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A NEW FAMILY OF MIXED FINITE ELEMENTS FOR THE LINEAR ELASTODYNAMIC PROBLEM*

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Abstract. We construct and analyze a new family of quadrangular (in two dimensions) or cubic (in three dimensions) mixed finite elements for the approximation of elastic wave equations. Our elements lead to explicit schemes (via mass lumping), after time discretization, including in the case of anisotropic media. Error estimates are given for these new elements.

Key words. mixed finite elements, mass lumping, elastodynamics

AMS subject classifications. 65M60, 65M12, 65M15, 65C20, 73D25

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1. Introduction.

In this paper we develop and analyze a new family of mixed finite elements for the velocity-stress formulation of elastodynamics.

We have recently developed an efficient numerical method for approximating the linear elastodynamic problem in complex media, such as anisotropic heterogeneous media with cracks of arbitrary shapes [10]. We use the fictitious domain method to take into account the boundary condition on the crack. This consists of extending the initial problem in a domain with a simple geometry (typically a rectangle in two dimensions) and considering the boundary condition as an equality constraint in a given functional space. The boundary condition on the crack is a free surface condition (the normal stress is zero). Thus, to express it as an equality constraint, the stress tensor has to be one of the unknowns. That is our first motivation for using the mixed velocity-stress formulation. The second one is that this formulation is compatible with a very efficient absorbing layer model [15], which generalizes to elastodynamics the perfectly matched layer introduced by Bérenger [11, 12].

We are concerned in this paper with the space discretization of this formulation. For efficiency reasons (to reduce the computation time and memory requests), we want to have mass lumping in order to get a really explicit method after time discretization.

Several families of mixed finite elements have been proposed in the literature for static plane elasticity. The main difficulty appearing in this problem is finding a way to take into account the symmetry of the stress tensor. Actually, taking into account the symmetry in the approximation space is not easy and can lead to numerical locking (this is what happens with Raviart–Thomas elements). The approach which is usually followed is the relaxed symmetry approach: the symmetry is imposed in a weak sense via a Lagrange multiplier [1, 2, 23, 24, 21]. Another approach using spaces of symmetric stress tensors, based on composite elements, was introduced in [19]. None of these, however, are adapted to mass lumping.

For this reason, we have introduced mixed finite elements (inspired by Nédélec's second family [22]), which appear to be new in spite of their simplicity. These new elements have two basic characteristics: they allow mass lumping and they use spaces

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of symmetric stress tensors. In [7], we have presented the lowest order element in two dimensions.

The outline of this paper is as follows. The first part (section 2) is devoted to the description of these elements in two and three space dimensions. In section 2.1, we define some notations and present the model evolution problem we shall consider, as well as its variational mixed formulation. In section 2.2.1, we present the construction of our lowest order elements in dimension 2 and show how we obtain mass lumping. Section 2.2.2 is devoted to the generalization to higher order elements and section 2.3 to the generalization to the three-dimensional case. The second part concerns the analysis of these new elements in two dimensions. The main difficulty is that the classical Babuska–Brezzi assumptions for the discrete problem (cf. [13, 5]) are not satisfied (defect of coerciveness). In a previous paper [9], we developed and analyzed a scalar version of these mixed finite elements. The approach is not directly adapted to the elastodynamic problem and we present here a different analysis. Section 3 is concerned with the mixed approximation of an elliptic problem, namely a stationary version of the evolution problem of section 2. In section 3.1, we explain why the new element does not fit the classical theory, and we announce a convergence result (Theorem 3). Sections 3.2 and 3.3 are devoted to the proof of this theorem, which requires in particular the use of a macroelement approach (cf. [26]). Section 4 is devoted to the analysis of the semidiscrete time dependent problem without mass lumping. This analysis, which essentially relies on energy estimates, relates the error estimates on the time domain solution to the error estimates obtained in the previous section on the elliptic problem.

2. Presentation of the new mixed finite elements.

2.1. The model problem: The two-dimensional elastic wave equation.

In the following, we identify the space of 2×2 tensors with the space $\mathcal{L}(\mathbb{R}^2)$ of linear applications from $\mathbb{R}^2$ to $\mathbb{R}^2$, in which we define the linear form

$$\mathrm{as}(\sigma) = \sigma_{12} - \sigma_{21}.$$

Let $\mathcal{L}^s(\mathbb{R}^2)$ be the subspace of symmetric tensors of $\mathcal{L}(\mathbb{R}^2)$, that is,

$$\mathcal{L}^s(\mathbb{R}^2) = \left\{ \sigma \in \mathcal{L}(\mathbb{R}^2) \ / \ \mathrm{as}(\sigma) = 0 \right\}.$$

The scalar product in $\mathcal{L}(\mathbb{R}^2)$ is defined (with summation convention) by

$$\sigma : \tau = \sigma_{ij}\tau_{ij} \quad \forall (\sigma, \tau) \in \mathcal{L}(\mathbb{R}^2), \quad |\sigma|^2 = \sigma : \sigma.$$

In what follows, a tensor σ is indifferently identified with an element of $\mathbb{R}^4$ or an element of $\mathbb{R}^2 \times \mathbb{R}^2$. In particular, we associate with each σ in $\mathcal{L}(\mathbb{R}^2)$ the two column vectors

$$\sigma_1 = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \end{bmatrix}. \tag{2.1}$$

If Ω is an open subspace of $\mathbb{R}^2$ and σ is a tensor field in Ω ($\sigma \in \mathcal{D}'(\Omega; \mathcal{L}(\mathbb{R}^2))$), we set

$$\mathrm{div}\ \sigma = \begin{bmatrix} \mathrm{div}\ \sigma_1 \\ \\ \mathrm{div}\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} \end{bmatrix}.$$

If $(u, v) \in (L^2(\Omega))^d$ $(d = 1, 2, 3)$, $(u, v)_\Omega$ denotes the usual inner product of $(L^2(\Omega))^d$.

2.1.1. The elastodynamic problem. We now consider that Ω is filled by an elastic material. We denote by $u(x,t)$ the displacement field in Ω at time t ($x \in \Omega, t > 0$). We associate with $u(x,t)$ the strain tensor $\varepsilon(u)$:

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In the following, we consider ε a linear operator from $\mathcal{D}'(\Omega;\mathbb{R}^2)$ into $\mathcal{D}'(\Omega;\mathcal{L}^s(\mathbb{R}^2))$. The displacement is governed by the following system:

$$\varrho\frac{\partial^2 u}{\partial t^2} - \text{div}\ \sigma(u) = f, \tag{2.2}$$

where $\varrho = \varrho(x)$ denotes the density which verifies

$$0 < \varrho_- \leq \varrho(x) \leq \varrho_+ < +\infty, \qquad \text{a.e. } x \in \Omega.$$

We should add to (2.2) the initial conditions, homogeneous for simplicity,

$$u(t=0) = 0; \qquad \frac{\partial u}{\partial t}(t=0) = 0,$$

and a boundary condition on $\partial\Omega$, for instance, the homogeneous Dirichlet condition

$$u = 0 \text{ on } \partial\Omega.$$

The stress tensor $\sigma(u)$ is related to the deformation tensor by Hooke's law:

$$\sigma(u)(x,t) = C(x)\varepsilon(u)(x,t),$$

where for all x in Ω, $C(x)$ is a 4×4 positive tensor having the usual symmetry properties [3, 4], which means that $C(x)$ is a positive symmetric definite operator in $\mathcal{L}^s(\mathbb{R}^2)$. We set $A(x) = C(x)^{-1}$ and assume that $x \to A(x)$ is measurable and satisfies the uniform inequalities (α and M are strictly positive real numbers)

$$0 < \alpha\,|\sigma|^2 \leq A(x)\sigma : \sigma \leq M\,|\sigma|^2 \quad \forall \sigma \in \mathcal{L}^s(\mathbb{R}^2),\ \text{a.e. } x \in \Omega. \tag{2.3}$$

The elastodynamic problem in Ω can be written as a first order hyperbolic system, the velocity-stress system, with the unknowns $v = \partial u/\partial t$ (the velocity vector) and $\sigma = \sigma(u)$ (the stress tensor):

$$\begin{cases} \varrho\dfrac{\partial v}{\partial t} - \text{div}\ \sigma = f, \\ A\dfrac{\partial \sigma}{\partial t} - \varepsilon(v) = 0, \end{cases} \tag{2.4}$$

subject to the initial conditions $v(0) = 0$, $\sigma(0) = 0$ and the boundary condition

$$v = 0 \text{ on } \partial\Omega. \tag{2.5}$$

2.1.2. Variational formulation in $\underline{\underline{X}}^{sym}$. We introduce the Hilbert spaces

$$\underline{M} = L^2(\Omega;\mathbb{R}^2), \quad \underline{\underline{H}} = L^2(\Omega;\mathcal{L}(\mathbb{R}^2)), \quad \text{and} \quad \underline{\underline{X}} = \left\{\sigma \in \underline{\underline{H}} \;/\; \text{div } \sigma \in \underline{M}\right\}.$$

We denote $\underline{\underline{X}}^{sym} \subset \underline{\underline{X}}$ the subspace of symmetric tensors in $\underline{\underline{X}}$:

$$\underline{\underline{X}}^{sym} = \left\{\sigma \in \underline{\underline{X}} \;/\; \text{as}(\sigma) = 0\right\}.$$

We can write now a mixed variational formulation of system (2.4), (2.5) in the form

$$(2.6) \qquad \begin{cases} \text{Find } (\sigma, v) : [0,T] \longmapsto \underline{\underline{X}}^{sym} \times \underline{M} \text{ such that} \\ \dfrac{d}{dt}\, a(\sigma(t),\tau) + b(v(t),\tau) = 0 \qquad \forall \tau \in \underline{\underline{X}}^{sym}, \\ \dfrac{d}{dt}\, c(v(t),w) - b(w,\sigma(t)) = (f,w) \quad \forall w \in \underline{M}, \end{cases}$$

where

$$(2.7) \qquad \begin{cases} a(\sigma,\tau) = \displaystyle\int_\Omega A\sigma : \tau dx \qquad \forall(\sigma,\tau) \in \underline{\underline{H}} \times \underline{\underline{H}}, \\ c(v,w) = \displaystyle\int_\Omega \varrho v \cdot w dx \qquad \forall(v,w) \in \underline{M} \times \underline{M}, \\ b(w,\tau) = \displaystyle\int_\Omega \text{div } \tau \cdot w dx \quad \forall(w,\tau) \in \underline{\underline{X}} \times \underline{M}. \end{cases}$$

