# Analytical 3d - Introduction

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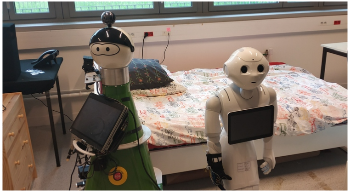
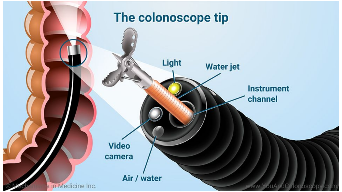
# Motivations: 3d Reconstruction from Videos

Reconstructing the scene geometry from videos is useful in many applications: Robot navigation (obstacle detection), Metrology, 3d Cartography, Medicine...






+ It is a cheap and flexible approach: One single passive camera, Adaptive baseline,...
– It strongly relies on scene structure (texture) and precise camera positioning.

# Presentation Outline

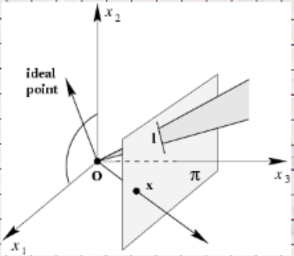
# Presentation Outline

1. Projective Geometry and Camera Matrices
   - Projective Geometry in $\mathbb{P}^2$
   - 2d Projective transformations
   - Projective Geometry in $\mathbb{P}^3$
2. Homographies: Practical cases
   - Rotation around the optical centre
   - Plane viewed from different poses
3. Estimation of a homography

# Projective Geometry in $\mathbb{P}^2$

- Homogeneous coordinates $\rightarrow$ additional component $\rightarrow$ non injective representation
- Affine transformations represented by linear functions $\rightarrow$ simpler operations
- Points and lines at infinity represented with finite coordinates

# Projective Geometry in $\mathbb{P}^2$




$$\mathbb{R}^2 \qquad\qquad \mathbb{P}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

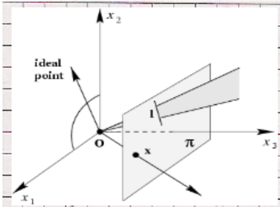
$$\begin{pmatrix} u/w \\ v/w \end{pmatrix} \longleftarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

- Equivalence Classes: $\forall \lambda \neq 0 \quad \lambda x \equiv x$
- Duality point / line:

$$m = (x, y, 1)^t \qquad l = (a, b, c)^t$$

- Ideal points: $(x, y, 0)^t$
- Line at infinity: $(0, 0, 1)^t$

# Projective Geometry in $\mathbb{P}^2$




Point $m$ belongs to line $\ell$

$$\Updownarrow$$

$$\boxed{m^t \ell = 0}$$

Point $m$ is at the intersection of lines $\ell$ and $\ell'$:

$$\boxed{\ell \times \ell' = m}$$

Line $\ell$ passes through points $m$ and $m'$:

$$\boxed{m \times m' = \ell}$$

Notation:

pre-vector product: $[U]_\times = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}$

with $U = (u, v, w)^t$

Then: $\boxed{U \times U' = [U]_\times U'}$

# Projective transformations

- A projective transformation $h$ of the plane is characterized by the fact that: if three point $m_1$, $m_2$ and $m_3$ are aligned, $h(m_1)$, $h(m_2)$ and $h(m_3)$ are aligned too.
- A function $h : \mathbb{P}^2 \mapsto \mathbb{P}^2$ is a projective transformation if and only if there exists a non singular $3 \times 3$ matrix $H$ such that $\forall m \in \mathbb{P}^2, h(m) = Hm$.

