THE STABILITY OF A HOMOPOLAR DYNAMO

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ABSTRACT. The stability of the self-exciting disk dynamo is considered. If there is no electrical load in parallel with the field coil and no friction at the axle, the dynamo can perform oscillations of constant amplitude about its state of steady motion. Viscous forces on the axle cause the dynamo to settle to a steady motion. A parallel load may have an effect similar to that of viscous forces or may cause the oscillations to grow without limit. With both a parallel load and viscous forces the amplitude of the oscillations is bounded.

Possible analogies between these results and the motion of a body of electrically conducting fluid in a magnetic field are discussed. The main applications are astronomical and geophysical.

1. *Introduction*. A homopolar dynamo is one in which a conductor moves steadily in a constant magnetic field and produces a direct current without the use of a commutator. The simplest example is a disk that rotates about its axis in a constant field parallel to the axis. Current can be drawn from two brushes, one of which rubs on the periphery of the disk and the other on the axle about which the disk rotates. If the current from the brushes is passed through a coil around the disk, as in Fig. 1, and itself produces the magnetic field, the dynamo is said to be 'self-exciting'. If a self-exciting dynamo rotates sufficiently rapidly, the state with no current is unstable and the smallest initial magnetic field or current grows.

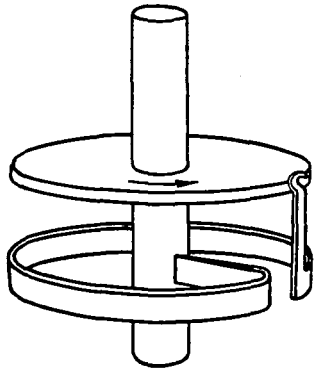

Fig. 1. Disk dynamo.

Homopolar dynamos are of little technical importance, since their e.m.f. is small and they only produce useful amounts of power when the current through the brushes is inconveniently large. There has, however, been a revival of interest in such systems recently, since they are analogous to the dynamos that have been supposed to be the cause of the magnetic fields of the earth, the sun and the stars. The theory of these 'homogeneous' dynamos is complicated, and even their properties in the steady state are imperfectly understood. Nothing has been published on their stability or their

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motion when disturbed, and an investigation of these questions would be of considerable difficulty. It seems likely that the behaviour of the disk dynamo may throw light on that of the more complicated systems; it has therefore been studied in some detail. In some circumstances it can have an unstable state of steady motion about which an infinitesimal disturbance will cause it to perform finite oscillations giving short bursts of current separated by much longer periods during which the current is small. The oscillations may settle to a steady amplitude or grow without limit in spite of ohmic dissipation.

In § 2 the axle is supposed to be devoid of friction and the dynamo to have no external electrical load. In § 3 friction is added and in § 4 a load. In § 5 the application of the results to homogeneous dynamos and to geophysical and astronomical problems is considered.

The phenomena described in this paper cannot be conveniently investigated in the laboratory, since the angular velocities required are excessive for systems of practicable dimensions. They could be reduced by using iron in the magnetic circuit, as is done in engineering dynamos, but this introduces hysteresis and other factors which would complicate the behaviour. It would also be difficult to make friction negligible and to provide a driving couple independent of speed of rotation.

2. *Frictionless dynamo.* Suppose the magnetic field, H, normal to the disk to be symmetrical about the axis of rotation. The induced e.m.f. is then radial and the potential difference between the brushes, V, is

$$V = \omega \int_b^a Hr\,dr,$$

where ω is the angular velocity of the disk, b is the radius of the axle and a that of the disk.*

If L is the self-inductance and R the resistance of the circuit, the current is given by

$$L\dot{I} + RI = V, \tag{1}$$

where the dot implies differentiation with respect to the time. The electromagnetic couple opposing the rotation of the disk is $I \int Hr\,dr$, and its equation of motion is

$$C\dot{\omega} = G - I \int_b^a Hr\,dr, \tag{2}$$

where C is the moment of inertia of the disk and G is the couple driving it. If the magnetic field is supplied from a steady external source, and not from the coil, the integral is a constant, say F. Elimination of ω then gives

$$LC\ddot{I} + RC\dot{I} + F^2 I = FG.$$

The system therefore performs damped oscillations and settles to a steady state with a constant current G/F and an angular velocity RG/F^2. The time for a disturbance to

* Throughout this paper induction by that part of the field normal to the disk due to the current passing through the disk is ignored. This is not justified for the system shown in Fig. 1, but would be if current were drawn from a large number of brushes spaced round the periphery of the disk or if the disk consisted of closely spaced radial spokes insulated from each other.

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fall to $1/e$ of its initial value is $2L/R$ and, if the damping is small, the period of the oscillations is $2\pi(LC)^{\frac{1}{2}}/F$.

If the dynamo is self-exciting, H and F are no longer constants, but are proportional to I. If the mutual inductance between the coil and the periphery of the disk is $2\pi M$,

$$F = \int_b^a Hr\,dr = MI = \tfrac{1}{2}(a^2-b^2)\,H_m, \tag{3}$$

where H_m is the mean field perpendicular to the disk and M is taken as positive when a positive ω produces an e.m.f. tending to cause a current that increases H. (1) then becomes

$$L\dot{I}+RI = M\omega I, \tag{4}$$

where L and R are now the inductance and resistance of the circuit consisting of the disk, brushes and coil. If the angular velocity is fixed, the current is given by

$$I = I_0\,e^{(M\omega-R)t/L},$$

where I_0 is the current when t is zero. If ω is less than R/M, the current decays to zero and the dynamo is not self-exciting; if ω is greater than R/M, the current increases without limit, no matter how small the initial current and field may be.

The dynamo driven at constant speed is of little interest for the study of stability, since only the current can change. If a constant driving couple is considered instead of a constant speed, a greater variety of behaviour is possible. The equation of motion is then

$$C\dot{\omega} = G - MI^2. \tag{5}$$

If the disk is started with a specified angular velocity ω_0 and a specified current I_0 at zero time, the subsequent motion and current are given by the solution of (4) and (5) with these initial conditions.