The bilinear form $a(\cdot,\cdot)$ (resp., $b(\cdot,\cdot)$) is continuous on $\underline{\underline{H}} \times \underline{\underline{H}}$ (resp., on $\underline{\underline{X}} \times \underline{M}$); thus we can define linear continuous operators $\mathcal{A} : \underline{\underline{H}} \to \underline{\underline{H}}'$ and $B : \underline{\underline{X}} \to \underline{M}'$ by

$$\langle \mathcal{A}\sigma, \tau\rangle_{H' \times H} = a(\sigma,\tau); \; \langle B\tau, w\rangle_{M' \times M} = b(w,\tau),$$

where H' holds for the dual space of H, and so on. We set $B^{sym} = B_{|\underline{\underline{X}}^{sym}} \in \mathcal{L}(\underline{\underline{X}}^{sym}, \underline{M}')$. The following properties are satisfied (e.g., [14, 18]):

(2.8)

(i) The continuous inf-sup condition

$$\exists\, \beta > 0 \;/\; \forall\; w \in \underline{M},\; \exists\; \tau \in \underline{\underline{X}}^{sym}, \tau \neq 0 \;/\; b(w,\tau) \geq \beta \left\|w\right\|_M \left\|\tau\right\|_X.$$

(ii) The coercivity of the form $a(\cdot,\cdot)$ on Ker B^{sym} = Ker $B \cap \underline{\underline{X}}^{sym}$

$$\exists\; \alpha > 0 \;/\; \forall\; \sigma \in \text{Ker } B^{sym},\; a(\sigma,\sigma) \geq \alpha \left\|\sigma\right\|_X^2,$$

with Ker $B^{sym} = \left\{\tau \in \underline{\underline{X}}^{sym} \;/\; b(w,\tau) = 0,\; \forall\; w \in \underline{M}\right\}$.

The mixed formulation (2.6) is the one we shall work with for the numerical approximation. Note that it is crucial to work in the space $\underline{\underline{X}}^{sym}$ of symmetric tensors because the operator $-\varepsilon$ is not the adjoint of div if one works in $\underline{\underline{X}}$.

2.2. Presentation of the finite element in two dimensions.

2.2.1. The lowest order finite element. We explain in this section the construction of the lowest order element (see also [7] where we include a dispersion analysis). In what follows, we shall use the following standard notation for spaces of polynomials of two variables. We denote by P_k the space of polynomials of degree less than k, and we define

$$P_{k,l} = \left\{ p(x_1, x_2) \;\mid\; p(x_1, x_2) = \sum_{i \le k, j \le l} a_{ij} x_1^i x_2^j \right\}$$

and $Q_k = P_{k,k}$. We suppose now that Ω is a union of rectangles in such a way that we can consider a regular mesh $(\mathcal{T}_h)$ with squares elements (K) of edge $h > 0$. To obtain our finite element spaces, we shall adopt a constructive approach which aims in particular to exploit the geometry of the mesh. We look for approximation spaces

$$\left\{ \begin{array}{l} \underline{M_h} = \{ v_h \in \underline{M} \;/\; \forall K \in \mathcal{T}_h,\; v_h|_K \in \mathcal{P} \}, \\[2ex] \underline{\underline{X_h}} = \left\{ \sigma_h \in \underline{\underline{X}} \;/\; \forall K \in \mathcal{T}_h,\; \sigma_h|_K \in \mathcal{Q} \right\}, \\[2ex] \underline{\underline{X_h}}^{sym} = \left\{ \sigma_h \in \underline{\underline{X_h}};\, /\ \mathrm{as}(\sigma_h) = 0 \right\}, \end{array} \right. \tag{2.9}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are finite-dimensional spaces of C^∞ functions—for instance, polynomials—to be determined. We can remark that $\underline{\underline{X_h}}^{sym}$ is sought as a subspace of $\underline{\underline{X}}^{sym}$: we want to take into account the symmetry condition in the strong sense. Then the approximate problem associated with the mixed velocity-stress system for elastodynamics can be written in the following form:

$$\left\{ \begin{array}{ll} \text{Find } (\sigma_h, v_h) : [0,T] \longmapsto \underline{\underline{X_h}}^{sym} \times \underline{M_h} \text{ such that} & \\[2ex] \dfrac{d}{dt} a(\sigma_h, \tau_h) + b(v_h, \tau_h) = 0 & \forall \tau_h \in \underline{\underline{X_h}}^{sym}, \\[2ex] \dfrac{d}{dt} c(v_h, w_h) - b(w_h, \sigma_h) = (f, w_h) & \forall w_h \in \underline{M_h} \end{array} \right. \tag{2.10}$$

with initial conditions $\sigma_h(0) = 0$, $v_h(0) = 0$.

Let us construct our lowest order spaces $\underline{\underline{X_h}}$ and $\underline{M_h}$. It is natural to approximate the velocity with piecewise constants

$$\underline{M_h} = \left\{ v_h \in \underline{M} \;/\; \forall K \in \mathcal{T}_h,\; v_h|_K \in (Q_0)^2 \right\}.$$

A possible choice for $\underline{\underline{X_h}}$ is the lowest order Raviart–Thomas element $RT_{[0]}$:

$$\underline{\underline{X_h}}^{RT} = \left\{ \sigma_h \in \underline{\underline{X}} \;/\; \forall K \in \mathcal{T}_h,\; (\sigma_1, \sigma_2)|_K \in (RT_{[0]})^2 \right\}, \quad \text{where } RT_{[0]} = P_{1,0} \times P_{0,1}.$$

However, this choice is not satisfactory, since the space $\underline{\underline{X_h}}^{RT} \cap \underline{\underline{X}}^{sym}$ is too small and thus cannot be considered as a good approximation space for $\underline{\underline{X}}^{sym}$. Indeed, if σ_h is a symmetric tensor in $\underline{\underline{X_h}}^{RT}$, then it is easy to see that σ_{12} is necessarily constant in Ω. That is a kind of numerical locking.

We shall construct $\underline{\underline{X_h}}^{sym}$ as a subspace of $\underline{\underline{X}}^{sym}$, guided by the following observation, which is true only because we consider a mesh with squares whose edges are parallel to the coordinate axis. (We omit the proof, which is trivial.)

THEOREM 1. *For all* $\sigma_h \in \underline{\underline{X_h}}$, *where* $\underline{\underline{X_h}}$ *is given by* (2.9), *we have*

$$\sigma_h \in \underline{\underline{X_h}}^{sym} \Longleftrightarrow \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \end{pmatrix} \in H(\mathrm{div}, \Omega) \quad \textit{and} \quad \sigma_{12} = \sigma_{21} \in H^1(\Omega).$$

Theorem 1 shows that σ_{12} must belong to an approximation space of H^1. Thus, in order to define the lowest order element, it is natural to choose

$$\sigma_{12} \in H^1(\Omega)/\forall K \in \mathcal{T}_h,\ \sigma_{12}\,|_K \in Q_1.$$

It now remains to define the approximation space of $H(\mathrm{div}, \Omega)$ in which the vector $(\sigma_{11}, \sigma_{22})^t$ belongs. Once again, a natural choice is the lowest order Raviart–Thomas element $RT_{[0]}$. We denote

$$\underline{\underline{\widetilde{X}_h}}^{sym} = \left\{\sigma_h \in \underline{\underline{X}}^{sym}/\forall K \in \mathcal{T}_h,\ \sigma_{12}\,|_K \in Q_1 \text{ and } (\sigma_{11}, \sigma_{22})\,|_K \in RT_{[0]}\right\}.$$

However, we can easily show that this choice does not permit us to get an explicit time discretization scheme. This was explained in [9] for the anisotropic wave equation. The reason is that the degrees of freedom for the stress tensor are associated either with a vertex of an element K (for σ_{12}) or with an edge (for $(\sigma_{11}, \sigma_{22})$). To obtain an explicit time discretization scheme, we want to use a mass lumping technique for the approximation of the mass matrix associated with the bilinear form $a(\sigma_h, \tau_h)$. (The reader can verify that the matrix associated with $c(v_h, w_h)$ is already diagonal in the usual basis of $\underline{M_h}$.) Thus, we are led to approximate the mass matrix $a(\sigma_h, \tau_h)$ by

$$a_h(\sigma_h, \tau_h) = \sum_{K \in \mathcal{T}_h} I_K(A\sigma_h : \tau_h),$$

where I_K is some quadrature formula, to be determined. The key point for choosing the approximation space of $H(\mathrm{div}, \Omega)$ is to regroup all the degrees of freedom at the nodes of the quadrature formula, namely the nodes of the mesh. Under this condition the adequate choice for $(\sigma_{11}, \sigma_{22})$ is the lowest order element of the second family of mixed finite elements proposed by Nédélec in [22], that is,

$$\left\{q_h \in H(\mathrm{div}, \Omega) \text{ such that } q_h\,|_K \in (Q_1(K))^2 \quad \forall K \in \mathcal{T}_h\right\},$$

and thus our choice for the space $\underline{\underline{X_h}}^{sym}$ can be written as

$$\begin{aligned} \underline{\underline{X_h}}^{sym} = \quad & \left\{\sigma_{12} \in H^1(\Omega)/\sigma_{12}\,|_K \in Q_1 \quad \forall K \in \mathcal{T}_h \text{ and}\right. \\ & \left.(\sigma_{11}, \sigma_{22}) \in H(\mathrm{div}, \Omega)/(\sigma_{11}, \sigma_{22})\,|_K \in (Q_1)^2 \quad \forall K \in \mathcal{T}_h\right\}. \end{aligned} \tag{2.11}$$

With this choice, the degrees of freedom for the stress tensor are all associated with the vertices of an element K, as we can see in Figure 1. (We choose point values instead of moments as in [22]; see also section 2.2.2 for a complete description of the degrees of freedom.) In this case the approximation of $a(\sigma_h, \tau_h)$ using the quadrature formula in K,

$$\int_k f dx \approx I_K(f) = \frac{h^2}{4} \sum_{M \in \{\text{vertices of } K\}} f(M) \qquad \forall f \in C^o(K),$$

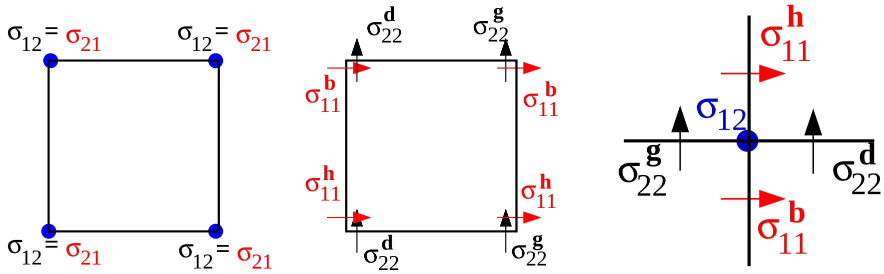

FIG. 1. *Degrees of freedom for the stress tensor.*

leads to a block diagonal mass matrix. Each block is associated with a node of the mesh and its dimension is equal to the number of degrees of freedom at this point (that is, 5; see Figure 1).

