^2 \cdot f(x)\ \mathrm{d}x + \int_{b}^{+c} x^2 \cdot f(x)\ \mathrm{d}x \\ &= 0 + \int_{a}^{b} x^2 \cdot f(x)\ \mathrm{d}x + 0 \\ &= \int_{a}^{b} \frac{x^2}{b-a}\ \mathrm{d}x \\ &= \left[\frac{x^3}{3 \cdot (b-a)}\right]_a^b \\ &= \frac{b^2 + a \cdot b + a^2}{3} \end{aligned}$$

Donc

$$\lim_{c \to +\infty} I_3(c) = \frac{b^2 + a \cdot b + a^2}{3} \in \mathbb{R}$$

Donc I_3 converge et $I_3 = \dfrac{b^2 + a \cdot b + a^2}{3}$.

(4) $\forall x \in \mathbb{R},\ F(x) = \displaystyle\int_{-\infty}^{x} f(t)\ \mathrm{d}t$.

Donc:

$$\begin{aligned}
x < a \Rightarrow \quad & F(x) &&= 0 \\
a \leq x \leq b \Rightarrow \quad & F(x) &&= \int_a^x \frac{1}{b-a}\ \mathrm{d}t \\
& &&= \frac{x-a}{b-a} \\
x > b \Rightarrow \quad & F(x) &&= 1
\end{aligned}$$