There is a steady state with angular velocity ω_c and current I_c given by

$$\omega_c = R/M, \quad I_c = \sqrt{(G/M)}. \tag{6}$$

The angular velocity ω_c can support any value of the current, and therefore any field, if the couple satisfying (6) is applied. The field and current may be in either direction, but there is only one direction of rotation and of the couple that will give a steady state.

If the initial values of ω and I are not those given by (6), the system will not be in a steady state and both ω and I will change with time. Elimination of ω between (4) and (5) gives

$$\omega = \frac{L}{MI}\frac{dI}{dt}+\frac{R}{M}, \tag{7}$$

$$\frac{d^2}{dt^2}(\ln I) = \frac{GM}{CL}\left(1-\frac{M}{G}I^2\right). \tag{8}$$

If GM is positive, put

$$\tau = \left(\frac{2GM}{CL}\right)^{\frac{1}{2}} t, \tag{9}$$

$$y = \ln(MI^2/G); \tag{10}$$

(8) then gives

$$\frac{d^2y}{d\tau^2} = 1-e^y, \tag{11}$$

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whence, if

$$x = \frac{dy}{d\tau}, \tag{12}$$

$$x^2 = A + 2(y - e^y), \tag{13}$$

where

$$A = x_0^2 - 2(y_0 - e^{y_0}), \tag{14}$$

x_0 and y_0 being the initial values of x and y, which from (7), (9) and (10) are given by

$$y_0 = \ln(MI_0^2/G), \quad x_0 = \left(\frac{2CM}{GL}\right)^{\frac{1}{2}}(\omega_0 - R/M).$$

By suitable choice of I_0 and ω_0, y_0 and x_0 can be made to take any positive or negative values no matter what are the dimension of the system or the driving couple. A can have any value that is not less than 2.

For a given A, (13) gives a closed curve in the x, y plane. These curves are symmetrical about the y axis; they are not symmetrical about the x axis, but the maximum of x is always at $y = 0$. If $A = 2$, the curve shrinks to a point at the origin; if A is slightly greater than 2, y is small and the curves are the circles

$$x^2 + y^2 = A - 2.$$

For A much greater than two, the extreme values of y are approximately $\ln \frac{1}{2}A$ and $-\frac{1}{2}A$; the curves therefore extend further along the negative y axis than they do along the positive axis. These points are illustrated in Fig. 2, which shows the curves for $A = 3, 5$ and 10.

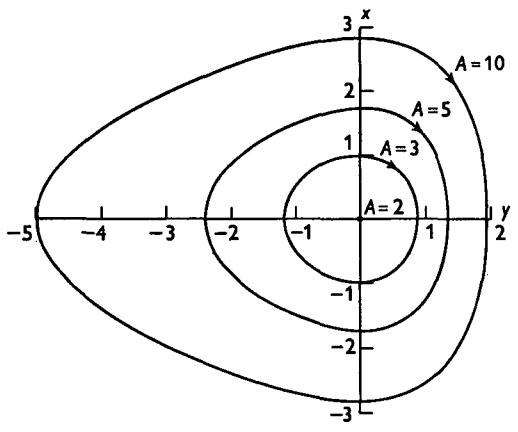

Fig. 2

As the curves are closed, the motion is periodic, but since they are not circles it is not simple harmonic (except for limitingly small amplitudes). The periodic time, T, is from (9) and (12)

$$T = \left(\frac{2CL}{GM}\right)^{\frac{1}{2}} \int_{-y_1}^{y_2} \frac{dy}{x},$$

where $-y_1$ and y_2 are the two values of y for which $x = 0$. Using (13), we find

$$T = \left(\frac{2CL}{GM}\right)^{\frac{1}{2}} \int_{-y_1}^{y_2} \frac{dy}{\sqrt{(A + 2y - 2e^y)}}.$$

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It is easily shown that the limits are the only singularities of the integrand and that the integral converges there. There is therefore a finite period for any A. If A only slightly exceeds its minimum value of 2, the integral can be evaluated and the period is found to be $\pi\sqrt{(2CL/GM)}$. The integral has also been evaluated numerically for $A = 3$, 5 and 10. The results are given in Table 1; they show that the period lengthens as A increases.

Table 1

A	Period*	Time* below critical speed	Time* above critical speed	$\sqrt{(M/G)}\, I$		$\sqrt{(2CM/GL)}\,(\omega-\omega_c)$ Max. and Min.
				Max.	Min.	
2	2π	π	π	1·000	1·000	0·000
3	6·55	3·96	2·59	1·535	0·549	$\pm$ 1·000
5	7·08	4·79	2·29	1·961	0·300	$\pm$ 1·732
10	8·39	6·43	1·96	2·634	0·0824	$\pm$ 2·828
Large	$2\left(\sqrt{A}+\frac{\log 2A}{\sqrt{A}}\right)$	$2\sqrt{A}$	$\frac{2\log 2A}{\sqrt{A}}$	$\sqrt{(\frac{1}{2}A)}$	$\exp(-\frac{1}{4}A)$	$\pm\sqrt{(A-2)}$

* Times in units of τ; to get seconds multiply by $\sqrt{(CL/2GM)}$.

When y is zero, the current has its equilibrium value $\sqrt{(G/M)}$. The variation of y between y_2 and $-y_1$ will cause the current to fluctuate between

$$\sqrt{\left(\frac{G}{M}\right)}e^{\frac{1}{2}y_2} \quad \text{and} \quad \sqrt{\left(\frac{G}{M}\right)}e^{-\frac{1}{2}y_1}.$$

It therefore remains always of the same sign, and neither the current nor the magnetic field can reverse. This also follows from (4), which shows that if I is zero so also is $\dot{I}$; repeated differentiation and substitution for ω from (5) then shows that all the differential coefficients of I are zero if I is. Thus if the current is not zero it cannot become so in any finite time.

The variation of the current with time can be found by numerical integration. Consider a motion for which y_0 is zero and I therefore has its equilibrium value at $t = 0$. Since all initial conditions lead to this value of the current at some time, this is no loss of generality. The value of y at any later time can then be found by integration of (13), which gives

$$\tau = \int_0^y \frac{dy}{\sqrt{(A+2y-2e^y)}}. \tag{15}$$

The corresponding values of t, I and ω are given by (9), (10) and (7). The calculations have been done for $A = 3$ and 10; the results are shown in Fig. 3. They exhibit a considerable departure from simple harmonic motion, the maxima of I being much more above I_c than the minima are below it (Table 1).

