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On the properties of the (4,8)-median axis

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Abstract. A new shape descriptor for binary images defined in the square grid is presented. The (4,8)-median axis is the mixed case of a generic median axis including the classical morphological skeletons for the two canonical distances of the square grid. We show that the (4,8)-median axis is the locus of the centers of the maximal representants from a collection of sets called (4,8)-fuzzy balls, which are formally defined.

1 Introduction

The discrete plane is assimilated with the *square grid* $\mathbb{Z}^2$, a (binary) X *image* is a subset of $\mathbb{Z}^2$. A *pixel* x is an element of $\mathbb{Z}^2$. The two canonical discrete distances of the square grid are respectively the *4-distance* d_4, and the *8-distance* d_8. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then $d_4(x, y) = |x_1 - y_1| + |x_2 - y_2|$ and $d_8(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$.
For K = 4 or 8, the *K-ball* of center x and radius n is defined as $B_K(x, n) = \{z \in \mathbb{Z}^2, d_K(x, z) \leq n\}$. A ball $B_K(x, n)$ is said to be *maximal* in the image X if $\forall (y, n') \in \mathbb{Z}^2 \times \mathbb{N}, B_K(x, n) \subset B_K(y, n') \subset X \Rightarrow (x, n) = (y, n')$. The *morphological skeleton* or *median axis* of image X associated with distance d_K is defined as the locus of the centers of maximal balls:

$$S_K(X) = \bigcup_{n \in \mathbb{N}} \{x \in X; B_K(x, n) \text{ is maximal in } X\} \tag{1}$$

Let $b \in \mathbb{Z}^2$. The *translated* of X by b is the set $X_b = \{x + b; x \in X\}$.
Let $B \subset \mathbb{Z}^2$. The *morphological dilation* of X by B is defined as:

$$X \oplus B = \bigcup_{b \in B} X_{-b} = \{z \in \mathbb{Z}^2; B_z \cap X \neq \emptyset\} \tag{2}$$

The *morphological erosion* of X by B is defined as:

$$X \ominus B = \bigcap_{b \in B} X_{-b} = \{z \in \mathbb{Z}^2; B_z \subset X\} \tag{3}$$

2 The (4,8)-median axis

Let K and P be equal to 4 or 8 and $K \leq P$. The (K, P)-*median axis* of image X is defined as:

$$S_K^P(X) = \bigcup \{x \in X; y \in B_P(x, 1) \cap X \Rightarrow d_K(x, X^c) \geq d_K(y, X^c)\} \tag{4}$$

For $K = P$, this set corresponds to the local maxima of distance d_K to the border. It coincides with the locus of the centers of maximal K-balls defined in (1). We are going to prove that the $(4,8)$-median axis corresponds to the centers of maximal $(4,8)$-*fuzzy balls* (see Figure 1), that are recursively defined as follows:

1. A $(4,8)$-fuzzy ball of radius 1 and center x $B_{(4,8)}(x,1)$ is any set verifying: $B_4(x,1) \subset B_{(4,8)}(x,1) \subset B_8(x,1)$.
2. A $(4,8)$-fuzzy ball of radius n+1 and center x is a set such that there exists F_n^x, a $(4,8)$-fuzzy ball of center x and radius n such that: $B_{(4,8)}(x,n+1) = \bigcup_{y \in F_n^x} B_{K_y}(y,1)$, where K_y is 4 or 8, depending on y.

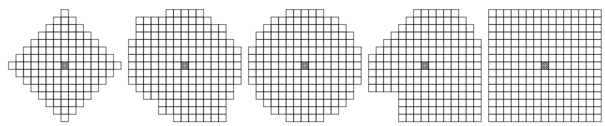


Fig. 1. Some $(4,8)$-fuzzy balls of radius 7. The extremal cases of $(4,8)$-fuzzy balls are respectively the 4-ball (on the left), and the 8-ball (on the right).

