Exercice 10.3
Soit:

$$I_n = \int_0^1 x^n \cdot \ln(1+x)\ \mathrm{d}x$$

(1)

$$\begin{aligned} I_0 = &\ \int_0^1 1 \cdot \ln(1+x)\ \mathrm{d}x \\ = &\ [x \cdot \ln(x+1)]_0^1 - \int_0^1 \frac{x}{1+x}\ \mathrm{d}x \\ = &\ \ln 2 - \int_0^1 \frac{x}{1+x}\ \mathrm{d}x \end{aligned}$$

Or:

$$\frac{x}{1+x} = \frac{x+1-1}{1+x} = 1 - \frac{1}{1+x}$$

Donc:

$$\begin{aligned} I_0 = &\ \ln 2 - \int_0^1 (1 - \frac{1}{1+x})\ \mathrm{d}x \\ = &\ \ln 2 - [x - \ln(x+1)]_0^1 \\ \text{Donc: } I_0 = &\ 2\ln 2 - 1 \end{aligned}$$

(2)

$$\begin{aligned} I_n = &\ \int_0^1 x^n \cdot \ln(1+x)\ \mathrm{d}x \\ = &\ [\frac{x^{n+1}}{n+1} \cdot \ln(1+x)]_0^1 - \int_0^1 \frac{x^{n+1}}{n+1} \cdot \frac{1}{x+1}\ \mathrm{d}x \\ = &\ \frac{\ln 2}{n+1} - \frac{1}{n+1} \cdot \int_0^1 \frac{x^{n+1}}{x+1}\ \mathrm{d}x \end{aligned}$$

(3)

$$\frac{x^2}{x+1} = \frac{x^2+1-1}{x+1} = x - 1 + \frac{1}{x+1}$$

(4)

$$
\begin{aligned}
I_1 &= \frac{\ln 2}{2} - \frac{1}{2} \cdot \int_0^1 \frac{x^2}{x+1}\ \mathrm{d}x \\
&= \frac{\ln 2}{2} - \frac{1}{2} \cdot \int_0^1 (x - 1 + \frac{1}{x+1})\ \mathrm{d}x \text{ d'aprés (2) avec } n = 1 \\
&= \frac{\ln 2}{2} - \frac{1}{2} \cdot [\frac{x^2}{2} - x + \ln(1+x)]_0^1 \\
I_1 &= \frac{1}{4}
\end{aligned}
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