The angular velocity is given by (7), which may be written

$$\omega = \left(\frac{LG}{2CM}\right)^{\frac{1}{2}} x + \omega_c.$$

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By (13) the two values of x for any y are equal and of opposite sign; ω therefore varies symmetrically above and below ω_c. It will reverse if the disk has an angular velocity greater than $2\omega_c$ when y is zero. The form of the oscillations for $A = 3$ and 10 are shown in Fig. 4.

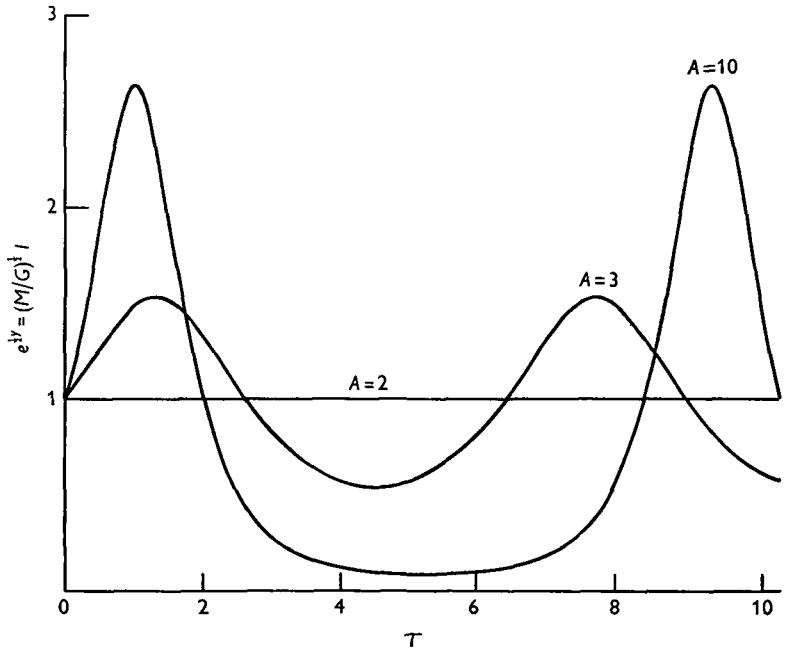

Fig. 3. Variation of current with time.

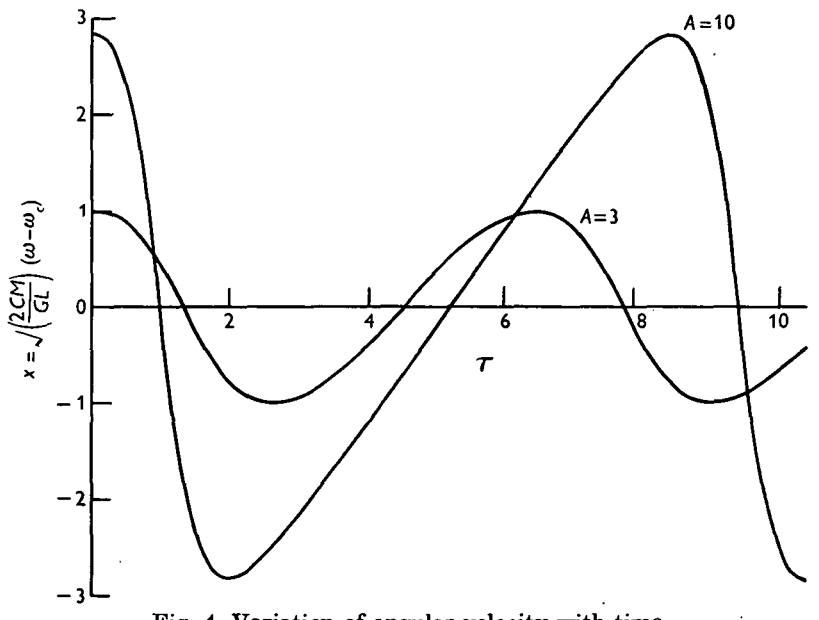

Fig. 4. Variation of angular velocity with time.

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If A is large, (15) can be evaluated approximately. The results are shown in the last line of Table 1 and diagrammatically in Fig. 5. As A increases the period and the part of the cycle for which the current is below its equilibrium value lengthen, whilst the part for which it is above shortens. For very large A the current is confined to bursts lasting for a small proportion of the time. The amplitudes of the peaks in $(M/G)^{\frac{1}{2}}I$ are approximately $\sqrt{(\frac{1}{2}A)}$, while in the intervals it falls off to $\exp(-\frac{1}{4}A)$. While the current is small the angular velocity accelerates almost uniformly till the disk is rotating above its critical speed. The current then grows rapidly and the resulting electromagnetic forces slow the disk till it is again rotating below the critical speed; the current then decreases and the cycle restarts (Fig. 5).

