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A Self-Consistent Treatment of Simple Dynamo Systems

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In Part I, the simple homopolar disc dynamo of Bullard (1955) is discussed, and it is shown that the conventional description is over-simplified and misleading in an important respect, in that it suggests the possibility of exponential growth of the magnetic field even in the limit of perfect disc conductivity, whereas, from fundamental considerations, it is known that the flux of magnetic field across the disc must in this limit remain constant. This contradiction is resolved through consideration of the effect of the azimuthal current distribution which is, in general, inevitably induced in the disc when the conditions for dynamo action are satisfied. By considering a refined model, it is shown that the field growth rate then tends to zero as the disc conductivity tends to infinity. The stability characteristics of this model are determined.

In Part II, an analogous contradiction arising in fluid dynamo theory is identified, viz. that whereas the α^2-dynamo in a spherical geometry suggests the possibility of exponential field growth even in the limit of perfect conductivity, fundamental considerations (Bondi and Gold, 1950) show that the dipole strength of the field is, in this limit, permanently bounded. Two possible ways of resolving this contradiction are discussed. The first, following a suggestion of Kraichnan (1979), involves consideration of an inhomogeneity layer on the surface of the sphere within which α (and the eddy diffusivity β) falls to zero; diffusion of flux across this layer is analogous to diffusion of flux across the rim of the disc in the simpler disc dynamo context. The second involves introduction of a time lag in the conventional linear relationship between mean field and mean electromotive force; this represents in a crude way the process by which flux has to diffuse into the interior of each helical eddy within which the fundamental field regeneration effect occurs. By either means, compatibility with the Bondi and Gold result may be achieved.

PART I: The self-exciting disc dynamo.

1. INADEQUACY OF THE CONVENTIONAL TREATMENT

The self-exciting disc dynamo (Bullard 1955, for a recent account, see Bullard 1978) has frequently been invoked as a simple prototype of

dynamo action, analogous to the dynamo process that is believed to operate in the liquid conducting core of the Earth and in the convective envelope of the Sun. The disc dynamo system, in its simplest form, is illustrated in figure 1; it consists of a solid conducting disc which rotates with angular velocity $2\pi\Omega\mathbf{k}$ (either constant, or a function of time) about its axis, and a wire twisted around the axle and making sliding contact with the rim of the disc (at the point A in the figure), and with the axle at the point B.

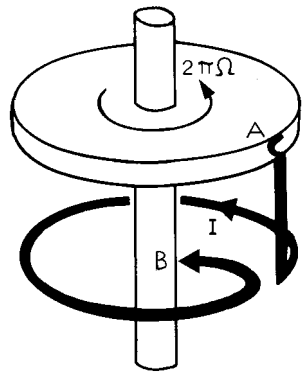

FIGURE 1 The self-exciting disc dynamo (Bullard 1955).

The conventional description of the dynamo process for this system is as follows. Suppose that a current $I(t)$ flows in the combined circuit C consisting of disc plus wire, the current in the disc being radial, and being "scooped off" at A. This current gives rise to an associated magnetic field $\mathbf{B}(\mathbf{x}, t)$, and the flux of this field across the disc is

$$\Phi(t) = \iint_{\text{disc}} B_z(r, \theta, 0, t) r\, dr\, d\theta, \tag{1.1}$$

where the plane of the disc is chosen to be $z = 0$. The induced electric field in the disc is

$$\mathscr{E} = \mathbf{u} \times \mathbf{B} = 2\pi\Omega(\mathbf{k} \times \mathbf{x}) \times \mathbf{B} = 2\pi\Omega \mathbf{x} B_z, \tag{1.2}$$

and so the potential difference between the axle and rim, averaged over θ, is given by

$$\Delta\phi = (2\pi)^{-1} \iint_{\text{disc}} \mathscr{E}_r\, dr d\theta = \Omega\Phi, \tag{1.3}$$

using (1.1) and (1.2). This potential difference drives the current I; and if R is the total resistance associated with the circuit C, and L its self-

inductance, the conventional equation for I is

$$LdI/dt + RI = \Omega\Phi. \tag{1.4}$$

Furthermore,

$$\Phi = MI, \tag{1.5}$$

where M is the mutual inductance between the "circuit" C and the rim of the disc. Hence

$$LdI/dt + RI = M\Omega I. \tag{1.6}$$

Note that L and M are determined wholly by the geometry of the system and do not depend on R.

Suppose now that the disc is driven at constant speed (i.e. Ω = constant); then clearly

$$I(t) = I(0)e^{pt} \quad \text{and} \quad \Phi(t) = \Phi(0)e^{pt}, \tag{1.7}$$

where

$$p = L^{-1}(\Omega M - R), \tag{1.8}$$

and we have dynamo action (i.e. spontaneous growth of current, magnetic field and associated magnetic energy) provided

$$\Omega M > R. \tag{1.9}$$

The inadequacy of this description, which has been reproduced and elaborated in many papers, is most clearly revealed by considering the limiting behaviour when $R \to 0$, a limit that is realised when the conductivity σ of the disc and wire tends to infinity. According to (1.7) and (1.8), we still have exponential growth of Φ with $p = \Omega M/L > 0$. But it is a fundamental result of electromagnetic theory that when $\sigma \to \infty$, the flux through any closed curve moving with the conductor is conserved; in the present context, when $R = 0$ (i.e. $\sigma = \infty$) the flux Φ through the closed curve consisting of the rim of the disc is therefore constant, in blatant contradiction with the "conclusion" that Φ increases exponentially.

This difficulty, that is so conspicuous in the limit $R \to 0$, was recognised in the brief description of the disc dynamo given in the introductory chapter of Moffatt 1978 (hereafter referred to as M78), but was not adequately resolved in that discussion. In Part I of the present paper, this omission is rectified, and the basic inconsistency in previous treatments of the disc dynamo is resolved.