# Projective transformations 1: Translations

$$H = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- with $\mathbf{t} = (t_x \ \ t_y)^T$ translation vector
- 2 degrees of freedom

# Projective transformations 2: Isometries

$$H = \begin{pmatrix} \cos(\theta) & -\varepsilon \sin(\theta) & t_x \\ \sin(\theta) & \varepsilon \cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- with $\mathbf{t} = (t_x \ t_y)^T$ translation vector
- $\theta$ rotation angle
- $\varepsilon = \pm 1 \rightarrow$ direct / indirect isometry
- 3 degrees of freedom
- *preserves:* angles, lengths, areas

# Projective transformations 3: Similarities

$$H = \begin{pmatrix} s\cos(\theta) & -s\sin(\theta) & t_x \\ s\sin(\theta) & s\cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

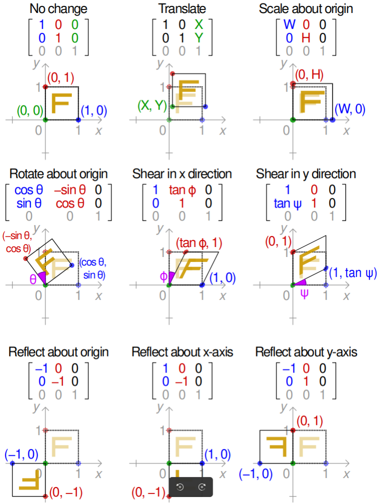
- with $\mathbf{t} = (t_x \;\; t_y)^T$ translation vector
- $\theta$ rotation angle
- $s$ homothety factor
- 4 degrees of freedom
- *preserves:* angles, ratios of lengths/areas, parallel lines

# Projective transformations 4: Affine transformations

$$H = \begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- 6 degrees of freedom
- *preserves:* ratios of areas, parallel lines

(Figure from Wikipedia)



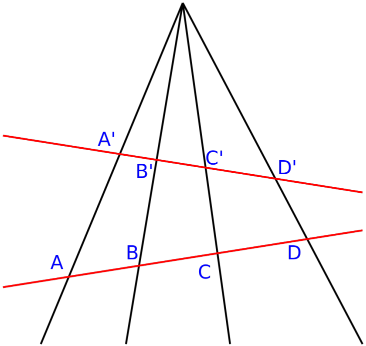
# Projective transformations 5: Homographies

$$H = \begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ v_1 & v_2 & 1 \end{pmatrix}$$

- $\mathbf{v} = (v_1 \; v_2)^T$ relates to the action on points/lines at infinity
- 8 degrees of freedom
- *preserves:* cross-ratios of four points on a line:

$$\frac{AC \times BD}{BD \times AC} = \frac{A'C' \times B'D'}{B'D' \times A'C'}$$

(Figure from Wikipedia)



# Homographies on points/lines at infinity

Consider a line at infinity $\mathbf{l}_\infty = (l_1 \; l_2 \; 0)^T$
When applied an affine transformation:

$$\begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 a_1^1 + l_2 a_1^2 \\ l_1 a_2^1 + l_2 a_2^2 \\ 0 \end{pmatrix}$$

A line at infinity remains at infinity!
When applied a general homography:

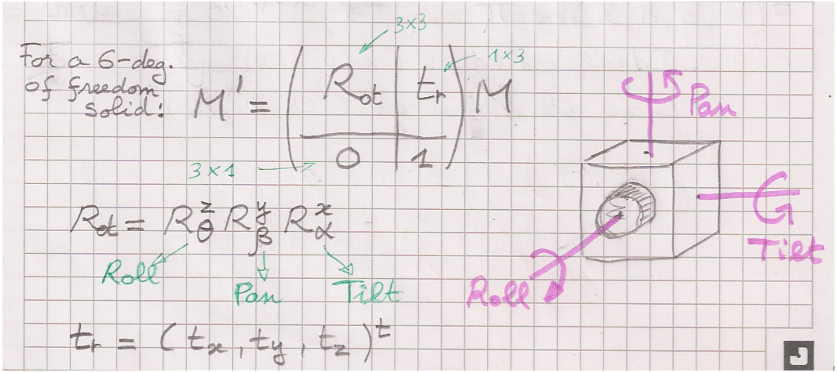
$$\begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ v_1 & v_2 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 a_1^1 + l_2 a_1^2 \\ l_1 a_2^1 + l_2 a_2^2 \\ l_1 v_1 + l_2 v_2 \end{pmatrix}$$

A line at infinity becomes finite!
This allows to observe vanishing points and horizon lines.