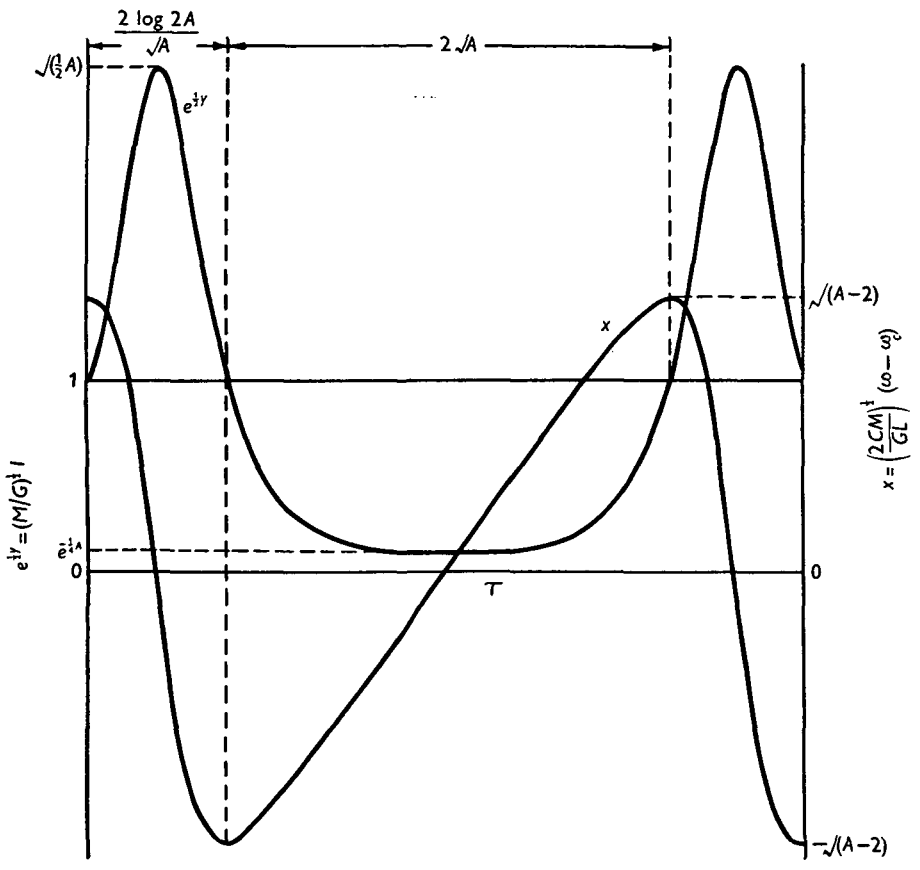

Fig. 5. Variation of current and angular velocity with time for large A.

The solution of the equations for given initial conditions is well-defined and unique, but the behaviour of the corresponding physical system may have a random element. For large A most of the time is spent with a very small current in the coil, which grows or decreases very slowly over most of the cycle. If some disturbance increases or decreases $(M/G)^{\frac{1}{2}}I$ by a few times $\exp(-\frac{1}{4}A)$ during the time when it is small, the time at which the next maximum will occur will be greatly affected or its sign reversed.

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The bursts of current will therefore occur at roughly the calculated intervals, but small disturbances may cause a large scatter in the times.

If GM is negative, the current in the coil produces a field opposed to the initial field and the system cannot act as a self-exciting dynamo. It is still governed by (4) and (5), but (9) and (10) must be replaced by

$$\tau = \left(-\frac{2GM}{CL}\right)^{\frac{1}{2}} t, \quad y = \ln(-MI^2/G).$$

These give
$$\frac{d^2y}{d\tau^2} = -(1+e^y) \tag{16}$$

and
$$x^2 = A_1 - 2(e^y + y), \tag{17}$$

where
$$A_1 = x_0^2 + 2(e^{y_0} + y_0).$$

(17) does not give closed curves in the (x, y)-plane, since for a given A_1 there is only one value of y corresponding to each value of x. At whatever point on one of these curves the system starts, y will eventually become large and negative and I will tend to zero. For large negative y, (16) gives

$$y = -\tfrac{1}{2}\tau^2 + \alpha\tau + \beta,$$

where α and β are constants. Thus for large times the current is given by

$$I = (-G/M)^{\frac{1}{2}} \exp\left[\frac{GM}{2CL}t^2 + \alpha\left(-\frac{GM}{2CL}\right)^{\frac{1}{2}} t + \beta\right]$$

and, since GM is negative, tends to zero like an exponential in $-t^2$. When the current has decayed there are no electromagnetic forces and the motion accelerates uniformly.

3. *Dynamo with viscous forces.* It is surprising to find that when $GM > 0$, undamped oscillations can be maintained by a steady couple in the presence of ohmic dissipation, and it is of interest to examine the effect of adding a viscous couple to the equation of motion. If the couple is $k\omega$ (k positive), (4) is unaffected, but the equation of motion (5) becomes

$$C\dot{\omega} + k\omega = G - MI^2. \tag{18}$$

As before, there is a steady state in which

$$\omega = \omega_c = R/M;$$

the current in this state is now not $\sqrt{(G/M)}$ but

$$I = \left(\frac{G - k\omega_c}{M}\right)^{\frac{1}{2}}. \tag{19}$$

The current is less than that in the steady state without viscous forces, since a part, $k\omega_c$, of the applied couple is used in overcoming these forces and the electromagnetic forces must be correspondingly reduced.

Elimination of ω between (18) and (4) gives

$$\frac{d^2}{dt^2}(\ln I) + \frac{k}{C}\frac{d}{dt}(\ln I) = \frac{GM}{CL}\left(1 - \frac{M}{G}I^2 - \frac{kR}{MG}\right) \tag{20}$$

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in place of (8). Suppose GM to be positive and kR/MG to be positive and less than one. Put

$$\tau' = \left(\frac{2GM}{CL}\right)^{\frac{1}{2}}\left(1-\frac{kR}{MG}\right)^{\frac{1}{2}} t$$

and

$$y' = \ln\frac{MI^2}{G-kR/M}.$$

With this substitution, (18) becomes

$$\frac{d^2y'}{d\tau'^2}+\gamma\frac{dy'}{d\tau'} = 1-e^{y'}, \tag{21}$$

where

$$\gamma = k\left\{\frac{L}{2C(MG-kR)}\right\}^{\frac{1}{2}}.$$

Putting

$$x' = \frac{dy'}{dt'},$$

we obtain from (18)

$$x'^2+2\gamma\int_0^{\tau'} x'^2 d\tau' = A'-2(e^{y'}-y'), \tag{22}$$

where

$$A' = x'^2+2(e^{y'_0}-y'_0).$$

If x' does not tend to zero for large τ', the left-hand side of (22) increases without limit as $\tau'\to\infty$; but the right-hand side is less than $(A'-2)$ and (22) cannot be satisfied. Thus x' must tend to zero for large τ' in such a way that the integral in (22) is bounded. If x' tends to zero, y' tends to a constant, y_∞, and $dy'/d\tau'$ to zero. Hence from (21), $d^2y'/d\tau'^2$ tends to $(1-\exp y_\infty)$. Thus if $dy'/d\tau'$ is to remain small, y_∞ must be zero and the current settles to its value in the steady state. The diagram relating x' and y' and corresponding to Fig. 2 would be a spiral converging on the origin.