2.2.2. Extension to higher orders and mass lumping. The natural generalization of the lowest order element presented in section 2.2.1 consists of taking

$$\text{(2.12)} \qquad \begin{cases} \underline{\underline{X_h}} = \left\{ \tau_h \in X \,/\, \forall K \in \mathcal{T}_h,\ \tau_h|_K \in (Q_{k+1})^4 \right\}, \\ \underline{\underline{X_h}}^{sym} = \underline{\underline{X_h}} \cap \underline{\underline{X}}^{sym}, \\ \underline{M_h} = \left\{ w_h \in \underline{M} \,/\ \forall K \in \mathcal{T}_h,\ w_h|_K \in (Q_k)^2 \right\}. \end{cases}$$

This will be referred as the $Q_{k+1}^{div} - Q_k$ element. The locations of the degrees of freedom for these elements correspond to tensor products of one-dimensional quadrature points associated with Gauss–Lobatto (for σ_h) or Gauss–Legendre (for v_h) quadrature formulas [16]. As an illustration, we represent the degrees of freedom of the element corresponding to $k = 1$ in Figure 2 and we also indicate the number of degrees of freedom per node. One can notice—this is general for all the higher order elements—that, for σ_h, there are three kind of nodes:

1. The nodes located at a vertex of an element, which are associated with 5 degrees of freedom: the value of σ_{12}, the two values (up and down) of σ_{11}, and the two values (left and right) of σ_{22}.
2. The nodes located on an edge, associated with 4 degrees of freedom:
   - for a vertical edge: the two values (left and right) of σ_{22}, the value of σ_{11}, and the value of σ_{12},
   - for a horizontal edge: the two values (up and down) of σ_{11}, the value of σ_{22}, and the value of σ_{12}.
3. The interior nodes located inside one element, associated with 3 degrees of freedom: the value of σ_{12}, the value of σ_{11}, and the value of σ_{22}.

Mass lumping can then be achieved using the same technique as for the anisotropic wave equation [9]. That consists of approximating the integrals in the mass matrices $a(\sigma_h, \tau_h)$ and $c(v_h, w_h)$ by adequate quadrature formulas. More precisely, we use the Gauss–Lobatto quadrature formulas to compute $a(\sigma_h, \tau_h)$. The resulting matrix is now block diagonal. Each block is associated with one quadrature point and its dimension is equal to the number of degrees of freedom at this point: from 3×3 for the interior nodes to 5×5 for the vertices of the mesh. Note that $c(v_h, w_h)$ leads naturally to a block diagonal matrix. However, using the Gauss–Legendre quadrature formula and appropriate degrees of freedom leads to a diagonal matrix.

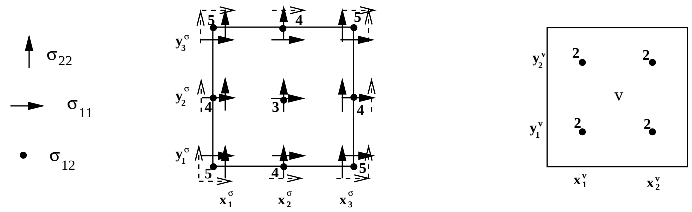

FIG. 2. *Degrees of freedom in the $Q_2 - Q_1$ element.*

2.2.3. On some properties of the $Q_{k+1}^{div} - Q_k$ element. We regroup in this section important theoretical properties of the spaces $\underline{\underline{X_h}}$ and $\underline{M_h}$ that will be useful for the convergence analysis. We introduced in [9] the spaces $\underline{X_h}$ and M_h defined by

$$(2.13)\qquad \begin{cases} \underline{X_h} = \left\{ q_h \in \underline{X} = H(\mathrm{div};\Omega) \,/\, \forall K \in \mathcal{T}_h,\ q_h|_K \in \underline{X_k} \right\}, \\ \underline{X_k} = Q_{k+1} \times Q_{k+1}, \\ M_h = \left\{ w_h \in L^2(\Omega) \,/\ \forall K \in \mathcal{T}_h,\ w_h|_K \in Q_k \right\}, \end{cases}$$

and we have shown that $\underline{X_h}$ admits the following orthogonal (in L^2) decomposition:

$$(2.14)\qquad \left| \begin{array}{l} \underline{X_h} = \underline{X_h}^s \oplus \underline{X_h}^r, \\ \underline{X_h}^s = \left\{ p_h \in H(\mathrm{div};\Omega) \,/\, \forall K \in \mathcal{T}_h,\ p_h|_K \in RT_{[k]} \right\}, \\ \underline{X_h}^r = \left\{ p_h \in H(\mathrm{div};\Omega) \,/\, \forall K \in \mathcal{T}_h,\ p_h|_K \in \Psi_k \right\}, \end{array} \right.$$

where Ψ_k is defined as the orthogonal complement in $\underline{X_k}$ of $RT_{[k]}$ (for the inner product of $L^2(K)$):

$$(2.15)\qquad \Psi_k(K) = \left\{ \psi \in \underline{X_k} \,/\, \int_K \psi\, \phi\, dx = 0\ \forall \phi \in RT_{[k]} \right\}.$$

We recall here the well-known property of Raviart–Thomas elements:

$$\forall p_h^s \in \underline{X_h}^s, \quad \mathrm{div} p_h^s \in M_h.$$

Thus, as

$$\underline{\underline{X_h}} = \underline{X_h} \times \underline{X_h},$$

it is straightforward that $\underline{\underline{X_h}}$ admits the orthogonal decomposition

$$(2.16)\qquad \underline{\underline{X_h}} = \underline{\underline{X_h}}^s \oplus \underline{\underline{X_h}}^r,$$

with

$$(2.17)\qquad \underline{\underline{X_h}}^s = \underline{X_h}^s \times \underline{X_h}^s \quad \text{and} \quad \underline{\underline{X_h}}^r = \underline{X_h}^r \times \underline{X_h}^r$$

and

$$(2.18)\qquad \forall \tau_h^s \in \underline{\underline{X_h}}^s, \quad \mathrm{div} \tau_h^s \in \underline{M_h}.$$

Moreover, one can prove easily the fundamental following properties of the space $\underline{\underline{X_h}}^r$ (see [9]):

$$(2.19)\quad \begin{cases} \text{(i)} & \forall \eta_h \in M_h\ \forall \tau_h^r \in \underline{\underline{X_h}}^r,\ ((\tau_h^r)_{ij}, \eta_h) = 0 \quad \forall i,j = 1,2, \\ \text{(ii)} & \forall K \in \mathcal{T}_h,\ \forall T_j,\ j = 1,\dots,4,\ \displaystyle\int_{T_j} \tau_h^r n.q\, d\gamma = 0 \quad \forall q \in (P_k(T_j))^2, \\ & \text{where } T_j \text{ are the edges of } K, \text{ i.e., } \partial K = T_1 \cup T_2 \cup T_3 \cup T_4, \\ \text{(iii)} & \forall w_h \in \underline{M_h},\ \forall \tau_h^r \in \underline{\underline{X_h}}^r,\ (\mathrm{div}\tau_h^r, w_h) = 0, \end{cases}$$

where property (iii) directly follows from properties (i) and (ii).

Finally, we give another characterization of the space $\underline{\underline{X_h}}^{sym}$ defined in (2.12) that follows from Theorem 1:

$$(2.20)\qquad \underline{\underline{X_h}}^{sym} = \left\{\sigma \in \underline{\underline{X}}^{sym} /\ \sigma_{12} = \sigma_{12} \in Q_h^{k+1},\ (\sigma_{11}, \sigma_{22}) \in \underline{X_h}\right\},$$

where Q_h^{k+1} is the space of continuous functions, locally Q_{k+1}:

$$(2.21)\qquad Q_h^{k+1} = \left\{\phi \in C^0(\overline{\Omega}),\ \phi|_K \in Q_{k+1}\ \forall K \in \mathcal{T}_h\right\}.$$

2.3. Extension to three dimensions. We shall present in this section only the lowest order three dimensions finite element. The extension to higher orders is similar to the two-dimensional case. Let Ω be a bounded domain of $\mathbb{R}^3$ that can be meshed with cubic elements (K) of edge $h > 0$. Applying the same approach as in the two-dimensional case, we remark that imposing the symmetry as a strong condition implies the following:

- $(\sigma_{11},\ \sigma_{22},\ \sigma_{33}) \in H(\mathrm{div}, \Omega)$ since σ_{11} is continuous only in the x_1 direction, σ_{22} in the x_2, and σ_{33} in the x_3 direction,
- σ_{12}, σ_{13}, and σ_{23} are continuous in two directions. More precisely, if i, j, k denotes a circular permutation of $1, 2, 3$,

$$x_i \to \sigma_{jk}(\cdot, x_i) \in L^2_{x_i}(H^1_{x_j,x_k}).$$

We introduce the approximation spaces $\underline{\underline{X_h}}^{sym} - \underline{M_h}$ defined by (2.9) with

$$\mathcal{Q} = (Q_1)^9, \qquad \mathcal{P} = (Q_0)^3.$$

We give in Figure 3 a schematic view of this element, via the degrees of freedom.