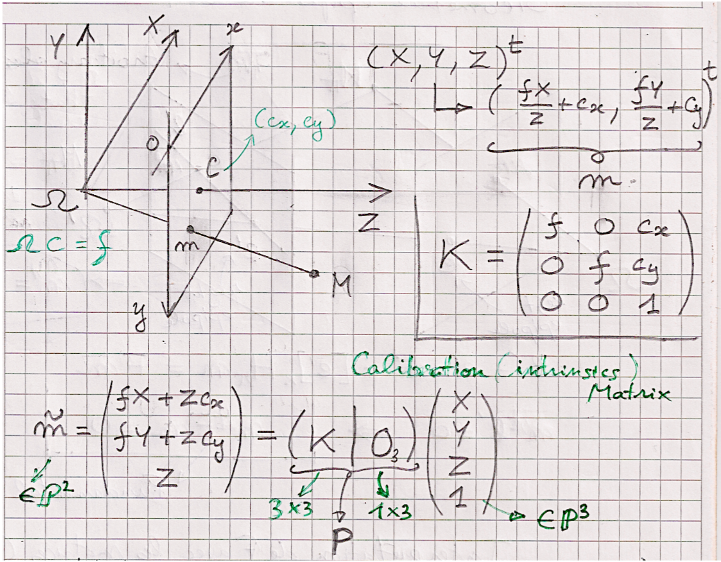
# Projective Geometry in $\mathbb{P}^3$

- $\mathbb{R}^3 \leftrightarrow \mathbb{P}^3$: $(X, Y, Z) \rightarrow (X, Y, Z, 1)$ ; $(u/h, v/h, w/h) \leftarrow (u, v, w, h)$
- Duality point / plane: $M = (X, Y, Z, 1)^t$ / $\Pi = (a, b, c, d)$.
- Lines are defined from 2 points or from 2 planes!

$\mathbb{P}^3$ allows to express linearly affine transformations:



# Camera (Calibration) Matrix: Intrinsics



# Projection and Back-Projection Matrices

$M = (X, Y, Z)^t \in \mathbb{R}^3$
$m = (x, y)^t \in \mathbb{R}^2$, and $\tilde{m} = (x, y, 1)^t \in \mathbb{P}^2$

## Camera (Projection) Matrix

$$m = \pi(M) = \left(f\frac{X}{Z} + c_x, f\frac{X}{Z} + c_x\right)$$

Equivalent to:

$$\tilde{m} = KM$$

with: $K = \begin{pmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{pmatrix}$

## Back-Projection Matrix

$$M = \pi^{-1}(m, Z) = \left(Z\frac{x-c_x}{f}, Z\frac{y-c_y}{f}, Z\right)$$

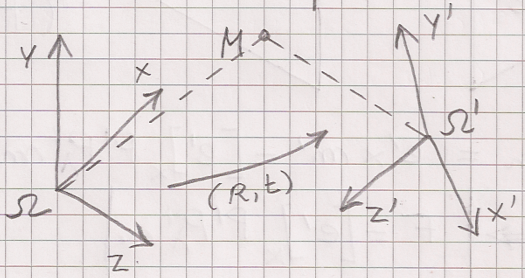
Equivalent to:

$$M = \underbrace{Z}_{\text{Depth}}\underbrace{K^{-1}\tilde{m}}_{\text{Direction}}$$

with: $K^{-1} = \begin{pmatrix} \frac{1}{f} & 0 & -\frac{c_x}{f} \\ 0 & \frac{1}{f} & -\frac{c_y}{f} \\ 0 & 0 & 1 \end{pmatrix}$