If y' is small, (21) becomes

$$\frac{d^2y'}{d\tau'^2}+\gamma\frac{dy'}{d\tau'}+y' = 0.$$

If γ is less than 2, this gives damped oscillations. The time for the amplitude to fall to $1/e$ is $2C/k$, which is twice the time required for the speed of rotation to fall to $1/e$ in the absence of a current or an applied couple. If γ is greater than 2, the system will settle to the steady state without oscillation. These results only apply to the final part of the settling process when the current and angular velocity are not very different from their equilibrium values; the earlier motion will be qualitatively similar, but the details could only be determined by numerical integration of (21).

If kR/MG is greater than one, the applied couple is insufficient to drive the disk at the critical speed R/M, and any current and field initially present will decay to zero and the angular velocity will settle to G/k.

The system provides an example of the exchange of stabilities. It has two equilibrium states, one with no current and angular velocity G/k and the other with current given by (19) and angular velocity R/M. With small viscous forces the first is unstable and the second stable. If k is increased through the value MG/R, the state with current

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ceases to be an equilibrium configuration and stability is transferred to the previously unstable state.

4. *Effect of an external load.* An electrical load of resistance R_1 and inductance L_1 connected in series with the coil has no effect except to alter R and L to $(R+R_1)$ and $(L+L_1)$. A similar load connected in parallel may either produce a damping similar to that due to a viscous retarding force or may render the steady state unstable.

If the resistance and inductance of the disk and brushes are neglected, so that R and L refer to the coil, equations (4) and (5) are replaced by

$$L\dot{I}+RI = M\omega I, \quad L_1\dot{I}_1+R_1I_1 = M\omega I, \tag{23}$$

$$C\dot{\omega} = G-MI(I+I_1), \tag{24}$$

where I is, as before, the current through the coil, and I_1 that through the load connected in parallel with it.

Two special cases give simple results. First suppose $L/R = L_1/R_1$. (23) then gives

$$I_1 = IR/R_1+I_0\exp(-Rt/L),$$

where I_0 is a constant. After the transient term has subsided, I_1 is equal to IR/R_1, and (23) and (24) behave like (4) and (5) with I replaced by $I(1+R/R_1)^{\frac{1}{2}}$. A parallel load with the same ratio of inductance to resistance as the coil therefore has only a minor effect on the system, which can still perform undamped oscillations of the same period as before.

If L_1 is zero, I_1 and ω may be eliminated from (23) and (24) to give

$$\frac{d^2}{dt^2}(\ln I) = \frac{GM}{CL}\left\{1-\frac{MI^2}{G}\left(1+\frac{R}{R_1}\right)\right\}-\frac{M^2I}{CR_1}\frac{dI}{dt}.$$

The method employed with (20) may be used to show that the solution of this settles to a steady value. A purely resistive load therefore damps out the oscillations.

It is difficult to investigate the solutions of (23) and (24) in the general case, where L_1 and R_1 have any values, unless the angular velocity and the currents are near their steady values, which are given by

$$\omega = \frac{R}{M}, \quad I^2 = \frac{GR_1}{M(R+R_1)}, \quad I_1^2 = \frac{GR^2}{M(R+R_1)R_1}.$$

If the departures of ω, I and I_1 from these values are small enough for their squares and products to be neglected, the equations are rendered linear. The solutions are then proportional to $\exp(\lambda t)$, where λ is a root of a cubic that may be written

$$\left(\lambda^2+\frac{(2R_1+R)MG}{(R+R_1)CL}\right)\left(\lambda+\frac{R_1}{L_1}\right)+\frac{MGR_1}{(R+R_1)CL_1}\left(\lambda+\frac{R}{L}\right) = 0$$

or

$$\lambda^3+\frac{R_1}{L_1}\lambda^2+\frac{MGR}{CL(R+R_1)}\left\{1+\left(2+\frac{L}{L_1}\right)\frac{R_1}{R}\right\}\lambda+\frac{2MGR_1}{CLL_1} = 0. \tag{25}$$

This always has a negative real root lying between $-R/L$ and $-R_1/L_1$. The solution corresponding to this root gives a transient which becomes small after an interval of

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a few times the larger of L/R and L_1/R_1. Call this root $-\sigma$ and the remaining roots $\mu+\nu i$, where i is $\sqrt{(-1)}$. Then (25) must be of the form

$$(\lambda+\sigma)(\lambda-\mu-\nu i)(\lambda-\mu+\nu i) = 0,$$

which gives
$$\lambda^3+(\sigma-2\mu)\lambda^2+(\mu^2+\nu^2-2\sigma\mu)\lambda+\sigma(\mu^2+\nu^2) = 0. \tag{26}$$

The stability of the motion in the neighbourhood of the steady state depends on the sign of μ. If μ is positive (which gives an unstable motion),

$$\frac{\sigma(\mu^2+\nu^2)}{\sigma-2\mu} > \mu^2+\nu^2-2\sigma\mu.$$

Comparing (26) with (25), this condition gives

$$\frac{R}{R+R_1}\left\{1+\left(2+\frac{L}{L_1}\right)\frac{R_1}{R}\right\} < 2,$$

which implies
$$\frac{L}{R} < \frac{L_1}{R_1}. \tag{27}$$

Thus, if L/R is less than L_1/R_1, a small disturbance about the state of steady motion will increase as an exponentially growing oscillation, but if L/R is greater than L_1/R_1 it will settle to a steady state.

I at first supposed that the non-linear terms in (23) and (24) would limit the growth of the oscillations and that, if L/R was less than L_1/R_1, the system would finally settle to a state in which it performed periodic, but not simple harmonic, oscillations about the steady motion. Failure to prove even that I and ω were bounded suggested that the system might behave in some unexpected way. A few solutions were therefore obtained on the NPL differential analyser. If

$$I = (G/M)^{\frac{1}{2}} I', \quad I_1 = (G/M)^{\frac{1}{2}} I_1', \quad \omega = (GL/CM)^{\frac{1}{2}}\omega', \quad t = (CL/GM)^{\frac{1}{2}} t',$$

(23) and (24) are reduced to

$$\frac{dI'}{dt'}+\frac{R}{L}\left(\frac{LC}{MG}\right)^{\frac{1}{2}} I' = \omega' I', \quad \frac{dI_1'}{dt'}+\frac{R_1}{L_1}\left(\frac{LC}{MG}\right)^{\frac{1}{2}} I_1' = \frac{L}{L_1}\omega' I', \quad \frac{d\omega'}{dt'} = 1-I'(I'+I_1').$$