3. Analysis of the new mixed finite element in the two-dimensional static case. For the study of the time dependent problem, we shall follow the approach of [9] concerning the anisotropic wave equation. It is made of two main steps. One step consists of relating, thanks to energy estimates, error estimates for the evolution problem to the estimation of the difference between the exact solution and its elliptic projection (that has to be cleverly defined). This step is rather standard and delayed to section 4. The second step, which amounts to analyzing the elliptic projection error, directly follows from the analysis of the approximation of the stationary problem associated with the evolution problem (2.10). This is the object of the present section.

In the following, in order to simplify the presentation, we will consider that Ω is a square in such a way that we can consider a regular mesh $(\mathcal{T}_h)$ composed by $N \times N$ squares elements (K) of edge $h = 1/N$. However, our results can be extended without any difficulty to any general structured mesh.

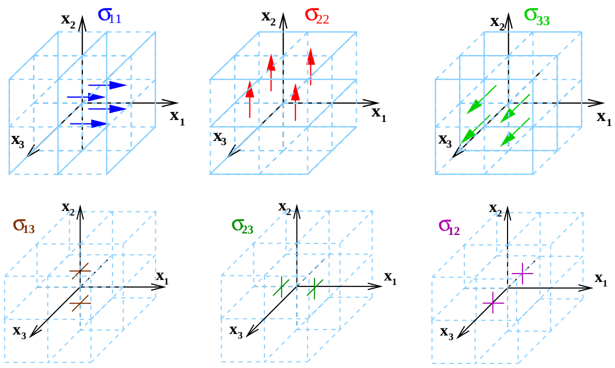

FIG. 3. *A schematic view of the lowest order element in three dimensions.*

3.1. The elliptic problem in $\underline{\underline{X}}^{sym}$ and its approximation. The stationary problem associated with (2.6) consists of finding $(\sigma, v) \in \underline{\underline{X}}^{sym} \times \underline{M}$ such that

$$(3.1)\qquad \left\{ \begin{aligned} a(\sigma,\tau) + b(v,\tau) &= 0 && \forall \tau \in \underline{\underline{X}}^{sym}, \\ b(w,\sigma) &= -(f,w) && \forall w \in \underline{M}, \end{aligned} \right.$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by (2.7) and satisfy properties (2.8). As a consequence [14], there exists a unique solution (σ, v) in $\underline{\underline{X}}^{sym} \times \underline{M}$ of problem (3.1). The corresponding discrete problem consists of finding $(\sigma_h, v_h) \in \underline{\underline{X_h}}^{sym} \times \underline{M_h}$ such that

$$(3.2)\qquad \left\{ \begin{aligned} a(\sigma_h,\tau_h) + b(v_h,\tau_h) &= 0 && \forall \tau_h \in \underline{\underline{X_h}}^{sym}, \\ b(w_h,\sigma_h) &= -(f,w_h) && \forall w \in \underline{M_h}. \end{aligned} \right.$$

The main difficulty in the convergence analysis of problem (3.2) is that the finite element spaces (2.12) do not satisfy the hypotheses of the classical theory [13, 5]. In particular, one can easily prove that the uniform discrete coercivity condition is not satisfied, because $\operatorname{div}(\underline{\underline{X_h}}^{sym}) \not\subset \underline{M_h}$ (see [9]). This difficulty was already present for the case of the scalar problem that we analyzed in a previous paper [9]: we proposed a modification of the classical theory based on the orthogonal decomposition (2.14). An important point in the proof is that the subspace $\underline{X_h}^s$ (see section 2.2.3) is a good approximation of space $\underline{X}$, in the sense that for all $p \in \underline{X}$

$$\lim_{h\to 0} \inf_{q_h^s \in \underline{X_h}^s} \|p - q_h^s\|_X = 0.$$

As we have noticed that in section 2.2.3 this implies the analogous decomposition (2.16) for space $\underline{\underline{X_h}}$. The difficulty is to find an appropriate decomposition for space $\underline{\underline{X_h}}^{sym}$. Actually a natural candidate for the decomposition would be

$$\underline{\underline{X_h}}^{sym} = (\underline{\underline{X_h}}^s \cap \underline{\underline{X}}^{sym}) \oplus (\underline{\underline{X_h}}^r \cap \underline{\underline{X}}^{sym}),$$

but this choice is not satisfactory since $\underline{\underline{X_h}}^s \cap \underline{\underline{X}}^{sym}$ is not an approximation space of $\underline{\underline{X}}^{sym}$ (numerical locking). We did not succeed in finding a more appropriate choice.

Thus, we develop a different convergence analysis in the L^2 framework for both unknowns σ and v. It is less obvious to obtain estimates of $\mathrm{div}(\sigma - \sigma_h)$ (see section 5 below). For this, we use only the coercivity of the form $a(.,.)$ on the space $\underline{\underline{H}}$

$$(3.3)\qquad \left|\ \exists \alpha > 0 \,/\, \forall\, \tau \in \underline{\underline{H}},\ a(\tau,\tau) \geq \alpha \, |\tau|^2_H \,.\right.$$

We can introduce, as in the case of the continuous problem, the operators B_h : $\underline{\underline{X_h}}^{sym} \to \underline{M_h}$ and $B_h^t : \underline{M_h} \to \underline{\underline{X_h}}^{sym}$ such that

$$(B_h \tau_h, w_h)_M = \big(\tau_h, B_h^t w_h\big)_X = b(w_h, \tau_h) \ \ \forall (\tau_h, w_h) \in \underline{\underline{X_h}}^{sym} \times \underline{M_h},$$

and we define

$$V_h(f) = \Big\{ \tau_h \in \underline{\underline{X_h}}^{sym} \,/\, b(w_h, \tau_h) = -(f, w_h) \ \ \forall w_h \in \underline{M_h} \Big\},$$

$$V_h = V_h(0) \equiv \ \mathrm{Ker} B_h.$$

For proving error estimates, we shall use the following lemma, which will be proved in section 3.2.

LEMMA 1. *One has the discrete uniform inf-sup condition*

$$(3.4)\qquad \left|\ \begin{aligned} &\exists \beta > 0 \,/\, \forall h, \forall w_h \in \underline{M_h},\ \exists\ \tau_h \in \underline{\underline{X_h}}^{sym},\ \ \tau_h \neq 0 \,/ \\ &b(w_h, \tau_h) \geq \beta \, \|w_h\|_M \, \|\tau_h\|_X \,. \end{aligned}\right.$$

Notation. For any integer $m \geq 0$, we introduce the space

$$(3.5)\qquad H^{m,m+1}(\Omega) = \{\psi \in H^m(\Omega), /\ \partial_2 \psi \in H^m(\Omega)\}$$

and similarly for $H^{m+1,m}(\Omega)$, and we set

$$(3.6)\qquad |\psi|_{m,m+1} = |\psi|_m + |\partial_2 \psi|_m \,, \quad |\psi|_{m+1,m} = |\psi|_m + |\partial_1 \psi|_m \,,$$

where $|.|_m$ denotes the seminorm on $H^m(\Omega)$.

As a consequence of (3.3) and of the continuity of the bilinear form $a(\cdot,\cdot)$ on $\underline{\underline{H}} \times \underline{\underline{H}}$, we deduce the following result (see [14, Prop. 2.13 and Rem. 2.14]).

THEOREM 2. *Problem* (3.2) *admits a unique solution* $(\sigma_h, v_h) \in \underline{\underline{X_h}}^{sym} \times \underline{M_h}$, *which satisfies*

$$(3.7)\qquad \begin{aligned} |\sigma - \sigma_h|_H + \|v - v_h\|_M \leq C \Bigg\{ &\inf_{\tau_h \in V_h(f)} |\sigma - \tau_h|_H + \inf_{w_h \in \underline{M_h}} \|v - w_h\|_M \\ &+ \inf_{w_h \in \underline{M_h}} \sup_{\tau_h \in V_h} \frac{b(v - w_h, \tau_h)}{\alpha \, |\tau_h|_H} \Bigg\}. \end{aligned}$$

This allows us to establish our convergence theorem.

THEOREM 3. *Let* $(\sigma, v) \in \underline{\underline{X}}^{sym} \times \underline{M}$ *be the solution of problem* (3.1) *and* $(\sigma_h, v_h) \in \underline{\underline{X_h}}^{sym} \times \underline{M_h}$ *be the solution of problem* (3.2). *If* $(\sigma_{11}, \sigma_{22}) \in H^{1,0}(\Omega) \times H^{0,1}(\Omega)$ *and* $\sigma_{12} \in H^1(\Omega)$, *then*

$$(3.8)\qquad |\sigma - \sigma_h|_H + \|v - v_h\|_M \longrightarrow 0 \quad \textit{when} \ \ h \to 0.$$

Furthermore, if $(\sigma_{11}, \sigma_{22}) \in H^{k+2,k+1}(\Omega) \times H^{k+1,k+2}(\Omega)$*,* $\sigma_{12} \in H^{k+2}(\Omega)$*, and* $v \in (H^{k+2}(\Omega))^2$*, then one has the error estimates*

$$\begin{aligned}|\sigma - \sigma_h|_H + \|v - v_h\|_M \leq Ch^{k+1} \Big(&|\sigma_{11}|_{k+2,k+1} + |\sigma_{22}|_{k+1,k+2} + |\sigma_{12}|_{k+2} \\ &+ |v|_{k+1} + |\nabla v|_{k+1} \Big).\end{aligned} \tag{3.9}$$

The proof of this theorem uses the following technical lemma, which will be proved in section 3.3.