2. SELF-CONSISTENT TREATMENT OF A "SEGMENTED" DISC DYNAMO

The inconsistency discussed above is associated with the oversimplified representation of the current field $\mathbf{j}(\mathbf{x},t)$ in terms of a single current loop $I(t)$. This is inadequate within the disc where current can flow around the axle as well as in the radial direction. When $I(t)$ is time-dependent, the associated flux has to diffuse across the rim of the disc, and this diffusion process is associated with an induced azimuthal current distribution in the disc. The instantaneous relationship (1.5), although correct in a static situation, is not correct when I is time-dependent.

To deal with this situation in a simple way, let us suppose that the disc is constructed in such a way as to permit azimuthal current flow only in the immediate neighbourhood of the rim $r=a$. This can be achieved by the insertion of $n(\gg 1)$ insulating strips (figure 2) at $\theta=2m\pi/n$ $(m=0,1,2,\ldots,n-1)$ along radii between $r=0$ and $r=a(1-\varepsilon)$ where ε is small; any azimuthal current is then concentrated in the region $a(1-\varepsilon)<r<a$. Let $J(t)$ be the total azimuthal current that flows in this region and let R' be the corresponding resistance.

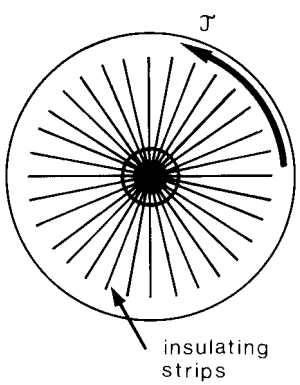

FIGURE 2 The segmented disc, which permits azimuthal current flow $J(t)$ only in the neighbourhood of the rim.

We now have two current loops $I(t)$ and $J(t)$. Let L, L' be the corresponding self-inductances, and M the mutual inductance†; then the fluxes Φ_1 and Φ_2 through the loops are

†If the wire has radius εa, and if the disc has thickness εa, then L and L' are both of order $\mu_0 a(\log 8\varepsilon^{-1}-2)$ (Abraham and Becker 1932, p. 166 *et seq.*). If the loop of the wire is separated from the disc by a distance $b(\ll a)$, then M is of order $\mu_0 a[\log(8b/a)-2]$.

$$\left.\begin{aligned}\Phi_1 &= LI + MJ,\\ \Phi_2 &= MI + L'J,\end{aligned}\right\} \tag{2.1}$$

and we have the inequality

$$LL' > M^2, \tag{2.2}$$

resulting from the positive-definiteness of the quadratic form representing the magnetic energy of the current system. The equations determining $I(t)$ and $J(t)$ are now

$$\left.\begin{aligned}RI &= \Omega\Phi_2 - d\Phi_1/dt,\\ R'J &= -d\Phi_2/dt,\end{aligned}\right\} \tag{2.3}$$

where R, as before, is the resistance in the current loop $I(t)$ (unaffected by the insertion of the insulating segments).

Note that, if the insulating segments extend to the rim of the disc (i.e. $\varepsilon = 0$), then $R' = \infty$ and $J = 0$; we then have $\Phi_1 = LI$, $\Phi_2 = MI$, and (2.3a) reduces to (1.6). Equation (1.6) does therefore correctly represent the situation when the current is totally *constrained* to flow in the radial direction (by essentially the imposition of a severely anisotropic conductivity).

When $\varepsilon > 0$, so that $R' < \infty$, (2.3b) and (2.1b) give

$$\frac{d\Phi_2}{dt} = -\frac{R'}{L'}(\Phi_2 - MI), \tag{2.4}$$

so that, as implied in the discussion above, there is indeed a lag between the current I and the flux Φ_2, the time constant being the natural decay time L'/R' for the current $J(t)$; this process of current decay is of course the same as the process of field diffusion across the rim of the disc, a process that becomes instantaneous only when $R' = \infty$. The need for the changing field to diffuse into the region where inductive action occurs is of fundamental importance for the dynamo process.

Equations (2.1) and (2.3) admit solutions of the form

$$(I, J, \Phi_1, \Phi_2) \propto e^{pt} \tag{2.5}$$

where p is a root of the quadratic equation

$$(LL' - M^2)p^2 + (RL' + R'L)p + R'(R - M\Omega) = 0. \tag{2.6}$$

By virtue of (2.2), the sum of the two roots is obviously negative. Also, if $\Omega M > R$, the product of the roots is negative, and so one is negative and

one positive. The condition (1.9) is still therefore the correct condition for dynamo action. The positive root is then given by

$$p_1 = [-(RL' + R'L) + \{(RL' + R'L)^2 + 4R'(M\Omega - R)(LL' - M^2)\}^{1/2}]/2(LL' - M^2). \tag{2.7}$$

As $R' \to 0$, we now have $p_1 \to 0$ (irrespective of the value of R); this is now consistent with the fact that the flux across the disc remains constant in the limit of infinite disc conductivity.†

If we remove the insulating segments, the azimuthal current $j_\theta(r, t)$ is no longer representable in terms of a single current loop $J(t)$, and a proper treatment of the problem would require consideration of the partial differential equations governing $\mathbf{j}$ and $\mathbf{B}$ within the disc. While such a treatment might be illuminating in some respects, it might also obscure the simple features that are best revealed by the above elementary analysis.