The form of the solution therefore depends on three non-dimensional parameters, which may be chosen to be

$$\frac{RL_1}{LR_1}, \quad \frac{R_1}{L_1}\left(\frac{LC}{MG}\right)^{\frac{1}{2}} \quad \text{and} \quad \frac{L}{L_1}.$$

(27) shows that the first of these indicates the degree of instability for small disturbances. For instability it must be greater than one; to ensure a marked instability it was taken as two for the solutions on the differential analyser. The second parameter is $\sqrt{2}\pi$ times the ratio of the period of the oscillations in the absence of a load to the upper limit of the decay time of the transient. As it is desirable that this should be fairly large, it was taken as two; L/L_1 was taken to be unity. The equations are then

$$\frac{dI'}{dt'}+4I' = \omega' I', \quad \frac{dI_1'}{dt'}+2I_1' = \omega' I', \quad \frac{d\omega'}{dt'} = 1-I'(I+I_1'). \tag{28}$$

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The cubic (25) then becomes $\lambda^3 + 2\lambda^2 + \frac{5}{3}\lambda + 4 = 0,$

whose roots are $-2{\cdot}1090$ and $(0{\cdot}0545 \pm 1{\cdot}3761i)$. The period for small disturbances is $4{\cdot}57$ in the t' units (in the absence of a shunt it would be $\sqrt{2}\pi = 4{\cdot}44$). Small oscillations grow by a factor e in $4{\cdot}02$ periods and the transient decays to $1/e$ in $0{\cdot}104$ of a period.

Solutions were obtained for a few sets of initial conditions and are all of the type shown in Fig. 6. They resemble those of Fig. 3, but have continually increasing amplitude and period. This suggests that the solutions of (23) and (24) may be con-

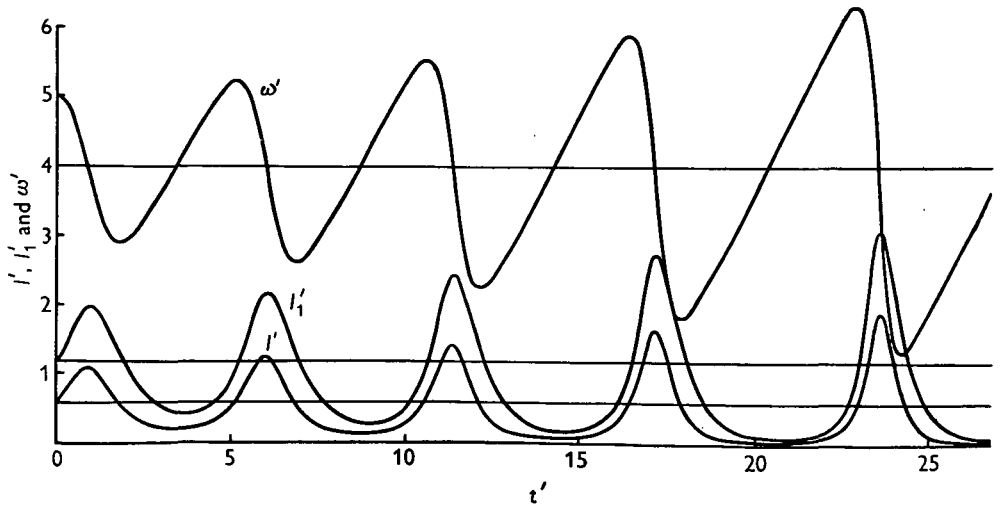

Fig. 6. Variation of current and angular velocity with time for a dynamo with a parallel load, showing increasing amplitude and period. The curves are solutions of (28) with the initial conditions $I = 1/\sqrt{3}$, $I_1 = 2/\sqrt{3}$, $\omega = 5$.

sidered to be given by (13) if A increases indefinitely with the passage of time. A rigorous analytical proof of this has not been found, but the conjecture from the numerical work may be rendered plausible as follows. Elimination of ω between (23) and (24) gives

$$\frac{d^2}{dt^2}(\ln I) = \frac{GM}{CL}\left\{1 - \frac{M}{G}(I^2 + II_1)\right\}.$$

Thus, if y and τ are defined, as before, by (9) and (10),

$$\frac{d^2y}{d\tau^2} = 1 - e^y - (M/G)\,II_1$$

and

$$\left(\frac{dy}{d\tau}\right)^2 = 2(y - e^y) + A_2 - \frac{M}{G}\int II_1\,dy, \tag{29}$$

where A_2 is a constant. Since $dy = 2dI/I,$

the integral is simply $\int I_1\,dI$ or the area enclosed by a plot of I_1 against I. If this becomes increasingly negative as time goes on, (29) will be equivalent to (13) with an increasing A. The plots of I_1 against I produced by the differential analyser are all similar to the example shown in Fig. 7; since the curves are described in a counter-

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clockwise direction, the integral is negative. I have not been able to prove that this must always be true, though the following argument makes it likely.

As in § 2, (23) shows that if I is once positive it will always remain so. Multiplying (23) by $\exp(R_1 t/L_1)$ and integrating, we find

$$I_1 = \frac{1}{L_1} e^{-R_1 t/L_1} \int_0^t e^{R_1 t/L_1} (L\dot{I} + RI)\, dt + B\, e^{-R_1 t/L_1}, \tag{30}$$

where B is a constant of integration. Integrating the term in $\dot{I}$ by parts, we find

$$I_1 = \frac{L}{L_1}\left[I + \left(\frac{R}{L} - \frac{R_1}{L_1}\right) e^{-Rt/L_1} \int_0^t e^{R_1 t/L_1} I\, dt\right] + B_1\, e^{-R_1 t/L_1},$$

where B_1 is a new constant. The term in B_1 is a transient which is negligible for times large compared to L_1/R_1. After it has decayed, I_1 is of the same sign as I and is greater than LI/L_1 provided (27) is satisfied and the system is unstable for small disturbances. The solution shown in Fig. 7 illustrates this. Integrating the I term in (30) by parts and neglecting the transient term, we obtain

$$I_1 = \frac{R}{R_1} I - \frac{L}{R_1}\left(\frac{R}{L} - \frac{R_1}{L_1}\right) e^{-R_1 t/L_1} \int_0^t e^{R_1 t/L_1} \dot{I}\, dt. \tag{31}$$

The second term is a weighted average of past values of $\dot{I}$ with most of the weight on values occurring during the period since a time $(t - L_1/R_1)$. If $\dot{I}$ has retained the same sign for a time large compared with L_1/R_1, the integral will have the sign of $\dot{I}$, and I_1 will be less than RI/R_1 while it is increasing and greater while it is decreasing. The plot of I_1 against I will therefore be described counter-clockwise, as in Fig. 7. The argument does not amount to a proof that the system will always behave as shown in Fig. 6 if (27) is satisfied, but it seems likely that this is so.