LEMMA 2. *For any* $w \in (H_0^1(\Omega))^2$*,*

$$\lim_{h\to 0} \inf_{w_h \in \underline{M_h}} \sup_{\tau_h \in V_h} \frac{b(w - w_h, \tau_h)}{\alpha\, |\tau_h|_H} = 0. \tag{3.10}$$

Furthermore, for all $w \in (H_0^1(\Omega))^2 \cap (H^{k+2}(\Omega))^2$*, there exists a* $w_h \in \underline{M_h}$ *such that*

$$\sup_{\tau_h \in V_h} \frac{b(w - w_h, \tau_h)}{\alpha\, |\tau_h|_H} \leq Ch^{k+1}\, |w|_{k+2,\Omega}\,. \tag{3.11}$$

Proof of Theorem 3. The proof consists of estimating each term of the right-hand side of (3.7).

- Using the inf-sup condition (3.4), we can bound the first term by (see [14])

$$\inf_{\tau_h \in V_h(f)} |\sigma - \tau_h|_H \leq \inf_{\tau_h \in V_h(f)} \|\sigma - \tau_h\|_X \leq C \inf_{\tau_h \in \underline{\underline{X_h}}^{sym}} \|\sigma - \tau_h\|_X\,.$$

Then from specific properties of the Raviart–Thomas approximation space $\underline{X_h}^s$ on regular meshes, combined with classical approximations properties of Q_h^{k+1}, one gets (see Appendix B), if $(\tau_{11}, \tau_{22}) \in H^{1,0} \times H^{0,1}$ and $\tau_{12} \in H^1$,

$$\lim_{h\to 0} \inf_{\tau_h \in \underline{\underline{X_h}}^{sym}} \|\tau - \tau_h\|_X = 0. \tag{3.12}$$

Furthermore, if $(\tau_{11}, \tau_{22}) \in H^{k+2,k+1} \times H^{k+1,k+2}$ and $\tau_{12} \in H^{k+2}$, we have

$$\inf_{\tau_h \in \underline{\underline{X_h}}^{sym}} \|\tau - \tau_h\|_X \leq Ch^{k+1}\, (|\tau_{11}|_{k+2,k+1} + |\tau_{22}|_{k+1,k+2} + |\tau_{12}|_{k+2}). \tag{3.13}$$

- The standard approximation property in $\underline{M_h}$ gives

$$\lim_{h\to 0} \inf_{w_h \in \underline{M_h}} \|v - w_h\|_M = 0 \quad \forall v \in \underline{M}.$$

Furthermore, for all $w \in (H^{k+1}(\Omega))^2$, we have

$$\inf_{w_h \in \underline{M_h}} \|w - w_h\|_M \leq\ C\ h^{k+1}\, |w|_{k+1}. \tag{3.14}$$

- Finally the last term is estimated thanks to Lemma 2. □

3.2. Proof of Lemma 1. To prove (3.4) we first construct a particular $\tau \in \underline{\underline{X}}^{sym}$ candidate for the continuous inf-sup condition.

LEMMA 3. *For all* $v \in \underline{M}$ *there exists a* $\tau \in \underline{\underline{X}}^{sym}$ *such that*

$$\text{(i)} \quad \tau \textit{ is diagonal (thus as}(\tau) = 0),$$

$$\text{(ii)} \quad \text{div } \tau = v,$$

$$\text{(iii)} \quad \|\tau\|_X \leq C_1 \|v\|_M.$$

Proof. In the case of a rectangular domain Ω it is easy to prove that for all $v = (v_1, v_2) \in (L^2(\Omega))^2 (= \underline{M})$ there exists a $\tau = (\tau_1, \tau_2) \in H(\text{div}, \Omega)(= \underline{\underline{X}})$ with

$$\tau_1 = \begin{pmatrix} \tau_{11} \\ 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 \\ \tau_{22} \end{pmatrix},$$

where τ_1, τ_2 verify

$$\text{div}\tau_1 = \frac{\partial \tau_{11}}{\partial x_1} = v_1 \text{ and } \text{div}\tau_2 = \frac{\partial \tau_{22}}{\partial x_2} = v_2.$$

Indeed, we define $\widetilde{v}_1$(resp., $\widetilde{v}_2$) the extension by zero of v_1(resp., v_2) at the exterior of Ω. We can define then τ_{11} (resp., τ_{22}) as a primitive of $\widetilde{v}_1$ (resp., $\widetilde{v}_2$):

$$\tau_{11}(x_1, x_2) = \int_0^{x_1} \widetilde{v}_1(s, x_2) ds, \qquad \tau_{22}(x_1, x_2) = \int_0^{x_1} \widetilde{v}_2(x_1, s) ds.$$

It is clear then that $\tau = (\tau_1, \tau_2)$ satisfy (i), (ii), and (iii) of Lemma 3. ◻

Then, by using the well-known properties of the Raviart–Thomas mixed finite element space, we are able to show the following lemma, which immediately implies Lemma 1.

LEMMA 4. *For all* $v_h \in \underline{M_h}$, *there exists* $\tau_h \in \underline{\underline{X_h}}^{sym}$ *such that*

$$\text{(i)} \quad (\text{div}\tau_h, w_h) = (v_h, w_h) \;\; \forall w_h \in \underline{M_h},$$

$$\text{(ii)} \quad \|\tau_h\|_X \leq C_1 \|v_h\|_M$$

with C_1 *a positive constant independent of* h.

Proof. Take v_h any element of $\underline{M_h}$. By Lemma 3, we can construct $\tau^h \in \underline{\underline{X}}^{sym}$ such that

$$\begin{aligned} &\text{(a1)} \quad \tau^h \text{ is diagonal } \Rightarrow \text{as}(\tau^h) = 0, \\ &\text{(a2)} \quad \text{div}\tau^h = v_h, \\ &\text{(a3)} \quad \|\tau^h\|_X \leq C_1 \|v_h\|_M. \end{aligned} \tag{3.15}$$

Let Π_h^s be the usual interpolation operator on the Raviart–Thomas space $\underline{\underline{X_h}}^s$ (see definition (2.17)). It is well known (see [14]) that

$$\begin{aligned} &\text{(b1)} \quad (\text{div}(\tau - \Pi_h^s \tau), w_h) = 0 \quad \forall (\tau, w_h) \in \underline{\underline{X}} \times \underline{M_h}, \\ &\text{(b2)} \quad \|\Pi_h^s \tau\|_X \leq C_2 \|\tau\|_X \quad \forall \tau \in \underline{\underline{X}}. \end{aligned} \tag{3.16}$$

Moreover, because of the particular mesh we work with, we have

$$\tau \text{ is diagonal } \Rightarrow \Pi_h^s \tau \text{ is diagonal, too, and thus } \Pi_h^s \tau \in \underline{\underline{X_h}}^{sym}. \tag{3.17}$$

Let us now take $\tau_h = \Pi_h^s \tau^h$ and check that it satisfies properties (i) and (ii) of Lemma 4. Indeed,

(i) is a consequence of (3.15(a2)) and (3.16(b1)).

(ii) is a consequence of (3.15(a3)) and (3.16(b2)). □

3.3. Proof of Lemma 2.

A macroelement partitioning. To prove Lemma 2, we will use a macroelement technique (cf. [25]). This technique will permit us to obtain a global estimate by simply adding together analogous local estimates. We need to introduce some notations.

We define $\{\tau^j,\ j = 1, N_\tau\}$ the basis function of $\underline{\underline{X_h}}^{sym}$ ($N_\tau = \dim \underline{\underline{X_h}}^{sym}$) associated with the degrees of freedom defined in section 2.2.2. It is easy to show that these basis functions satisfy

(3.18)

(a) $\left\|\tau^j\right\|_{L^\infty} = 1,$

(b) $\exists\, C > 0,\quad \text{C independent of } h,\quad \forall j,\quad \left\|\text{div}\tau^j\right\|_{L^2} \le C,$

(c) $\exists\, C_1 > 0$ and $C_2 > 0$, C_1 and C_2 independent of h,

$$\forall\ \tau_h = \sum_j \alpha_j \tau_j, \quad C_1\ h \left(\sum_j \alpha_j^2\right)^{1/2} \le |\tau_h|_H \le C_2\ h \left(\sum_j \alpha_j^2\right)^{1/2}.$$

We shall define a macroelement M_e associated with each vertex of the mesh $\mathcal{T}_h$, S_e $(1 \le e \le Ne = (N+1)^2)$ as the union of the elements having S_e as a common node; see Figure 4. We obviously have

$$\Omega = \bigcup_{e=1,\dots,N_e} M_e \tag{3.19}$$

and the finite overlapping property (that is essential for property (c))

(3.20) each element $K \in \mathcal{T}_h$ is included in at most 4 macroelements.

For each macroelement M_e ($e = 1, \dots, N_e$), we define the basis functions τ^j, $j \in \mathcal{J}(e)$, of the macroelement as the τ^j which have a support included in M_e. The number of basis functions in the macroelements is bounded, the bound depending only on the order of the element:

$$\exists J_k > 0 \text{ independent of } h \text{ such that } \text{card}(\mathcal{J}(e)) \le J_k. \tag{3.21}$$

One can notice that each basis function belongs to at least one macroelement and at most four macroelements. That is why we define weighting coefficients λ_{je} for each basis function τ^j, $j = 1, \dots, N_\tau$, associated with a degree of freedom located at a node R_j (the nodes are the locations of the degrees of freedom; see section 2.2.2):

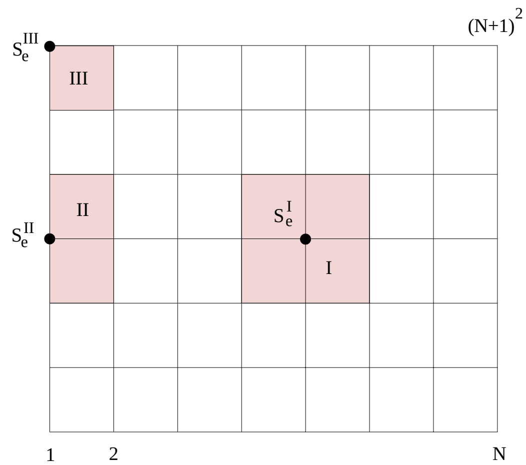

FIG. 4. *Definition of the macroelement:* I *for an interior node,* II *for a node on the boundary (distinct of a vertex),* III *for a vertex.*