3. DYNAMICAL CHARACTERISTICS OF THE SEGMENTED DISC DYNAMO

Suppose now that the disc is driven by a prescribed steady torque $2\pi G$. There is also an electromagnetic torque given by

$$\mathbf{G}_e = \int_{\text{disc}} \mathbf{x} \times (\mathbf{j} \times \mathbf{B}) dV = -\mathbf{k} \int B_z (\mathbf{x} \cdot \mathbf{j}) dV. \tag{3.1}$$

Now B_z is approximately axisymmetric on the disc, and

$$\iint_{\text{disc}} r j_r \, d\theta dz = I. \tag{3.2}$$

Hence

$$\mathbf{G}_e \approx -\mathbf{k} I \int B_z r \, dr \approx -\mathbf{k} I \Phi_2 / 2\pi. \tag{3.3}$$

Let C be the moment of inertia of the disc and axle about the z-axis. Then the equation of motion of the disc is

$$C d\Omega/dt = G - I\Phi_2, \tag{3.4}$$

and we have now to consider this equation in conjunction with (2.1) and

†If $R = R'$ and $L = L'$, and if $N = 1 - M^2/L^2$ is small, then it is easily shown from (2.7) that p_1 is maximal as a function of the magnetic Reynolds number $R_m = M\Omega/R$ at a value $R_m = O(N^{-1/3})$, and there $p_1 = O(\Omega N^{-1/3})$.

(2.3). We regard Φ_1, Φ_2 and Ω as the basic variables, I and J being then given by

$$\begin{pmatrix} I \\ J \end{pmatrix} = \begin{pmatrix} L & M \\ M & L' \end{pmatrix}^{-1} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{1}{LL'-M^2} \begin{pmatrix} L' & -M \\ -M & L \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{3.5}$$

There are two equilibrium states ($S_\pm$ say) of this third order system, viz.

$$\Phi_1 = \pm L(G/M)^{1/2}, \qquad \Phi_2 = \pm (GM)^{1/2}, \qquad \Omega = R/M. \tag{3.6}$$

Defining non-dimensional variables

$$\tau = \frac{R}{L} t, \quad X = \frac{\Phi_2}{(GM)^{1/2}}, \quad Y = \frac{\Phi_1}{L(G/M)^{1/2}}, \quad Z = \frac{M\Omega}{R}, \tag{3.7}$$

equations (2.1), (2.3) and (3.4) reduce to†

$$\left.\begin{aligned} \dot{X} &= r(Y-X), \\ \dot{Y} &= mX - (1+m)Y + XZ, \\ \dot{Z} &= g(1+mX^2 - (1+m)XY), \end{aligned}\right\} \tag{3.8}$$

where

$$r = \frac{R'L^2}{R(LL'-M^2)}, \quad m = \frac{M^2}{LL'-M^2}, \quad g = \frac{G}{C}\left[\frac{LM}{R^2}\right], \tag{3.9}$$

and the equilibrium states are now given by

$$X = Y = \pm 1, \qquad Z = 1. \tag{3.10}$$

The stability of these states may be investigated by a standard small perturbation approach. Let

$$X = \pm 1 + \xi, \qquad Y = \pm 1 + \eta, \qquad Z = 1 + \zeta, \tag{3.11}$$

substitute in (3.8) and linearise in ξ, η, ζ; we obtain

†This third order system is similar, but not identical, to the system studied by Lorenz (1963). The difference however is not one that can be eliminated by any simple change of dependent variables. The general system of which (3.8) is an example may clearly be written in the compact form

$$\dot{X}_i = a_i + b_{ij}X_j + c_{ijk}X_jX_k.$$

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} -r & r & 0 \\ (1+m) & -(1+m) & 1 \\ -g(1+m) & -g(1+m) & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}. \tag{3.12}$$

This linear system has solutions $(\xi, \eta, \zeta) \propto e^{pt}$ provided p satisfies the cubic equation

$$f(p) \equiv p^3 + p^2(r+1+m) + pg(1+m) + 2rg = 0. \tag{3.13}$$

This equation has roots $-\sigma$, $\mu \pm i\nu$, where $\sigma > 0$, and the condition for linear instability ($\mu > 0$) is

$$\frac{2rg}{r+1+m} > g(1+m),$$

i.e.

$$r > \frac{(1+m)^2}{1-m}, \quad \text{or equivalently} \quad \frac{R'}{R} > \frac{L^2}{LL' - 2M^2}. \tag{3.14}$$

The corresponding region of the (r, m) plane (with $r > 0$, $m > 0$) is indicated in figure 3.

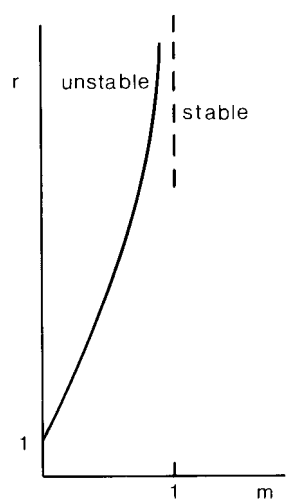

FIGURE 3 Stable and unstable regions of the (r, m) plane determined by small perturbation analysis, as given by (3.14).

A solution of the system (3.8) may be represented by the motion of a point $P(\tau)$ with Cartesian coordinates $[X(\tau), Y(\tau), Z(\tau)]$ (as in the tradi-

tional treatment of the coupled disc dynamo system whose stability was analysed by Cook and Roberts 1970—see M78, §12.4). The phase space 'velocity field' is given by

$$\mathbf{u}=(\dot{X},\dot{Y},\dot{Z})=[r(Y-X),mX-(1+m)Y+XZ,g(1+mX^2-(1+m)XY], \tag{3.15}$$

which has uniform negative divergence

$$\nabla\cdot\mathbf{u}=\frac{\partial u_1}{\partial X}+\frac{\partial u_2}{\partial Y}+\frac{\partial u_3}{\partial Z}=-(1+m+r). \tag{3.16}$$

When $r<(1-m)^{-1}(1+m)^2$, it seems probable that all trajectories tend to one of the two equilibrium points given by (3.10). When $r>(1-m)^{-1}(1+m)^2$, all trajectories in the phase space presumably tend to a limit set of points which may be a surface, or possibly a strange attractor [see e.g. Marzek and Spiegel, 1978].

