Exercice 10.4**: Intégration**

Soit:

$$I = \int_1^{+\infty} \frac{\ln x}{1+x^2}\,\mathrm{d}x,\ \ J = \int_0^{+\infty} \left|\frac{\ln x}{1+x^2}\right|\,\mathrm{d}x \text{ et } K = \int_0^{+\infty} \frac{\ln x}{1+x^2}\,\mathrm{d}x.$$

(1) Posons

$$\begin{aligned} u = \frac{1}{x} \longrightarrow\ & \text{bornes: } u : \frac{1}{1} = 1 \to \frac{1}{+\infty} = 0, \\ & du = -\frac{1}{x^2}\mathrm{d}x, \\ & \frac{\ln x}{1+x^2}\,\mathrm{d}x = \frac{\ln x}{1+\dfrac{1}{x^2}}\,\frac{\mathrm{d}x}{x^2} = \frac{\ln \frac{1}{u}}{1+u^2}\,(-\mathrm{d}u) = \frac{\ln u}{1+u^2}\mathrm{d}u \end{aligned}$$

Donc:

$$\begin{aligned} I &= \int_1^0 \frac{\ln u}{1+u^2}\mathrm{d}u \\ I &= -\int_0^1 \frac{\ln u}{1+u^2}\mathrm{d}u \end{aligned}$$

(2) $\forall x \in [0;+\infty[$,

$$\begin{aligned} \left|\frac{\ln x}{1+x^2}\right| &= \frac{|\ln x|}{1+x^2} \qquad \text{car } 1+x^2 > 0 \\ &= \begin{cases} -\dfrac{\ln x}{1+x^2}, & \text{si } 0 \le x \le 1 \\ \dfrac{\ln x}{1+x^2}, & \text{si } 1 \le x \end{cases} \end{aligned}$$

Donc:

$$\begin{aligned} J = \int_0^{+\infty} \left|\frac{\ln x}{1+x^2}\right|\,\mathrm{d}x &= \int_0^1 \left|\frac{\ln x}{1+x^2}\right|\,\mathrm{d}x + \int_1^{+\infty} \left|\frac{\ln x}{1+x^2}\right|\,\mathrm{d}x \\ &= \int_0^1 -\frac{\ln x}{1+x^2}\,\mathrm{d}x + \int_1^{+\infty} \frac{\ln x}{1+x^2}\,\mathrm{d}x \\ &= I + I \text{ d'aprés } (1) \\ &= 2 \cdot I \end{aligned}$$

(3)

$$\begin{aligned} K = \int_0^{+\infty} \frac{\ln x}{1+x^2}\ \mathrm{d}x &= \int_0^1 \frac{\ln x}{1+x^2}\ \mathrm{d}x + \int_1^{+\infty} \frac{\ln x}{1+x^2}\ \mathrm{d}x \\ &= -I + I \text{ d'aprés } \ (1) \\ &= 0 \end{aligned}$$