- If R_j is a vertex of the mesh, there exists a unique $e(j) \in 1, \dots, N_e$ such that $R_j = S_{e(j)}$,

$$\lambda_{jf} = \delta_{fe(j)}, \quad f = 1, \dots, N_e.$$

- If R_j is located on an edge of the mesh, there exists a unique $(e_1(j), e_2(j))$ such that $R_j \in [S_{e_1(j)}, S_{e_2(j)}]$,

$$\lambda_{jf} = \frac{1}{2}\delta_{fe_1(j)} + \frac{1}{2}\delta_{fe_2(j)}, \quad f = 1, \dots, N_e.$$

- If R_j is an interior node, there exists a unique $(e_1(j), e_2(j), e_3(j), e_4(j))$ such that R_j belongs to the element of the mesh whose vertices are $S_{e_m(j)}$, $m = 1, \dots, 4$,

$$\lambda_{jf} = \frac{1}{4}\delta_{fe_1(j)} + \frac{1}{4}\delta_{fe_2(j)} + \frac{1}{4}\delta_{fe_3(j)} + \frac{1}{4}\delta_{fe_4(j)}, \quad f = 1, \dots, N_e.$$

By construction, one has the property

$$\sum_{e=1}^{N_e} \lambda_{je} = 1 \quad \forall j = 1, \dots, N_\tau. \tag{3.22}$$

DEFINITION 1. *For each macroelement M_e ($e = 1, \dots, N_e$), and for any function $w \in \underline{M}$, we define the polynomial function w_e as $w_e = 0$ outside M_e, while inside M_e it is defined as follows:*

- *If S_e is an interior node, w_e is the $L^2(M_e)$ projection of w on $(P_{k+1}(M_e))^2$, the space of functions which are polynomial of degree $k+1$;*

$$P_{k+1}(M_e) = \left\{ p(x_1, x_2) \,/\, p(x_1, x_2) = \sum_{i+j \le k+1} a_{ij} x_1^i x_2^j \right\}.$$

- *If S_e is a boundary node, distinct of a vertex, w_e is the $L^2(M_e)$ projection of w on to the subspace $(P^t_{k+1}(M_e))^2$ of $(P_{k+1}(M_e))^2$, defined by functions which are polynomial of degree k in the variable tangent to the boundary. For instance, in the case of a vertical boundary (the tangential direction being $t = x_2$), the polynomial space $(P^t_{k+1}(M_e))^2$ is defined by*

$$P^t_{k+1}(M_e) = P^{x_2}_{k+1}(M_e) = \left\{ p(x_1,x_2) \,/\, p(x_1,x_2) = \sum_{\substack{i+j \le k+1,\\ j \le k}} a_{ij} x_1^i x_2^j \right\}.$$

- *If S_e is a vertex of the domain, w_e is zero.*

We can then prove the following lemmas.

LEMMA 5. *For all $w \in (H^1_0(\Omega))^2 \cap (H^{k+2}(\Omega))^2$, one has*

$$\|w - w_e\|_{0,M_e} \le C h^{k+2} \, |w|_{k+2,M_e}. \tag{3.23}$$

Proof. If S_e is an interior node, the result is standard and can be obtained using Bramble–Hilbert's lemma.

For a node S_e located on the boundary, one uses the fact that w vanishes on a part of the boundary of M_e. One can first prove (see Appendix A) that for a reference macroelement $\widehat{M}_e$, if $\widehat{w} \in (H^1(\widehat{M}_e))^2 \cap (H^{k+2}(\widehat{M}_e))^2$, and $\widehat{w}$ vanishes on a vertical edge of $\widehat{M}_e$ for instance, then there exists a constant $C > 0$ such that

$$\|\widehat{w} - \widehat{w}_e\|_{0,\widehat{M}_e} \le C \, |\widehat{w}|_{k+2,\widehat{M}_e}, \tag{3.24}$$

$\widehat{w}_e$ being the $L^2(\widehat{M}_e)$ projection of $\widehat{w}$ onto $(P^t_{k+1}(\widehat{M}_e))^2$. Coming back on the current macroelement M_e gives the adequate power of h in estimate (3.23). ∎

The key point of the analysis is the following nice result, which has been pointed out to us by one of the referees of this paper.

LEMMA 6. *For all $w \in \underline{M}$, for all macroelement M_e, and for every basis function of the macroelement τ^j, $j \in \mathcal{J}(e)$, one has*

$$\int_{M_e} (P_{M_h} w_e - w_e) \text{ div } \tau^j dx = 0, \tag{3.25}$$

where $P_{M_h} : \underline{M} \to \underline{M_h}$ denotes the orthogonal projection onto $\underline{M_h}$ and w_e is defined from w by Definition 1.

Proof. For each $e = 1, \dots, N_e$ and for all $j \in \mathcal{J}(e)$, we set

$$I = \int_{M_e} (P_{M_h} w_e - w_e) \text{ div } \tau^j dx.$$

For proving this result, one uses the orthogonal decomposition of space $\underline{\underline{X_h}}$ given in (2.16), which shows that

$$\text{div } \tau^j = \text{div } (\tau^j)^s + \text{div } (\tau^j)^r \quad \text{with} \quad (\tau^j)^s \in \underline{\underline{X_h}}^s \quad \text{and} \quad (\tau^j)^r \in \underline{\underline{X_h}}^r.$$

From property (2.18), one deduces that $\text{div}(\tau^j)^s \in \underline{M_h}$; therefore, from the definition of P_{M_h}, one has

$$\int_{M_e} (P_{M_h} w_e - w_e) \text{ div } (\tau^j)^s dx = 0.$$

Thus

$$I = \int_{M_e} (P_{M_h} w_e - w_e) \text{ div } (\tau^j)^r dx.$$

One now uses property (2.19(iii)) which implies

$$\int_{M_e} P_{M_h} w_e \text{ div } (\tau^j)^r dx = 0.$$

Finally, by Green's formula, we have

$$I = -\int_{M_e} w_e \text{ div } (\tau^j)^r dx = \int_{M_e} \nabla w_e \ : \ (\tau^j)^r dx - \int_{\partial M_e} (\tau^j)^r n.w_e \ d\gamma.$$

The vector function w_e belongs to $(P_{k+1}(M_e))^2$, so that ∇w_e belongs to M_h and, using (2.19(i)), the first term of the right-hand side vanishes.

The term on the boundary also vanishes, if S_e is an interior node, since $(\tau^j)^r \in H(\text{div}, \Omega)$ is supported in M_e. If S_e is a vertex, it also vanishes since $w_e = 0$. For a node S_e on the boundary, distinct of a vertex of Ω—for instance, on a vertical boundary—the normal component of $(\tau^j)^r$ does not vanish any more on $\partial\Omega \cap \partial M_e$. However, as $w_e \in P_{k+1}^{x_2}$, its restriction to $\partial\Omega \cap \partial M_e$ is a polynomial of degree k in x_2, and property (2.19(ii)) permits us to conclude. □

We can now prove the following lemma which will imply in particular the estimate (3.11) of Lemma 2.

LEMMA 7. *For all* $w \in (H_0^1(\Omega))^2 \cap (H^{k+2}(\Omega))^2$ *and for all* $\tau_h \in \underline{\underline{X_h}}^{sym}$, *we have*

$$b(w - P_{M_h} w, \tau_h) \leq C h^{k+1} |\tau_h|_H \, |w|_{k+2,\Omega}\,, \tag{3.26}$$

where P_{M_h} *is defined as in Lemma* 6 *and* $C > 0$ *is a constant independent of* h.

Proof. We can decompose τ_h on the basis $\tau_h = \sum_{j=1}^{N_\tau} \alpha_j \tau^j$, which can be written, using property (3.22), as

$$\tau_h = \sum_{j=1}^{N_\tau} \sum_{e=1}^{N_e} \lambda_{je} \alpha_j \tau^j = \sum_{e=1}^{N_e} \sum_{j=1}^{N_\tau} \lambda_{je} \alpha_j \tau^j = \sum_{e=1}^{N_e} \sum_{j \in \mathcal{J}(e)} \lambda_{je} \alpha_j \tau^j.$$

We then obtain

$$b(w - P_{M_h} w, \tau_h) = \sum_{e=1}^{N_e} \sum_{j \in \mathcal{J}(e)} \lambda_{je} \alpha_j \int_{M_e} (w - P_{M_h} w) \text{ div}\tau^j dx.$$

From (3.25) in Lemma 6, this can be written as

$$\begin{aligned} b(w - P_{M_h} w, \tau_h) &= \sum_{e=1}^{N_e} \sum_{j \in \mathcal{J}(e)} \lambda_{je} \alpha_j \int_{M_e} (w - w_e + P_{M_h}(w_e - w)) \text{ div}\tau^j dx \\ &\leq \sum_{e=1}^{N_e} \sum_{j \in \mathcal{J}(e)} \lambda_{je} |\alpha_j| \left(\|w - w_e\|_{0,M_e} + \|P_{M_h}(w - w_e)\|_{0,M_e} \right) \left\| \text{div}\tau^j \right\|_{0,M_e}. \end{aligned}$$

Using that P_{M_h} is a projection operator, property (3.18(b)) of the basis functions and estimate (3.23) of Lemma 5, one deduces that

$$
\begin{aligned}
b(w - P_{M_h} w, \tau_h) &\leq C h^{k+2} \sum_{e=1}^{N_e} \sum_{j \in \mathcal{J}(e)} \lambda_{je} \, |\alpha_j| \, |w|_{k+2,M_e} \\
&\leq C h^{k+2} \left(\sum_{e=1}^{N_e} \sum_{j \in \mathcal{J}(e)} \lambda_{je} \, |\alpha_j|^2 \right)^{1/2} \left(\sum_{e=1}^{N_e} \sum_{j \in \mathcal{J}(e)} \lambda_{je} \, |w|^2_{k+2,M_e} \right)^{1/2} \\
&\leq C h^{k+2} \left(\sum_{j=1}^{N_\tau} \left(\sum_{e=1}^{N_e} \lambda_{je} \right) |\alpha_j|^2 \right)^{1/2} \left(\sum_{e=1}^{N_e} \left(\sum_{j \in \mathcal{J}(e)} \lambda_{je} \right) |w|^2_{k+2,M_e} \right)^{1/2} .
\end{aligned}
$$