The Bullard limit is obtained by letting $r\to\infty$ for fixed m and g. The three roots of (3.13) are then

$$-r+O(1),\quad \pm i(2g)^{1/2}+\frac{g(1-m)}{2r}+O\left(\frac{1}{r^2}\right). \tag{3.17}$$

When $r=\infty$, we apparently have neutral stability, but for any large but finite r, the states $S_\pm$ both remain unstable for $m<1$ (although the growth rate of the instability is small). The flow of azimuthal current in the disc therefore has an important influence (both quantitative *and* qualitative) on the dynamical stability characteristics of the system.

PART II: Fluid dynamos in the high conductivity limit.

4. MEAN FIELD ELECTRODYNAMICS AND THE α-EFFECT

In the following sections, we shall focus attention on a very puzzling contradiction that arises when the now standard methods of mean-field electrodynamics are pushed to the high conductivity limit $\sigma\to\infty$. The contradiction is in some respects analogous to the contradiction described (and resolved) in Part I above in the disc dynamo context. In order to provide a convincing discussion, it will be necessary first to review some of the salient points of dynamo theory, if only to pinpoint later possible sources of the contradiction.

First, let $\mathbf{u}(\mathbf{x},t)$ be a turbulent velocity field in a fluid of infinite extent, statistically steady, homogeneous and invariant under rotations of the frame of reference, (but not under the parity transformation $\mathbf{x}'=-\mathbf{x}$). The turbulence is then isotropic in a weak sense, but pseudoscalar mean quantities, such as the mean helicity $\langle \mathbf{u}\cdot\nabla\wedge\mathbf{u}\rangle$, are not in general zero. The properties of such turbulence are discussed at length in (M78).

Let $\mathbf{B}(\mathbf{x},t)$ be a magnetic field evolving according to the induction equation

$$\partial\mathbf{B}/\partial t=\nabla\times(\mathbf{u}\wedge\mathbf{B})+\lambda\nabla^2\mathbf{B}, \tag{4.1}$$

where $\lambda=(\mu_0\sigma)^{-1}$ is the magnetic diffusivity of the fluid, and suppose that $\mathbf{B}(\mathbf{x},0)$ is non-random and varies on a scale L much larger than any scale l_0 characteristic of the turbulence. For $t>0$, we may write

$$\mathbf{B}(\mathbf{x},t)=\langle\mathbf{B}(\mathbf{x},t)\rangle+\mathbf{b}(\mathbf{x},t), \tag{4.2}$$

where the angular brackets indicate averaging over any scale l_a satisfying

$$l_0\ll l_a\ll L. \tag{4.3}$$

The mean field evolves according to the equation

$$(\partial/\partial t)\langle\mathbf{B}\rangle=\nabla\times\mathscr{E}+\lambda\nabla^2\langle\mathbf{B}\rangle \tag{4.4}$$

where $\mathscr{E}=\langle\mathbf{u}\times\mathbf{b}\rangle$. The residual equation for $\mathbf{b}$ establishes a linear relationship between $\mathbf{b}$ and $\langle\mathbf{B}\rangle$ and so between $\mathscr{E}$ and $\langle\mathbf{B}\rangle$, which is customarily developed in the form

$$\mathscr{E}=\alpha\langle\mathbf{B}\rangle-\beta\nabla\times\langle\mathbf{B}\rangle+\ldots, \tag{4.5}$$

where α and β are determined in principle solely by the statistical properties of the turbulence and the parameter λ. Here we are particularly concerned with the limiting behaviour when $\lambda\to 0$.

Exact expressions for α and β (when $\lambda=0$) were obtained by Moffatt (1974) by an analysis which has been confirmed and extended by Kraichnan (1976a, b). Let $\mathbf{v}_\mathbf{a}(t)$ be the velocity of the fluid particle which passes through the point $\mathbf{a}$ at time $t=0$, and let

$$\alpha_1(t)=-\tfrac{1}{3}\int_0^t\langle\mathbf{v}_\mathbf{a}(t)\cdot\nabla_a\times\mathbf{v}_\mathbf{a}(\tau)\rangle d\tau, \tag{4.6}$$

$$\begin{aligned}\beta_1(t)=\tfrac{1}{3}\int_0^t\langle\mathbf{v}_\mathbf{a}(t)\cdot\mathbf{v}_\mathbf{a}(\tau)\rangle d\tau+\int_0^t\alpha_1(t)\alpha_1(\tau)d\tau\\ +\tfrac{1}{6}\int_0^t\int_0^t\{\langle\mathbf{v}_\mathbf{a}(t)\cdot\mathbf{v}_\mathbf{a}(\tau_2)\nabla_a\cdot\mathbf{v}_\mathbf{a}(\tau_1)\rangle\\ -\langle\mathbf{v}_\mathbf{a}(t)\cdot[\nabla_\mathbf{a}\mathbf{v}_\mathbf{a}(\tau_1)]\cdot\mathbf{v}_\mathbf{a}(\tau_2)\rangle\}d\tau_1 d\tau_2.\end{aligned} \tag{4.7}$$

Then the relevant values of α and β are given by

$$\alpha = \lim_{t\to\infty} \alpha_1(t), \qquad \beta = \lim_{t\to\infty} \beta_1(t), \tag{4.8}$$

provided these limits exist. Evidence for the convergence of both expressions in a time of order l_0/u_0 (where u_0 is the rms velocity) is provided by Kraichnan's (1976b) computer experiments involving numerical simulation of 'maximally helical' homogeneous turbulence. The convergence of the expression (4.6) as $t\to\infty$ is certainly plausible. The convergence of the expression (4.7) is harder to accept since it requires cancellation of the positive infinity appearing in the second term (when $\alpha\neq 0$) by an equal negative infinity in the third term, which seems intrinsically unlikely. Kraichnan has nevertheless argued that this is what happens in general, and his computation of $\beta_1(t)$ (Kraichnan 1976b, figure 2) certainly supports this conclusion. Support for this conclusion is also given by earlier Markovian model studies of Vainshtein (1970).