Lemma 1. *If $B_{(4,8)}(x,n)$ is a $(4,8)$-fuzzy ball of radius n and center x, and if $y \in B_8(x,1)$, then $[B_{(4,8)}(x,n) \cup B_4(y,n+1)]$ is a $(4,8)$-fuzzy ball of radius $n+1$ and center y.*

Preliminary remark: it is clear, by the definition of fuzzy balls, that if F_n^x is a $(4,8)$-fuzzy ball of center x and radius n, then any set S verifying:

$$F_n^x \oplus B_4(0,1) \subset S \subset F_n^x \oplus B_8(0,1) \tag{5}$$

is a $(4,8)$-fuzzy ball of center x and radius $n+1$. Now we prove the lemma by induction on n. If $n = 0$, $B_{(4,8)}(x,0) = \{x\}$. If $y \in B_8(x,1)$, $B_4(y,1) \subset \{x\} \cup B_4(y,1) \subset B_8(y,1)$, so $[B_{(4,8)}(x,0) \cup B_4(y,1)]$ is a $(4,8)$-fuzzy ball of radius 1 and center y.
Now suppose the lemma true for radii less or equal to $(n-1)$. Let $B_{(4,8)}(x,n)$ be a $(4,8)$-fuzzy ball of radius n and center x. By definition, there exists F_{n-1}^x, a $(4,8)$-fuzzy ball of center x and radius $(n-1)$ such that:

$$B_{(4,8)}(x,n) = \bigcup_{z \in F_{n-1}^x} B_{K_z}(y,1) \tag{6}$$

and

$$F_{n-1}^x \oplus B_4(0,1) \subset B_{(4,8)}(x,n) \subset F_{n-1}^x \oplus B_8(0,1) \tag{7}$$

Let $y \in B_8(x, 1)$. By induction hypothesis, $G_n^y = F_{n-1}^x \cup B_4(y, n)$ is a $(4, 8)$-fuzzy ball of radius n and center y.

$$G_n^y \oplus B_4(0,1) = (F_{n-1}^x \oplus B_4(0,1)) \cup (B_4(y,n) \oplus B_4(0,1)) \quad (8)$$
$$= (F_{n-1}^x \oplus B_4(0,1)) \cup B_4(y, n+1) \quad (9)$$

So, from (7), we get:

$$G_n^y \oplus B_4(0,1) \subset [B_{(4,8)}(x,n) \cup B_4(y,n+1)] \quad (10)$$

On the other hand, we have:

$$G_n^y \oplus B_8(0,1) = (F_{n-1}^x \oplus B_8(0,1)) \cup (B_4(y,n) \oplus B_8(0,1)) \quad (11)$$

and as

$$B_4(y, n+1) \subset (B_4(y,n) \oplus B_8(0,1)) \quad (12)$$

from (7), we get:

$$[B_{(4,8)}(x,n) \cup B_4(y,n+1)] \subset G_n^y \oplus B_8(0,1) \quad (13)$$

Finally, as G_n^y is a $(4, 8)$-fuzzy ball of radius n and center y, we conclude thanks to (10) and (13) that $[B_{(4,8)}(x, n) \cup B_4(y, n + 1)]$ is a $(4, 8)$-fuzzy ball of radius $(n + 1)$ and center y.

□

Theorem 1. *$S_{(4,8)}(X)$ is the locus of the centers of maximal $(4, 8)$-fuzzy balls in X.*