It is remarkable that a simple system driven by a constant couple and governed by equations as apparently straightforward as (28) should exhibit this rather subtle 'cat and mouse' behaviour. For large times the disk accelerates almost uniformly till it is suddenly stopped and reversed by a burst of current; it then repeats the same behaviour over a longer interval, reaching a higher angular velocity than any it has previously had, only to be stopped again by a larger burst of current than any preceding one.

Such a behaviour seems 'unnatural' and would be prevented in a real system by features that have been neglected in the above discussion. One of these is friction in the bearings in which the axle runs. This may be introduced by adding a term $k\omega$ to (24), which then becomes

$$L\dot{\omega} + k\omega = G - MI(I + I_1). \tag{32}$$

Small oscillations about the steady state are easily investigated. As would be expected, a system that satisfies (32) and (27) and has a sufficiently small value of k remains unstable for small disturbances. A larger k causes any disturbance to decay to a steady state. The behaviour of a system that is unstable for small disturbances has been studied with the differential analyser. The constants chosen were the same as those in (28) with the addition of a term $\omega/20$ to the left-hand side of the third equation. The system settled to steady oscillations of finite amplitude resembling those of Fig. 6, except that the amplitude and period did not increase with time.

A further factor limiting the growth of oscillations in a physical system is the

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inevitable decrease in the driving couple at high speeds. If the couple is zero at all speeds above some speed ω_1 which is greater than ω_c, the system may still be unstable for small disturbances, but in each oscillation the rise of ω will stop short of ω_1. The system has not been investigated numerically, but is believed to settle to a periodic motion.

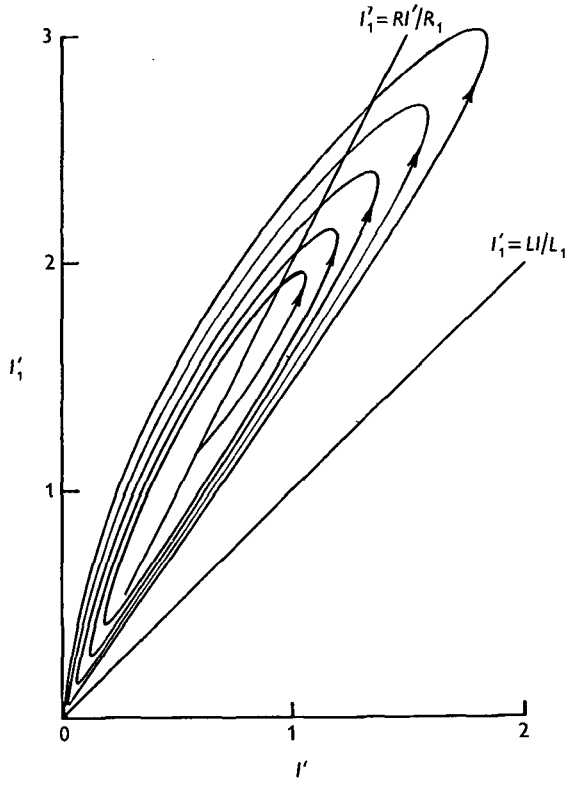

Fig. 7. Relation between I' and I_1' for a dynamo with a parallel load, the same solution as in Fig. 6.

A disk driven at constant power W, that is with W/ω in place of G in (24), settles to steady rotation with angular velocity R/M.

The results of §§2, 3 and 4 are summarized in Table 2.

5. *Discussion.* The main interest of the results obtained in §§2, 3 and 4 is that they may give some hint of the behaviour of a conducting fluid when disturbed from a state of steady motion in a magnetic field.

The period, T, of the small oscillations of a non-self-exciting disk dynamo is $2\pi(LC)^{\frac{1}{2}}/F$. If H_m is the mean magnetic field perpendicular to the disk and the radius of the axle is negligible compared to that of the disk,

$$F = \tfrac{1}{2}a^2H_m \quad \text{and} \quad T = \frac{4\pi(LC)^{\frac{1}{2}}}{a^2H_m}. \tag{33}$$

For a self-exciting dynamo, the period of small oscillations is $\pi(2LC/GM)^{\frac{1}{2}}$. Expressing G in terms of the mean equilibrium field, H_m, by (3) and (6), we see that this differs

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from (33) only by a factor $1/\sqrt{2}$. Thus the periods of the externally excited and self-excited dynamos are not greatly different; for moderate amplitudes of oscillation the difference will be reduced, since the period of the externally excited dynamo is independent of the amplitude of the oscillations whilst that of the self-excited dynamo increases with amplitude. From Fig. 5 and Table 1 it can be seen that the amplitude would have to be very great for the period of the self-excited dynamo to exceed that of

Table 2

	Viscous forces	Parallel load	
$MG>0$	No	No	Oscillations of constant amplitude, no reversal of I, ω may reverse
	$\gamma<2$	No	Damped oscillations settling to steady motion and current
	$\gamma>2$	No	Aperiodic settling to a steady motion and current
	$k>GM/R$	Yes or No	Speed too low for regeneration, current decays to zero and disk accelerates to angular velocity G/k
	No	$L/R=L_1/R_1$	Oscillations of constant amplitude, no reversal of I, ω may reverse
	No	$L/R>L_1/R_1$	Damped oscillations settling to steady motion and current
	No	$L/R<L_1/R_1$	Oscillations growing indefinitely in amplitude and period
	$k<GM/R$	$L/R<L_1/R_1$	Oscillations settling to a finite amplitude for small k, or settling to a steady motion for larger k
$MG<0$	No	Yes or No	The current and field decay to zero and the disk accelerates uniformly

the externally excited one. It might have been thought that the period could not have been less than L/R, the time taken for a current to decay to $1/e$ in the non-rotating disk and coil; since (33) does not contain R, it is clear that this is not so. From Fig. 5 it is seen that at large amplitudes of oscillation the decrease of the current during one half of the cycle is not due to ohmic dissipation, but to the disk rotating in a direction producing an opposing e.m.f.