Let us then for the moment suppose that the expressions (4.8) do converge, and that, when $\lambda=0$, α and β have orders of magnitude

$$\alpha \sim u_0, \qquad \beta \sim u_0 l_0 \tag{4.9}$$

(independent of λ). If we retain only the first two terms of the series (4.5) (on the grounds that the scale of $\langle \mathbf{B}\rangle$ is very large), then (4.4) becomes

$$(\partial/\partial t)\langle \mathbf{B}\rangle = \alpha \nabla\times\langle \mathbf{B}\rangle + \beta\nabla^2\langle \mathbf{B}\rangle, \tag{4.10}$$

(since $\nabla\cdot\langle \mathbf{B}\rangle = 0$). This equation is known to have exponentially growing solutions in an infinite medium (M78, §9.2). It also has exponentially growing solutions in a spherical geometry; to be specific, if it is supposed that $\langle \mathbf{B}\rangle$ satisfies (4.10) for $|\mathbf{x}|<a$ and that $\langle \mathbf{B}\rangle$ matches to a potential field with no external sources for $|\mathbf{x}|>a$, then exponentially growing solutions of dipole symmetry exist for

$$|\alpha| a/\beta > 4.49, \tag{4.11}$$

(Krause and Steenbeck, 1967; M78, §9.4). (Of course, the "homogeneous turbulence" description is now tenable only if $l_0 \ll a$.) In particular, the dipole moment $\boldsymbol{\mu}$ of the field, which is a linear functional of $\mathbf{B}$, [see eqn. (5.6) below] grows exponentially, provided (4.11) is satisfied.

It is at this point that we encounter the contradiction referred to at the outset. For it is a fundamental result of dynamo theory (Bondi and Gold 1950; M78, §6.3) that, when $\lambda=0$, the dipole moment of a current distribution confined to a sphere of conducting fluid is *permanently*

bounded, no matter how complicated or artificial the velocity field in the fluid may be. A proof of this result, which is more complete than that of Bondi and Gold, is given in the following section.

5. BOUNDEDNESS OF THE DIPOLE MOMENT, WHEN $\lambda = 0$.

As is well known, the dipole moment $\boldsymbol{\mu}(t)$ of a current distribution $\mathbf{j}(\mathbf{x}, t) = \mu_0^{-1}\nabla \times \mathbf{B}$ in a conducting sphere $V: |\mathbf{x}| < a$ is given by

$$\boldsymbol{\mu}(t) = (\mu_0/8\pi)\int_V \mathbf{x} \times \mathbf{j} dV = (8\pi)^{-1}\int_V \mathbf{x} \times (\nabla \times \mathbf{B}) dV. \tag{5.1}$$

If $\mathbf{B} = -\nabla\Psi$ for $|\mathbf{x}| > a$, then

$$\Psi = -(\boldsymbol{\mu} \cdot \nabla)\frac{1}{r} + \sum_{n=2}^{\infty} A_n \frac{S_n(\theta, \phi)}{r^{n+1}}, \tag{5.2}$$

where $r = |\mathbf{x}|$, and $S_n(\theta, \phi)$ is a surface harmonic of degree n.

The integral (5.1) can be manipulated by the divergence theorem to give

$$8\pi\boldsymbol{\mu}(t) = \int_S \mathbf{x} \times (\mathbf{n} \times \mathbf{B}) dS + 2\int_V \mathbf{B} dV, \tag{5.3}$$

where S is the surface $r = a$. Now $\mathbf{n} \wedge \mathbf{B}$ is continuous across S, and so the surface integral becomes

$$-\int_S \mathbf{x} \times (\mathbf{n} \times \nabla\Psi) = -a\int_S \mathbf{n}(\partial\Psi/\partial r) dS + a\int \nabla\Psi dS. \tag{5.4}$$

Clearly, the second term on the right-hand side vanishes. Moreover, only the dipole term of (5.2) contributes to the first integral on the right of (5.4) [by virtue of the orthogonality of the surface harmonics $S_1(\theta, \phi)$ and $S_n(\theta, \phi)$, $(n > 1)$]. Hence

$$\int_S \mathbf{x} \times (\mathbf{n} \times \mathbf{B}) dS = -a \int \mathbf{n} \frac{\partial}{\partial r} \mu r^{-2} \cos\theta \, dS = \frac{8\pi}{3}\boldsymbol{\mu}, \tag{5.5}$$

and so (5.3) yields

$$\boldsymbol{\mu}(t) = (3/8\pi)\int_V \mathbf{B}(\mathbf{x}, t) dV. \tag{5.6}$$

We note in passing that when $\mathbf{B}$ can be decomposed [equation (4.2)] into a mean and fluctuating part, the fluctuating part must integrate to zero over the fluid domain, and so (5.6) gives equivalently

$$\boldsymbol{\mu}(t) = (3/8\pi)\int_V \langle \mathbf{B} \rangle dV. \tag{5.7}$$

A further use of the divergence theorem shows that (5.6) is equivalent

to

$$\boldsymbol{\mu}(t) = (3/8\pi)\int_S \mathbf{x}(\mathbf{B}\cdot\mathbf{n})dS. \tag{5.8}$$

This form for $\boldsymbol{\mu}(t)$ permits the simplest proof of the Bondi and Gold result. Let S_+ be that part of S on which $\mathbf{B}\cdot\mathbf{n}>0$, and S_- that part on which $\mathbf{B}\cdot\mathbf{n}<0$, and let

$$\boldsymbol{\mu}_\pm = (3/8\pi)\int_{S_\pm} \mathbf{x}(\mathbf{B}\cdot\mathbf{n})dS, \tag{5.9}$$

so that $\boldsymbol{\mu}=\boldsymbol{\mu}_+ + \boldsymbol{\mu}_-$. We then have

$$|\boldsymbol{\mu}_+| \leqq 3a\Phi/8\pi, \qquad |\boldsymbol{\mu}_-| \leqq 3a\Phi/8\pi, \tag{5.10}$$

where

$$\Phi = \int_{S_+}(\mathbf{n}\cdot\mathbf{B})dS = -\int_{S_-}(\mathbf{n}\cdot\mathbf{B})dS. \tag{5.11}$$