# Displacement Matrix: Extrinsics

$$\tilde{m}' = \underbrace{\left(K \,\middle|\, 0_3\right) \left(\begin{array}{c|c} R & -Rt \\ \hline 0_3^t & 1 \end{array}\right)}_{P'} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$




$m = PM$

$m' = P'M$

# Presentation Outline

# Homographies

- **Homography** $\rightarrow$ Most general case of 2d projective transformation

$$\tilde{m}' = H\tilde{m}$$

- 8 degrees of freedom $\rightarrow$ At least four non colinear 2d points!
- Corresponds to 2 particular cases of image pairs:
  - 3d scene viewed under pure rotation around the optical centre ($\mathbf{t} = O_3$).
  - Same plane viewed under two different 3d poses.

# Rotation around the optical centre

In the case of a pure rotation around the optical centre ($\mathbf{t} = O_3$), the projected image transformation is a homography:

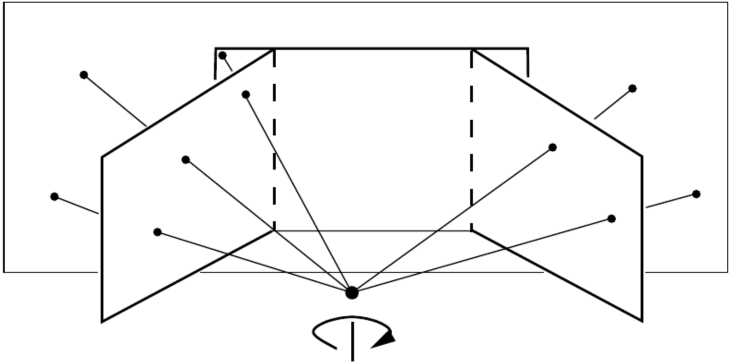


Figure from [**Hartley and Zisserman 2004**]

# Rotation around the optical centre

Since $\mathbf{t} = O_3$ we get:

$$\tilde{m} = \left( \begin{array}{c|c} K & O_3 \end{array} \right) \tilde{M}$$

$$\tilde{m}' = \left( \begin{array}{c|c} K & O_3 \end{array} \right) \left( \begin{array}{c|c} R & \color{red}{O_3} \\ \hline O_3^t & 1 \end{array} \right) \tilde{M}$$

which can be written more simply:

$$\begin{array}{rcl} \tilde{m} & = & KM \\ \tilde{m}' & = & KRM = \underbrace{KRK^{-1}}_{H} \tilde{m} \end{array}$$

# Rotation around the optical centre

Note the difference between rotation around the optical centre ((a) to (b)), and translation ((a) to (c)):

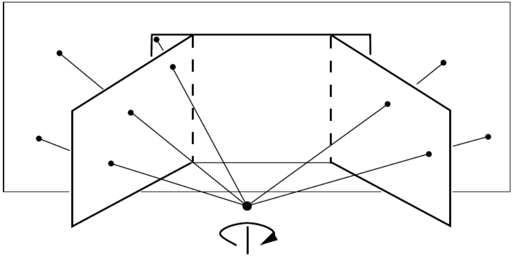
a b c

Images from [**Hartley and Zisserman 2004**]

# Rotation around the optical centre

Since there is no parallax, the images can be stitched to form a mosaic:

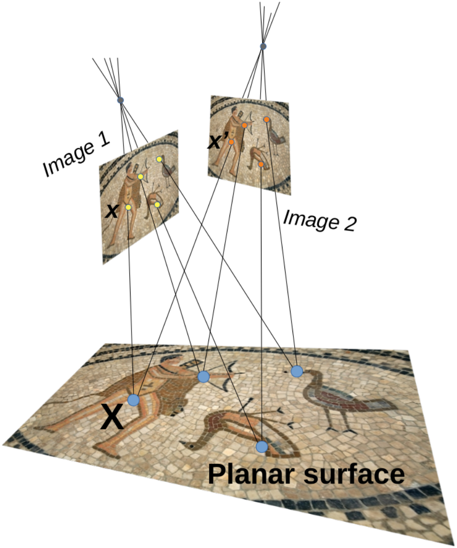


# Plane viewed from different poses

$$\tilde{x} = H_{\pi,1} X$$

$$\tilde{x}' = H_{\pi,2} X$$

$$\tilde{x}' = H_{\pi,2} H_{\pi,1}^{-1} \tilde{x} = H_{\pi} \tilde{x}$$



# Plane viewed from different poses

Let us first assume that $K = I_3$ (i.e. $f = 1, c_x = c_y = 0$). Then if the pose of the right camera is given by rotation matrix $R$ and translation vector $\mathbf{t}$, we get:

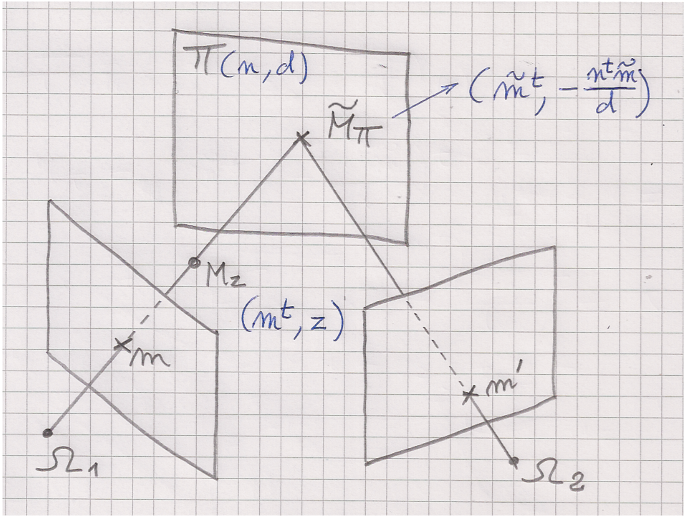
$$\tilde{m} = P\tilde{M} = \left( \begin{array}{c|c} I_3 & O_3 \end{array} \right) \tilde{M}$$

$$\tilde{m}' = P'\tilde{M} = \left( \begin{array}{c|c} R & \mathbf{t} \end{array} \right) \tilde{M}$$

Every point on the ray $M_z = (m^t, z)$ (parameterized by $z$) projects on $m$.
If the point $M_z$ is on the plane $\pi$, it must satisfy: $\pi^t.\tilde{M}_z = 0$.
If the coordinates of the plane are given as $\pi = (\mathbf{n}^t, d)^t$, so that for points $M$ on the plane, we have: $\mathbf{n}^t M + d = 0$,
then the point of the ray backprojected from $m$ and intersecting plane $\pi$ is:

$$\tilde{M}_\pi = \left( \tilde{m}^t, -\frac{\mathbf{n}^t \tilde{m}}{d} \right)^t$$

# Plane viewed from different poses



# Plane viewed from different poses

The point of the ray backprojected from $m$ and intersecting plane $\pi$ is:

$$\tilde{M}_\pi = \left(\tilde{m}^t, -\frac{\mathbf{n}^t \tilde{m}}{d}\right)^t$$

And then:

$$\begin{aligned} \tilde{m}' &= P'\tilde{M}_\pi = \left(\begin{array}{c|c} R & \mathbf{t} \end{array}\right) \tilde{M}_\pi \\ &= R\tilde{m} - \frac{\mathbf{t}\mathbf{n}^t}{d}\tilde{m} \\ &= \underbrace{\left(R - \frac{\mathbf{t}\mathbf{n}^t}{d}\right)}_{H_\pi} \tilde{m} \end{aligned}$$

Finally, by considering the internal parameter matrix $K$ of a single camera moved with rotation $R$ and translation $\mathbf{t}$, the homography related to the plane $\pi = (\mathbf{n}^t, d)^t$ is given by:

$$H = K\left(R - \frac{\mathbf{t}\mathbf{n}^t}{d}\right)K^{-1}$$

# Presentation Outline

# Estimation of a Homography

Now we wish to estimate the parameters of a homography using a set of correspondances from a pair of images:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ v_1 & v_2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- In the following practical session we will use a the Direct Linear Transform (DLT) resolved by Singular Values Decomposition (SVD).
- The next slides are adapted from **Gianni Franchi**'s 2022 course.