(1) Right inclusion. Let x be the center of a $(4, 8)$-fuzzy balls $B_{(4,8)}(x, n)$ that is maximal in X. Now suppose that it exists $y \in (B_8(x, 1) \cap X)$ such that $d_4(y, X^c) > d_4(x, X^c)$. Then we must have $B_4(y, n + 1) \subset X$. And so:

$$[B_{(4,8)}(x,n) \cup B_4(y,n+1)] \subset X \quad (14)$$

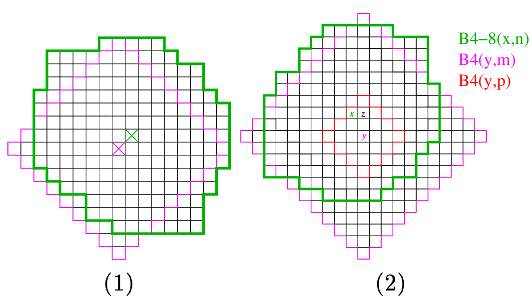


Fig. 2. (1) The center of a maximal $(4, 8)$-fuzzy ball is an element of $S_{(4,8)}(X)$. (2) An element of $S_{(4,8)}(X)$ is center of a maximal $(4, 8)$-fuzzy ball.

But from lemma 1, $[B_{(4,8)}(x,n) \cup B_4(y,n+1)]$ is a $(4,8)$-fuzzy ball of radius $n+1$ (see Figure 2(1)), which is in contradiction with the maximality of $B_{(4,8)}(x,n)$.
(2) Left inclusion. Let $x \in S_{(4,8)}(X)$. Let $B_{(4,8)}(x,n)$ be the biggest $(4,8)$-fuzzy ball of center x contained in X. We are going to prove that $B_{(4,8)}(x,n)$ is maximal in X.
Suppose that there exists a $(4,8)$-fuzzy ball F^y_m of center y and radius m such that $(y,m) \neq (x,n)$ and $B_{(4,8)}(x,n) \subset F^y_m \subset X$. We have:

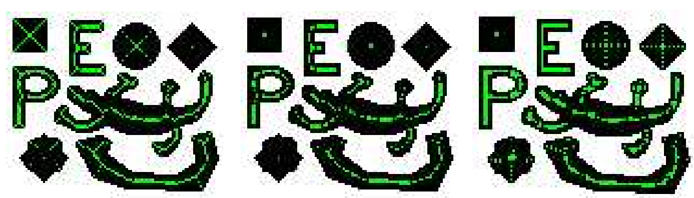
$$B_4(y,m) \subset F^y_m \tag{15}$$

$$B_4(x,n) \subset B_{(4,8)}(x,n) \tag{16}$$

The erosion of the ball $B_4(y,m)$ by $B_4(0,n)$ is a ball $B_4(y,p)$ containing x.
But $x \notin B_4(y,p) \ominus B_4(0,1)$, otherwise it would mean that $B_4(x,n+1) \subset F^y_m \subset X$, and then $B_{(4,8)}(x,n) \cup B_4(x,n+1)$ would be a $(4,8)$-fuzzy ball of center x and radius $(n+1)$ (see lemma 1) contained in X, which is in contradiction with the fact that $B_{(4,8)}(x,n)$ is the biggest $(4,8)$-fuzzy ball of center x contained in X.
So there must exist $z \in B_4(x,1)$ (see Figure 2(2)) such that $z \in B_4(y,p) \ominus B_4(0,1)$. Then $B_4(z,n+1) \subset F^y_m \subset X$, and so $d_4(z,X^c) > d_4(x,X^c)$. As $z \in B_4(x,1)$, we get $x \notin S_{(4,8)}(X)$, which is in contradiction with our hypothesis.

□

3 Conclusion



(1) (2) (3)

Fig. 3. (1) $S_4(X)$ the morphological skeleton based on d_4. (2) $S_{(4,8)}(X)$ the mixed median axis. (3) $S_8(X)$ the morphological skeleton based on d_8.

The mixed median axis $S_{(4,8)}(X)$ has been presented. Its relation with a particular class of sets, the $(4,8)$-fuzzy balls has been formally identified. The fuzzy ball is a new shape description tool, which interest lies in the robustness of morphological or connected skeletons defined in the square grid. Figure 3 shows an example of (4,8)-median axis, compared with the morphological skeletons defined for the canonical distances.