Now, when $\lambda=0$, Φ is constant, since flux through every closed material circuit is conserved, and so

$$|\boldsymbol{\mu}| \leqq |\boldsymbol{\mu}_+| + |\boldsymbol{\mu}_-| \leqq 3a\Phi/4\pi, \tag{5.12}$$

the maximum being attained only when the flux is entirely concentrated at poles at opposite ends of a diameter of the sphere.†

There can therefore be no doubt that, when $\lambda=0$, exponential increase of the dipole moment is impossible, no matter what the complexity (laminar or turbulent) of the velocity field in V may be. This is in stark contrast with the inferences that may be drawn from equation (4.10), if the estimates (4.9) are valid as $\lambda\to 0$. We must therefore re-examine the steps described in §4, with a view to pinpointing the source of this crucial contradiction.

6. THE INHOMOGENEITY LAYER $a-r=O(l_0)$‡

The first weak point in the discussion of §4 is the implicit assumption that results derived from consideration of a field of homogeneous turbulence can be applied without qualification to a turbulent motion inside a sphere,

†An ingenious use of the conservation of Φ as given by (5.11) has recently been proposed by Hide (1978); see also Hide (1979). Note that the condition $\mathbf{u}\cdot\mathbf{n}=0$ on S is implicit in the result $\Phi=$const.

‡I am greatly indebted to Dr. R. H. Kraichnan, whose comments on an earlier version of this paper led to the discussion as now presented in §6—see Kraichnan (1979). I am indebted also to the referee of the earlier version whose comments stimulated an important change of emphasis in the presentation.

which by virtue of the effect of the boundary, is *necessarily* inhomogeneous. Clearly, if $\mathbf{u}\cdot\mathbf{n}=\mathbf{0}$ on $r=a$, then there must be an inhomogeneity layer on $r=a-$, of thickness δ say, and it is to be expected that $\delta=O(l_0)$. The turbulent intensity may be expected to decrease as $r\uparrow a$, (and it will decrease to zero if the no-slip condition $\mathbf{u}=0$ is satisfied on $r=a$). Likewise, the electromotive force $\mathscr{E}$ and the associated coefficients α, β, ... may be expected to be smaller within this layer than in the core region $a-r\gg l_0$.

The important effect that this may have on the growth rate of unstable modes of equation (4.4) may be appreciated by considering the extreme situation in which α and β are both zero within the layer $L_\delta: a-\delta<r<a$. Let C_μ be the circle $r=a-\mu\delta$ $(0<\mu<1)$ in the equatorial plane $z=0$ and let S_μ be the disc spanning C_μ. Then, from (4.4), with $\mathscr{E}=0$ on C_μ, we have

$$(d/dt)\int_{S_\mu}\langle\mathbf{B}\rangle\cdot d\mathbf{S}=-\lambda\int_{C_\mu}(\nabla\times\langle\mathbf{B}\rangle)\cdot d\mathbf{x}. \tag{6.1}$$

Now as $\lambda\to 0$, it is to be expected that $|\nabla\times\langle\mathbf{B}\rangle|$ will be of order $B/\delta\sim B/l_0$ (independent of λ) at interior points of L_δ, so that

$$(d/dt)\int_{S_\mu}\langle\mathbf{B}\rangle\cdot d\mathbf{S}=O(\lambda)\quad\text{as}\quad\lambda\to 0. \tag{6.2}$$

This suggests that the growth rate p of any unstable mode of (4.4) must then satisfy

$$p=O(\lambda)\quad\text{as}\quad\lambda\to 0. \tag{6.3}$$

This is to be contrasted with the conventional α^2-dynamo behaviour for which $p=O(\lambda^0)$ as $\lambda\to 0$. The effect of the inhomogeneity layer is to change the structure of the eigenfunctions in such a way as to reduce the growth rate to a level compatible with flux diffusion across L_δ.

It seems reasonable to conjecture that, as for the disc dynamo problem considered in Part I, an inhomogeneity layer of the above type will not affect the criterion for dynamo action (e.g. $R_\alpha>4.49$ for the α^2-dynamo in a sphere), but *will* in general have an important effect on the growth rate of unstable modes, and equally on the frequency of oscillation of oscillatory modes. There would seem to be a good case for analytical and numerical investigation of this effect on the various α^2- and $\alpha\omega$-dynamous that have been described in the literature.†

It must be noted however that the Bondi and Gold result described in

†One example of an $\alpha\omega$-dynamo for which the growth rates are fully determined is described in M78, §9.9. In this example, the α-effect is concentrated in layers embedded in a slab of diffusivity λ, and the growth rates (real or complex) of the eigenmodes all have the asymptotic behaviour $p\sim\lambda(\log\lambda)^2$ as $\lambda\to 0$.

§5 requires only that $\mathbf{u}\cdot\mathbf{n}=0$ on $r=a$ [and not the stronger condition $\mathbf{u}=0$]. Moreover, the coefficients α, β,... are not in general purely *local* properties of the turbulence, but depend on the statistics of particle paths (see 4.6–4.8) which determine the field $\mathbf{b}$ of magnetic perturbations. Hence it is by by no means obvious that α, β, ... *must* decrease to zero on $r=a$ (although more detailed analyses may reveal that this is in fact the case).