# Estimation by Direct Linear Transformation (DLT)

Let us rearrange the equation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

we use auxiliary $1 \times 3$ vectors $\mathbf{h_1}$, $\mathbf{h_2}$ and $\mathbf{h_3}$:

$$\mathbf{x}' = \begin{bmatrix} \mathbf{h_1} \\ \mathbf{h_2} \\ \mathbf{h_3} \end{bmatrix} \mathbf{x}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \mathbf{h_1 x} \\ \mathbf{h_2 x} \\ \mathbf{h_3 x} \end{bmatrix}$$

# Estimation by Direct Linear Transformation (DLT)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \mathbf{h_1 x} \\ \mathbf{h_2 x} \\ \mathbf{h_3 x} \end{bmatrix}$$

$$x' = \frac{u'}{w'} = \frac{\mathbf{h_1 x}}{\mathbf{h_3 x}}$$

$$y' = \frac{v'}{w'} = \frac{\mathbf{h_2 x}}{\mathbf{h_3 x}}$$

# Estimation by Direct Linear Transformation (DLT)

We can rewrite the equations:

$$\begin{cases} -\mathbf{h_1x} & & +x'\mathbf{h_3x} = 0 \\ & -\mathbf{h_2x} & +y'\mathbf{h_3x} = 0 \end{cases}$$

we want to estimate $\mathbf{h_1}$, $\mathbf{h_2}$ and $\mathbf{h_3}$

# Estimation by Direct Linear Transformation (DLT)

Let us write $\mathbf{h} = \begin{bmatrix}\mathbf{h_1} & \mathbf{h_2} & \mathbf{h_3}\end{bmatrix}^t$. $\mathbf{h}$ is a vector of size $9 \times 1$.
We can rewrite the previous system with $\mathbf{h}$, as follows:

$$\begin{cases} \mathbf{a}_x^t \mathbf{h} = 0 \\ \mathbf{a}_y^t \mathbf{h} = 0 \end{cases}$$

with

$$\mathbf{a}_x^t = \begin{bmatrix}-\mathbf{x}^t & \mathbf{0}_3^t & x'\mathbf{x}^t\end{bmatrix}$$

$$\mathbf{a}_x^t = \begin{bmatrix}-x & -y & -1 & 0 & 0 & 0 & x'x & x'y & x'\end{bmatrix}$$

$$\mathbf{a}_y^t = \begin{bmatrix}\mathbf{0}_3^t & -\mathbf{x}^t & y'\mathbf{x}^t\end{bmatrix}$$

$$\mathbf{a}_y^t = \begin{bmatrix}0 & 0 & 0 & -x & -y & -1 & y'x & y'y & y'\end{bmatrix}$$

# Estimation by Direct Linear Transformation (DLT)

Now let us consider that we have multiple pairs of points indexed by $i$:

$$\mathbf{a}_{x_i}^t = \begin{bmatrix} -\mathbf{x}_i^t & \mathbf{0}^t & x_i'\mathbf{x}_i^t \end{bmatrix}$$

$$\mathbf{a}_{y_i}^t = \begin{bmatrix} \mathbf{0}^t & -\mathbf{x}_i^t & y_i'\mathbf{x}_i^t \end{bmatrix}$$