Using properties (3.21) and (3.22) we obtain

$$
b(w - P_{M_h} w, \tau_h) \leq J_k \; h^{k+2} \left(\sum_{j=1}^{N_\tau} |\alpha_j|^2 \right)^{1/2} \left(\sum_{e=1}^{N_e} |w|^2_{k+2,M_e} \right)^{1/2} .
$$

We conclude, thanks to the finite overlapping property (3.20) and to inequality (3.18(c)), that

$$
b(w - P_{M_h} w, \tau_h) \leq J_k / C_1 \; h^{k+1} \, |\tau_h|_H \, |w|_{k+2,\Omega} . \qquad \square
$$

End of the proof of Lemma 2. It remains to prove (3.10) by the density of $(H_0^1(\Omega))^2 \cap (H^{k+2}(\Omega))^2$ in $(H_0^1(\Omega))^2$: for any $w \in (H_0^1(\Omega))^2$ and for any $\varepsilon > 0$, there exists $w_\varepsilon \in (H_0^1(\Omega))^2 \cap (H^{k+2}(\Omega))^2$ such that

$$
|w - w_\varepsilon|_1 \leq \varepsilon/2.
$$

We then decompose

$$
\frac{b(w, \tau_h)}{|\tau_h|_H} = \frac{b(w - w_\varepsilon, \tau_h)}{|\tau_h|_H} + \frac{b(w_\varepsilon, \tau_h)}{|\tau_h|_H}.
$$

Since $w_\varepsilon \in (H_0^1(\Omega))^2 \cap (H^{k+2}(\Omega))^2$, we can apply Lemma 7; thus, for all $\tau_h \in V_h$ ($b(P_{M_h} w_\varepsilon, \tau_h) = 0$), we have

$$
\frac{b(w_\varepsilon, \tau_h)}{|\tau_h|_H} \leq C h^{k+1} \, |w_\varepsilon|_{k+2} .
$$

For the first term, since $w - w_\varepsilon \in (H_0^1(\Omega))^2$, by an integration by parts, one has

$$
b(w - w_\varepsilon, \tau_h) = -(\tau_h, \nabla(w - w_\varepsilon)) \leq |\tau_h|_H \, |w - w_\varepsilon|_1 \quad \forall \tau_h \in \underline{\underline{X_h}}^{sym}.
$$

Thus, for all $\tau_h \in V_h$,

$$
\frac{b(w, \tau_h)}{|\tau_h|_H} \leq |w - w_\varepsilon|_1 + C h^{k+1} \, |w_\varepsilon|_{k+2} \leq \varepsilon/2 + C h^{k+1} \, |w_\varepsilon|_{k+2} .
$$

Then choosing h_0 such that $C h_0^{k+1} \, |w_\varepsilon|_{k+2} \leq \varepsilon/2$ we get

$$
\forall h \leq h_0, \quad \frac{b(w, \tau_h)}{|\tau_h|_H} \leq \varepsilon,
$$

which implies (3.10). $\square$

4. Error estimates for the evolution problem. To study the error between the solution (σ, v) of (2.6) and the solution (σ_h, v_h) of (2.10), we follow the same technique as in [9, 17, 20]. We first define an elliptic projection and relate error estimates for the evolution problem to the projection error, thanks to energy techniques.

REMARK 1. *Although we have constructed the new family of mixed finite elements in order to be able to do mass lumping, we analyze here the error for the discrete problem without mass lumping. Of course, when doing mass lumping, one should add to this error the quadrature error due to the numerical integration (see* [28, 6, 27]*).*

In this section, we shall use the notation ∂_t for the time derivative and introduce the spaces (r and m are integers, and T is positive and fixed)

$$\begin{aligned} &C^{r,m} = C^r(0,T;H^m(\Omega)), \\ (4.1)\quad &C^{r,(m,m+1)} = C^r(0,T;H^{m,m+1}(\Omega)), \qquad C^{r,(m+1,m)} = C^r(0,T;H^{m+1,m}(\Omega)), \end{aligned}$$

and we introduce

$$\|u\|_{C^{r,m}} = \sup_{t\in[0,T]} \sup_{q\le r} |\partial_t^q u|_m \ \text{ and } \ \|u\|_{C^{r,(m,m+1)}} = \sup_{t\in[0,T]} \sup_{q\le r} |\partial_t^q u|_{m,m+1}\,.$$

We shall make on the solution (σ, v) of (2.6) regularity assumptions which are sufficient to obtain the optimal rate of convergence when using elements of order k:

$$(4.2)\quad \sigma_{11} \in C^{1,(k+2,k+1)}, \quad \sigma_{22} \in C^{1,(k+1,k+2)}, \quad \sigma_{12} \in C^{1,k+2}, v \in (C^{1,k+2})^2.$$

4.1. The elliptic projection error. Following [20, 17] we introduce

$$(\widehat{\sigma}_h(t), \widehat{v}_h(t)) \in \underline{\underline{X_h}}^{sym} \times \underline{M_h}, \quad \widehat{\sigma}_h(0) = 0,$$

and for each $t > 0$, $(\partial_t\widehat{\sigma}_h(t), \widehat{v}_h(t))$ is the unique solution of (cf. Theorem 2)

$$(4.3)\quad \begin{cases} a(\partial_t\widehat{\sigma}_h, \tau_h) + b(\widehat{v}_h, \tau_h) = 0 & \forall \tau_h \in \underline{\underline{X_h}}^{sym}, \\ b(w_h, \partial_t\widehat{\sigma}_h) = b(w_h, \partial_t\sigma) = (\mathrm{div}\partial_t\sigma, w_h) & \forall w_h \in \underline{M_h}, \end{cases}$$

so that

$$(4.4)\quad \begin{cases} \dfrac{d}{dt} a(\widehat{\sigma}_h, \tau_h) + b(\widehat{v}_h, \tau_h) = 0 & \forall \tau_h \in \underline{\underline{X_h}}^{sym}, \\ b(w_h, \widehat{\sigma}_h) = b(w_h, \sigma) & \forall w_h \in \underline{M_h}. \end{cases}$$

As a direct application of the Theorem 3, we can state the following result.

LEMMA 8. *We have the following error estimates:*

$$\begin{aligned} |(\partial_t\sigma - \partial_t\widehat{\sigma}_h)(t)|_H + \|(v - \widehat{v}_h)(t)\|_M \le\ & C\, h^{k+1}\, (|\partial_t\sigma_{11}(t)|_{k+2,k+1} + |\partial_t\sigma_{22}(t)|_{k+1,k+2} \\ & + |\partial_t\sigma_{12}(t)|_{k+2} + |v(t)|_{k+1} + |\nabla v(t)|_{k+1}). \end{aligned}$$

LEMMA 9. *Setting*

$$C(v,\sigma) = |\sigma_{11}|_{C^{1,(k+2,k+1)}} + |\sigma_{22}|_{C^{1,(k+1,k+2)}} + |\sigma_{12}|_{C^{1,k+2}} + |v|_{C^{1,k+1}} + |\nabla v|_{C^{1,k+1}},$$

we have

$$\|v - \widehat{v}_h\|_{C^0(0,T;M)} \leq C(v,\sigma)\ h^{k+1},$$

$$|\sigma - \widehat{\sigma}_h|_{C(0,T;H)} \leq C(v,\sigma)\ T\ h^{k+1}.$$

REMARK 2. *Obviously, the same estimates hold for* $\partial_t^p(v - \widehat{v}_h)$ *and* $\partial_t^p(\sigma - \widehat{\sigma}_h)$ *if one replaces* $C(v,\sigma)$ *with* $C(\partial_t^p v, \partial_t^p \sigma)$.

4.2. The global error estimates.

LEMMA 10. *There exists a constant* C_1*, independent of* h*, such that* $\forall t \in [0,T]$

$$|(\widehat{\sigma}_h - \sigma_h)(t)|_H + \|(\widehat{v}_h - v_h)(t)\|_M \leq C_1 \int_0^t \|\partial_t(v - \widehat{v}_h)(s)\|_M ds. \tag{4.5}$$

Proof of Lemma 10. By the difference between (2.10) and (4.4) and using the second equation of (2.6), we observe that

$$\begin{aligned} &\text{(i)} \quad a(\partial_t(\widehat{\sigma}_h - \sigma_h), \tau_h) + b(\widehat{v}_h - v_h, \tau_h) = 0 \quad \forall \tau_h \in \underline{\underline{X_h}}^{sym}, \\ &\text{(ii)} \quad b(w_h, \widehat{\sigma}_h - \sigma_h) = c(\partial_t(v - v_h), w_h) \quad \forall w_h \in \underline{M_h}. \end{aligned} \tag{4.6}$$

Taking $\tau_h = \widehat{\sigma}_h - \sigma_h$ and $w_h = \widehat{v}_h - v_h$, we get

$$a(\partial_t(\widehat{\sigma}_h - \sigma_h), \widehat{\sigma}_h - \sigma_h) = c(\partial_t(v_h - v), \widehat{v}_h - v_h),$$

or equivalently, setting $2E_h = a(\widehat{\sigma}_h - \sigma_h, \widehat{\sigma}_h - \sigma_h) + c(\widehat{v}_h - v_h, \widehat{v}_h - v_h)$,

$$\frac{dE_h}{dt} = c(\partial_t(\widehat{v}_h - v), \widehat{v}_h - v_h),$$

and Gronwall's lemma leads to (4.5). ◻

Joining Lemmas 9 and 10 and Remark 2, we easily prove our final result.

THEOREM 4. *Let* (σ, v) *be the solution of* (2.6) *and* (σ_h, v_h) *be the solution of the approximate problem* (2.10). *We have the following error estimates:*

$$\|v - v_h\|_{C^0(0,T;M)} \leq (C(v,\sigma) + T\ C(\partial_t v, \partial_t \sigma))\ \ h^{k+1},$$

$$|\sigma - \sigma_h|_{C(0,T;H)} \leq T\ \ (C(v,\sigma) + C(\partial_t v, \partial_t \sigma))\ \ h^{k+1}.$$

5. Conclusion. For both the stationary and the evolution problem, we have obtained the convergence of the solution in the L^2 norms. This result is based on the coerciveness of the bilinear form $a(\cdot,\cdot)$ on the space $\underline{\underline{H}}$ and requires a minimum regularity $H^2(\Omega)$ for v, the solution of the continuous problem. Moreover, the results are valid for any order k of the finite element space. The generalization of these results to the three-dimensional case is straightforward.