7. POSSIBLE DIVERGENCE OF THE SERIES (4.5)

The second weak point of the discussion of §4 concerns the neglect of all terms indicated by ... in the series (4.5). This neglect is plausible when λ is bounded away from zero; however, when $\lambda=0$, the multiple integrations over Lagrangian particle paths which appear in the determination of successive coefficients in the series (4.5) (of which eqn. (4.7) provides only a foretaste!) may well lead to numerical divergence of the coefficients, despite the helpful factor $(l_0/L)^n$ in the nth term.†

If the series (4.5) in fact diverges when $\lambda=0$, then it is necessary to restore the effects of weak molecular diffusivity. We have already noted in the simpler disc dynamo context the crucial need for the changing magnetic field to diffuse into the region in which the primitive induction process occurs (i.e. into each helical eddy in the turbulent context). This suggests that we may be able to accelerate the convergence of the series (4.5) by incorporating a time-lag in the linear relationship between $\langle\mathbf{B}\rangle$ and $\mathscr{E}$. The simplest such relationship is

$$\partial\mathscr{E}/\partial t=-(1/t_\lambda)(\mathscr{E}-\alpha\langle\mathbf{B}\rangle+\beta'\nabla\times\langle\mathbf{B}\rangle-\cdots), \tag{7.1}$$

or equivalently

$$\mathscr{E}(\mathbf{x},t)=(1/t_\lambda)\int_{-\infty}^{t}(\alpha'-\beta'\nabla\times+\ldots)\langle\mathbf{B}\rangle(\mathbf{x},\tau)e^{-(t-\tau)/t_\lambda}d\tau. \tag{7.2}$$

Formal manipulations of (4.4) and (4.5) provide the relations‡

†Kraichnan (1979) has shown that the expansion converges (when $\lambda=0$) for a particular "kinematically possible" field of turbulence consisting of statistically independent but identically distributed fields on successive time intervals, provided the turbulence is "not too intermittent". It is not as yet clear whether this result holds for general dynamically realisable (Navier–Stokes) turbulence.

‡More generally, if we write

$$\mathscr{E}=\sum_{n=0}^{\infty}\alpha^{(n)}(-\nabla\times)^n\langle\mathbf{B}\rangle,$$

$$\alpha' = \alpha, \qquad \beta' = \beta - t_\lambda \alpha^2, \cdots \tag{7.3}$$

The coefficient β' may be regarded as a "renormalised" eddy diffusivity. The time t_λ is here unspecified, but should simply have the property, characteristic of a diffusive time-scale, that

$$t_\lambda \to \infty \quad \text{as} \quad \lambda \to 0. \tag{7.4}$$

The relationship $\beta = \beta' + t_\lambda \alpha^2$ is immediately suggestive in comparison with (4.7). Incorporation of a "diffusion" factor $\exp[-(t-\tau)/t_\lambda]$ under the second integral of (4.7) in fact yields the integral

$$\int_0^t \alpha_1(t)\alpha_1(\tau)e^{-(t-\tau)/t_\lambda} d\tau,$$

which converges to $t_\lambda \alpha^2$ as $t \to \infty$. The effects of molecular diffusivity are of course extremely subtle, and nothing approaching an adequate theory is as yet available. Removal of a potential infinity in one of the terms contributing to β in (4.7) by the renormalisation process suggested above, does however seem to provide additional motivation for the procedure.

At any rate, we shall now examine the consequences of (7.1), in conjunction with (4.4), and under the assumption that†

$$\alpha' \sim u_0, \qquad \beta' \sim u_0 l_0, \tag{7.5}$$

as $\lambda \to 0$. Again we shall neglect the effect of terms in the series in (7.1) represented by $\cdots$. For simplicity, we first examine the growth rate of

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then

$$\partial \mathscr{E}/\partial t = -(1/t_\lambda)\left[\mathscr{E} - \sum_{n=0}^{\infty} \alpha^{(n)'}(-\nabla \times)^n \langle \mathbf{B} \rangle\right],$$

where

$$\begin{cases} \alpha^{(0)'} = \alpha^{(0)}, \\ \alpha^{(n)'} = \alpha^{(n)} - t_\lambda[\alpha^{(0)}\alpha^{(n-1)} + \alpha^{(1)}\alpha^{(n-2)} + \ldots + \alpha^{(n-1)}\alpha^{(0)}], \quad (n = 1, 2, \ldots). \end{cases}$$

Another possible modification of (4.5) (suggested to me by Professor J. T. Stuart) which makes more explicit appeal to the diffusion process is

$$\partial \mathscr{E}/\partial t - \lambda \nabla^2 \mathscr{E} = -(1/t_\lambda)\left(\mathscr{E} - \sum_{n=0}^{\infty} \alpha^{(n)''}(-\nabla \times)^n \langle \mathbf{B} \rangle\right).$$

In this case

$$\alpha^{(n)''} = \begin{cases} \alpha^{(n)'} & (n = 0, 1) \\ \alpha^{(n)'} + \lambda t_\lambda \alpha^{(n-2)} & (n = 2, 3, \ldots). \end{cases}$$

†This assumption requires that $\beta \to \infty$ as $t_\lambda \to \infty$, in contrast with Kraichnan's (1976b) conclusion.

"free modes" in an infinite medium (cf. M78, §9.2). These have a "force-free" structure satisfying

$$\nabla \times \langle \mathbf{B} \rangle = K \langle \mathbf{B} \rangle, \qquad \nabla \times \mathscr{E} = K \mathscr{E}, \tag{7.6}$$

so that (4.4) and (7.1) become

$$\left.\begin{aligned} (\partial/\partial t)\langle \mathbf{B} \rangle &= K\mathscr{E} - \lambda K^2 \langle \mathbf{B} \rangle, \\ \partial \mathscr{E}/\partial t &= -(1/t_\lambda)(\mathscr{E} - \alpha' \langle \mathbf{B} \rangle + \beta' K \langle \mathbf{B} \rangle). \end{aligned}\right\} \tag{7.7}$$

These equations admit solutions $(\mathscr{E}, \langle \mathbf{B} \rangle) \propto e^{pt}$ where p satisfies the quadratic equation

$$(p + \lambda K^2)(p + t_\lambda^{-1}) - K t_\lambda^{-1}(\alpha' - \beta' K) = 0. \tag{7.8}$$

Provided

$$(\lambda + \beta')K^2 < K\alpha', \tag{7.9}$$

there is a positive real root, namely

$$p_1 = -(\lambda K^2 + t_\lambda^{-1}) + \{(\lambda K^2 + t_\lambda^{-1})^2 + 4t_\lambda^{-1}(K\alpha' - (\lambda + \beta')K^2)\}^{1/2}, \tag{7.10}$$

and this indicates dynamo action. Now however, as $\lambda \to 0$ (so that $t_\lambda^{-1} \to 0$ also), we have $p_1 \to 0$ and the resulting field structure is therefore steady (rather than exponentially growing) in the limit $\lambda = 0$.