This consideration suggests a conceivable way of escape from the principal difficulty in the theory of the magnetic field of sunspots. It is likely that the field from a pair of sunspots is part of a ring of field encircling the sun's axis. If the flux in such a ring is supposed to change only in times of the order of L/R it cannot change appreciably in the whole history of the solar system, but in fact the field reverses in each successive sunspot cycle of about 11 years. I have attempted to explain this by assuming two oppositely directed rings of field in each hemisphere (9). An alternative would be to suppose that the interactions of the motion and the field suppressed one ring and produced the other towards the end of each sunspot cycle. Both these views are entirely speculative and would be most difficult to investigate in detail; the existence of two rings seems the more probable hypothesis.

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The expression (33) for the period may be related to the work of Schwarzschild (8), Cowling (5) and Plumpton and Ferraro (7) on the oscillations of a fluid conducting sphere in an externally applied field. The moment of inertia of the disk, C, is $\frac{1}{2}\pi\rho s a^4$, where s is its thickness. The inductance of the coil is equal to its radius multiplied by a function of the ratio of the radius of the coil to the radius of the wire used to wind it. Thus (33) gives

$$T = \frac{Ka\rho^{\frac{1}{2}}}{H_m}, \tag{34}$$

where K is a function of the ratios of the thickness of the disk and the dimensions of the coil to the radius of the disk. For a coil of round wire of radius r bent into a single turn of radius a_1,

$$K = \pi^2(s/a)^{\frac{1}{2}}[4\ln(8a_1/r) - 7]^{\frac{1}{2}}.$$

An expression of the form (34) might be expected to apply to the oscillations of a sphere of conducting fluid in a field, K being a numerical constant depending on the form of the field and motion considered. This expectation is strengthened by observing that the time taken for a magneto-hydrodynamic wave, travelling along the lines of force of a field H, to cross a sphere of radius a is $4\pi^{\frac{1}{2}}a\rho^{\frac{1}{2}}/H$. The longest period found by Schwarzschild (8) for a sphere oscillating in a uniform field is $20{\cdot}3a\rho^{\frac{1}{2}}/H$. Later work gives similar results (Plumpton and Ferraro (7)).

For the sun, the radius a is 7×10^{10} cm., the mean density is $1{\cdot}4$ g./cm.^{3}, and the central density about 90 g./cm.^{3}. The period of oscillation may therefore be expected to be some moderate multiple of $10^4/H$ years. The field within the sun is probably similar to that in sunspots, which is about 2000 gauss. The period of the oscillations due to the coupling of the magnetic field and motions in the sun would therefore be expected to be a few years. It seems quite possible that the sunspot period of about eleven years is controlled by such processes, in fact Alfvén's theory of sunspots is of this kind, though the details of the mechanism proposed are open to various objections (Cowling (6)). The much shorter periods of a few days observed by Babcock in magnetically variable stars cannot be due to this mechanism and are presumably due to modes of motion giving strong gravitational restoring forces, as suggested by Cowling (5).

For the earth's core, $a = 3{\cdot}4\times10^8$ cm., $\rho = 11$ g./cm.^{3}, and the period will therefore be a moderate multiple of $30/H$ years. If H is a few hundred gauss, this is considerably less than the typical periods of the secular change, which are of the order of a hundred years. The time scale of the secular variation is therefore longer than would be expected from (34) with a moderate value of K. It is difficult to think of any other process that will give a period or a relaxation time of the order of a hundred years and it seems possible that (34) is of the right form, but that K may have a very large value in systems in which Corioli's forces are important (compare, for example, the lengthening of the period of a pendulum when its bob contains a gyroscope). The absence of Corioli's forces from the system considered in this paper may seriously interfere with any application of the results to the earth, sun or stars.

It is not known whether the undamped oscillations of the disk dynamo have an analogy in the motions of a fluid sphere in a magnetic field. In astronomical applications the effects of viscosity are likely to be so small as to have little influence in 10^9 years.

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It is probable that current paths not contributing to the dynamo action will act like the parallel load of § 3. If this is so, they may have either a stabilizing or a destabilizing effect, and the result will depend on the details of the motion. An unpublished investigation by Rikitake suggests that a dynamo whose steady state has been considered by Bullard and Gellman (2) is unstable for small disturbances. In the light of the discussion at the end of § 4, this probably means, not that the system will depart indefinitely from its steady state, but that it will perform finite oscillations in which the motion, the current and the field will all fluctuate by amounts depending on how the driving forces vary with the velocity of the motion. Thompson (9) and Chandrasekhar (3) have shown that in the presence of a magnetic field thermal instability of a fluid heated from below may start by growing oscillations. Chandrasekhar (4) has shown that the same thing is true in a rotating fluid even in the absence of a magnetic field.

It has been suggested that sudden reversals of the earth's field have frequently occurred in the past. The disk dynamo does not behave in this way; it can produce a field in either direction, but it does not give reversals once a field has been established in a given direction. A field which reverses could be produced by a coil coupled inductively to the coil of the dynamo, but it is not possible to produce in this way a field that remains roughly constant for a long time and then suddenly reverses and again remains constant in its new direction. It does not follow, of course, that more complicated systems cannot behave as suggested.

The arguments of this section assume an analogy between the solution of a set of two or three ordinary differential equations, such as (4) and (5) or (23) and (24), and the partial differential equations that control a continuous system (Maxwell's equations and the equations of hydrodynamics). Such analogies are a little precarious, but may serve to suggest qualitatively how the solutions of the equations of magneto-hydrodynamics behave. The most important result is the possibility of self-maintained oscillations even in the presence of ohmic dissipation and small viscous forces.

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