With another approach (rather technical) presented in [8], one can obtain the convergence of the stress tensor in the $H(\mathrm{div})$ norm. The error estimates obtained for the elliptic problem give the same convergence rate, assuming less regularity for the solution and convergence is obtained with a regularity $H^1(\Omega)$ for v. However, the error estimates in $C(0,T;H(\mathrm{div}))$ require more regularity in time for the solution of the evolution problem (see [8] for more details).

Appendix A. Proof of (3.24). Consider a node S_e located on a vertical edge of the boundary and let $\widehat{M}_e$ be the reference macroelement. If $\widehat{w} \in (H^1(\widehat{M}_e))^2 \cap (H^{k+2}(\widehat{M}_e))^2$ and $\widehat{w}$ vanishes on $\Gamma_v = \partial \widehat{M_e \cap \partial\Omega}$ (a vertical edge of $\widehat{M}_e$), then there exists a constant $C > 0$ such that

$$\|\widehat{w} - \widehat{w}_e\|_{0,\widehat{M}_e} \leq C\, |\widehat{w}|_{k+2,\widehat{M}_e}\,, \tag{A.1}$$

$\widehat{w}_e$ being the $L^2(\widehat{M}_e)$ projection of $\widehat{w}$ onto $(P^t_{k+1}(\widehat{M}_e))^2$.

Proof. This result will be obtained by contradiction. We use that

$$\widehat{w} \longrightarrow \|\widehat{w}\|_{0,\widehat{M}_e} + |\widehat{w}|_{k+2,\widehat{M}_e}$$

is a norm equivalent to the usual norm in $(H^{k+2}(\widehat{M}_e))^2$.

Assume that (A.1) is not true; then one can construct a sequence $\widehat{w}^n$:

$$\begin{aligned}
&\text{(i)} \quad \|\widehat{w}^n - \widehat{w}^n_e\|_{0,\widehat{M}_e} = 1,\\
&\text{(ii)} \quad |\widehat{w}^n - \widehat{w}^n_e|_{k+2,\widehat{M}_e} \leq \frac{1}{n},\\
&\text{(iii)} \quad \widehat{w}^n\,|_{\Gamma_v} = 0.
\end{aligned}$$

Now define e^n by

$$e^n = \widehat{w}^n - \widehat{w}^n_e;$$

this sequence is bounded in $(H^{k+2}(\widehat{M}_e))^2$ and therefore converges:

$$e^n \rightharpoonup e \quad \text{in } (H^{k+2}(\widehat{M}_e))^2 \text{ weakly and } (H^1(\widehat{M}_e))^2 \text{ strongly.}$$

Taking the limit of (ii) yields $|e|_{k+2,\widehat{M}_e} = 0$, i.e., $e \in P_{k+1}(\widehat{M}_e)$. We can thus write

$$e(x_1, x_2) = \alpha_{0,k+1} x_2^{k+1} + e_1(x_1, x_2), \quad e_1 \in P^t_{k+1}(\widehat{M}_e). \tag{A.2}$$

The restriction of e on Γ_v is therefore a polynomial in x_2 of degree $k+1$, $e\,|_{\Gamma_v} \in P_{k+1}(x_2)$. On the other hand, (iii) shows that $e^n\,|_{\Gamma_v} = -\widehat{w}^n_e\,|_{\Gamma_v} \in P_k(x_2)$, which implies that $e\,|_{\Gamma_v} \in P_k(x_2)$ and thus $\alpha_{0,k+1} = 0$ and $e = e_1 \in P^t_{k+1}(\widehat{M}_e)$. From the definition of $\widehat{w}^n_e$, one has

$$(e^n, q)_{L^2(\widehat{M}_e)} = 0 \quad \forall q \in P^t_{k+1}(\widehat{M}_e) \Longrightarrow (e, q)_{L^2(\widehat{M}_e)} = 0 \quad \forall q \in P^t_{k+1}(\widehat{M}_e),$$

and thus $e = 0$, which is in contradiction with (i). $\square$

A similar proof can be used to prove (3.24) when S_e is a vertex of the domain.

Appendix B. Approximation properties (3.12) and (3.13).

B.1. On the Raviart–Thomas elements on a rectangular mesh. To get the approximation property (3.12), we use a specific property of the Raviart–Thomas elements on a rectangular mesh, whose proof is easy and left to the reader.

LEMMA 11. *Let $\mathcal{T}_h$ be a regular mesh constituted of rectangular elements K. For any K, if $q = (q_1, q_2) \in H^{1,0}(\Omega) \times H^{0,1}(\Omega)$, the local Raviart–Thomas interpolant $\Pi_K^{RT_k} = (\Pi_1^{RT_k}, \Pi_2^{RT_k})$ can be defined as in* [14]. *Moreover, one has*

$$\left(\frac{\partial}{\partial x_1}\Pi_1^{RT_k} q, \frac{\partial}{\partial x_2}\Pi_2^{RT_k} q\right) = \left(\Pi_K^{Q_k}\frac{\partial q_1}{\partial x_1}, \Pi_K^{Q_k}\frac{\partial q_2}{\partial x_2}\right), \tag{B.1}$$

where $\Pi_K^{Q_k}$ is the orthogonal projection on $Q_k(K)$.

As a consequence, if $\Pi_h^s = (\Pi_1^s, \Pi_2^s)$ denotes the standard Raviart–Thomas global interpolant [14], one has the following property specific to our regular mesh (the proof is left to the reader).

LEMMA 12. *If $q \in H^{1,0}(\Omega) \times H^{0,1}(\Omega)$, then*

$$\left(\frac{\partial}{\partial x_1}\Pi_1^s q, \frac{\partial}{\partial x_2}\Pi_2^s q\right) = \left(\Pi_h^{Q_k}\frac{\partial q_1}{\partial x_1}, \Pi_h^{Q_k}\frac{\partial q_2}{\partial x_2}\right), \tag{B.2}$$

where $\Pi_h^{Q_k}$ is the orthogonal projection on Q_h^k defined in (2.21).

Using classical estimates for the orthogonal projection on Q_h^k, it is then straightforward to obtain the following.

LEMMA 13. *On a rectangular mesh, if $q \in (H^{1,0}(\Omega) \times H^{0,1}(\Omega))$, then*

$$\|q - \Pi_h^s q\|_{L^2} + \left\|\frac{\partial}{\partial x_1}(q_1 - \Pi_1^s q)\right\|_{L^2} + \left\|\frac{\partial}{\partial x_2}(q_2 - \Pi_2^s q)\right\|_{L^2} \to 0.$$

Furthermore, if $q \in H^{k+2,k+1}(\Omega) \times H^{k+1,k+2}(\Omega)$, then

$$\begin{aligned} &\|q - \Pi_h^s q\|_{L^2} \leq C h^{k+1} |q|_{k+1}, \\ &\|\partial_1(q_1 - \Pi_1^s q)\|_{L^2} + \|\partial_2(q_2 - \Pi_2^s q)\|_{L^2} \leq C h^{k+1}\left(\left|\frac{\partial q_1}{\partial x_1}\right|_{k+1} + \left|\frac{\partial q_2}{\partial x_2}\right|_{k+1}\right). \end{aligned} \tag{B.3}$$

B.2. Approximation property for $\underline{\underline{X_h}}^{sym}$. In order to establish approximation properties for the space $\underline{\underline{X_h}}^{sym}$, we will use the characterization (2.20) and the fact that $\underline{X_h} \supset \underline{X_h}^s$, where $\underline{X_h}^s$ is the Raviart–Thomas space (see (2.14)). The idea is then to obtain an approximation of (τ_{11}, τ_{22}) in $\underline{X_h}^s$ and an approximation of $\tau_{12} = \tau_{21}$ in Q_h^{k+1}. If we assume that $\tau \in \underline{\underline{X}}^{sym}$ satisfies $(\tau_{11}, \tau_{22}) \in H^{1,0} \times H^{0,1}$, and $\tau_{12} = \tau_{21} \in H^1$, then by observing that

$$\begin{aligned} \|\tau - \tau_h\|_X \leq &\left\|\tau_{12} - \tau_{12}^h\right\|_{H^1} + \left\|\tau_{11} - \tau_{11}^h\right\|_{L^2} + \left\|\tau_{22} - \tau_{22}^h\right\|_{L^2} \\ &+ \left\|\partial_1(\tau_{11} - \tau_{11}^h)\right\|_{L^2} + \left\|\partial_2(\tau_{22} - \tau_{22}^h)\right\|_{L^2}, \end{aligned}$$

it is clear from Lemma 13 and from classical approximation results of H^1 with Q_h^{k+1} that, if $(\tau_{11}, \tau_{22}) \in H^{k+2,k+1} \times H^{k+1,k+2}$ and $\tau_{12} \in H^{k+2}$, then

$$\inf_{(\tau_{11}^h, \tau_{22}^h) \in \underline{X_h}^s,\ \tau_{12}^h = \tau_{21}^h \in Q_h^{k+1}} \|\tau - \tau_h\|_X \leq C h^{k+1}(|\tau_{11}|_{H^{k+2,k+1}} + |\tau_{22}|_{H^{k+1,k+2}} + |\tau_{12}|_{H^{k+2}})$$

and by standard density arguments that, if $(\tau_{11}, \tau_{22}) \in H^{1,0} \times H^{0,1}$ and $\tau_{12} \in H^1$,

$$\lim_{h \to 0} \inf_{(\tau_{11}^h, \tau_{22}^h) \in \underline{X_h}^s,\ \tau_{12}^h = \tau_{21}^h \in Q_h^{k+1}} \|\tau - \tau_h\|_X = 0.$$

These results imply (3.12) and (3.13) since $\underline{X_h}^s \subset \underline{X_h}$. □

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