A similar property obviously holds for dynamo-type solutions of (4.4) and (7.1) in a spherical geometry. In this case, the relevant solutions do not satisfy the force-free conditions (7.6), but they *do* satisfy the higher order Beltrami conditions

$$\nabla \times \nabla \times \langle \mathbf{B} \rangle = K \nabla \times \langle \mathbf{B} \rangle, \qquad \nabla \times \nabla \times \mathscr{E} = K \nabla \times \mathscr{E}, \tag{7.11}$$

(cf. the discussion of M78, §9.4) and it is not difficult to show that the growth rate is still given by (7.10) where $|K|a$ satisfies

$$J_{n+1/2}(|K|a) = 0, \qquad (n = 1, 2, \ldots), \tag{7.12}$$

$J_m(x)$ denoting the Bessel function of order m. The smallest root (with $n = 1$) is $|K|a \approx 4.49$, and the inequality (7.9) is satisfied provided

$$a|\alpha'| > 4.49(\lambda + \beta'). \tag{7.13}$$

Again, however, $p_1 \to 0$ as $\lambda \to 0$, and in particular the dipole moment of the resulting field structure is constant (rather than exponentially increasing) in the limit $\lambda = 0$. The model described by equations (4.4) and (7.1) therefore shows no inconsistency with the Bondi and Gold constraint.

In the above dicussion we have concentrated on the α^2-dynamo in a spherical geometry with constant α. This is in fact a rather artificial situation, but it has the advantage of being amenable to simple analysis. In the solar context (in which the behaviour as $\lambda \to 0$ is of critical importance), the $\alpha\omega$-model (Steenbeck and Krause 1969) with a distribution of α that is antisymmetric about the equatorial plane is more relevant. It is known that the growth rate for this type of dynamo has the form

$$p = \frac{\lambda+\beta}{a^2} F\left(\frac{\omega_0'|\alpha|a^4}{(\lambda+\beta)^2}\right) \sim \frac{\beta}{a^2} F\left(\frac{\omega_0'|\alpha|a^4}{\beta^2}\right) \quad \text{as} \quad \lambda \to 0, \tag{7.14}$$

where ω_0' is a typical value of the gradient of angular velocity, and that dynamo action occurs ($\operatorname{Re} p > 0$) If $X = \omega_0'|\alpha|a^4/\beta^2$ exceeds a critical value X_c of order unity (M78, §9.12). Here again there is a conflict with the Bondi and Gold result whenever $a > (\beta^2 X_c/\omega_0'|\alpha|)^{1/4}$. There seems little doubt that this conflict can again be resolved by replacement of (4.5) by (7.1), in conjunction with the mean induction equation.

8. DISCUSSION

This paper has been concerned with the growth rates (p) of magnetic instabilities in simple dynamo systems, and in particular with the dependence of these growth rates on magnetic diffusivity λ in the limit $\lambda \to 0$. In Part I, we showed that the traditional description of the self-excited disc dynamo leads to the conclusion $p = O(\lambda^0)$ as $\lambda \to 0$, in conflict with the fundamental property of flux conservation across the disc in this limit. This conflict was resolved by allowing for an azimuthal current in the disc, (or equivalently, for diffusion of axial flux across the rim of the disc); the criterion for dynamo instability is then unchanged, but the growth rate satisfies $p = O(\lambda)$ as $\lambda \to 0$, consistent with the flux conservation principle.

In Part II, an analogous conflict relating to traditional dynamos of α^2- and $\alpha\omega$-type is considered. If, as is widely believed, the generation coefficient α and the eddy diffusivity β tend to non-zero finite values as $\lambda \to 0$, then the growth rates of these dynamos are again $O(\lambda^0)$ as $\lambda \to 0$, and this is in conflict with the exact result of Bondi and Gold (1950) that the dipole moment is in this limit permanently bounded (a result closely dependent on the flux conservation principle). Two possible means of escape from this conflict were discussed in §§6 and 7: (i) The growth rates may be strongly affected by the (necessary) presence of an inhomogeneity layer within which $|\alpha|$ and β are greatly reduced, and across which flux associated with a dynamo instability must diffuse through the molecular diffusion process alone; it seems likely that this effect will reduce growth

rates to $O(\lambda)$ (an effect that is obviously important in the solar context). (ii) Regarding each helical eddy of the turbulence as analogous to a miniature disc dynamo, it is suggested in §7 that in the customary linear relationship between $\mathscr{E}$ and $\langle \mathbf{B} \rangle$, it is physically realistic to incorporate a time lag t_λ (where $t_\lambda \to \infty$ as $\lambda \to 0$) between the mean flux across an eddy and the emf generated [eqns. (7.1) and (7.2)]. This leads to the appearance of a "renormalised" eddy diffusivity β' [eqn. (7.3)] and, more importantly, to model equations for the mean field which are entirely compatible with the Bondi and Gold result in the limit $\lambda \to 0$.

These considerations suggest that it would be useful to re-examine the various α^2- and $\alpha\omega$-dynamos that have been described in the literature and, in particular, to determine the influence of (i) an inhomogeneity layer and (ii) a time-lag in the relationship between $\mathscr{E}$ and $\langle \mathbf{B} \rangle$, on the growth rates of unstable modes in such models.

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