We can rewrite the previous system for the $N$ pairs of points:

$$\begin{cases} \mathbf{a}_{x_1}^t \mathbf{h} = 0 \\ \mathbf{a}_{y_1}^t \mathbf{h} = 0 \\ \vdots \\ \mathbf{a}_{x_N}^t \mathbf{h} = 0 \\ \mathbf{a}_{y_N}^t \mathbf{h} = 0 \end{cases}$$

Collecting everything together we have:

$$\underbrace{\mathbf{A}}_{2N\times 9}\underbrace{\mathbf{h}}_{9\times 1} = \underbrace{\mathbf{0}}_{9\times 1}$$

# Estimation by Direct Linear Transformation (DLT)

- if we use $N = 4$ then we have an exact solution
- if we use $N > 4$ then we have an **over-determined solution**. There are no exact solution, hence we need to find approximate solution.
- Additional constraint is needed to avoid 0, e.g. $\|\mathbf{h}\|_2^2 = 1$

# Estimation of $\mathbf{h}$: Minimisation

In the case of redundant observations we get inconsistencies (due to the noise).
Let us write $\mathbf{Ah} = \mathbf{w}$.
Our goal is to find $\mathbf{h}$ such that:

$$\hat{\mathbf{h}} = \arg\min_{\mathbf{h}} \mathbf{w}^t\mathbf{w}$$

$$\hat{\mathbf{h}} = \arg\min_{\mathbf{h}} \mathbf{h}^t\mathbf{A}^t\mathbf{A}\mathbf{h}$$

with $\|h\|_2^2 = 1$
How do we minimize the loss?

# Estimation of $\mathbf{h}$: Singular Value Decomposition

The eigenvector belonging to the smallest eigenvalue of $\mathbf{A}^t\mathbf{A}$ provides the solution of the over-determined, constrained system of linear equations:

$$\underbrace{\mathbf{A}}_{2N\times 9} = \underbrace{\mathbf{U}}_{2N\times 9}\underbrace{\mathbf{S}}_{9\times 9}\underbrace{\mathbf{V}}_{9\times 9} = \sum_{i=1}^{9} s_i \mathbf{u}_i \mathbf{v}_i^t$$

with $\mathbf{U}^t\mathbf{U} = \mathbf{I}_9$ and $\mathbf{V}^t\mathbf{V} = \mathbf{I}_9$

The vector $v_i$ are orthonormal since

$$v_i v_j^t = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

So, $\mathbf{h}$ is equal to $v_9$, with $s_9$ the smallest eigen value.

# Estimation of $\mathbf{h}$: Singular Value Decomposition

The estimate of $\mathbf{h}$ is given by

$$\hat{\mathbf{h}} = \begin{bmatrix} \hat{\mathbf{h_1}} & \hat{\mathbf{h_2}} & \hat{\mathbf{h_3}} \end{bmatrix}^t = v_9$$

This leads to the estimated projection matrix.

**No solution if too many points $\mathbf{x}_i$ are on a line.**

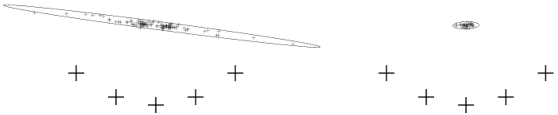
# DLT + SVD algorithm

**Objective:**
Given $N \geq 4$ 2d to 2d point correspondences $(\mathbf{x}_i, \mathbf{x}'_i)$, determine the 2d homography matrix $\mathbf{H}$ such that $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$.
**Algorithm:**

- For each correspondence $(\mathbf{x}_i, \mathbf{x}'_i)$ compute $\mathbf{A}_i$. Usually only two first rows needed.
- Assemble N $2 \times 9$ matrices $\mathbf{A}_i$ into a single $2N \times 9$ matrix $\mathbf{A}$
- Obtain SVD of $\mathbf{A}$. Solution for $\mathbf{h}$ is the last line of $\mathbf{V}$
- Determine $\mathbf{H}$ from $\mathbf{h}$

# Estimation of $\mathbf{h}$: Data ranges

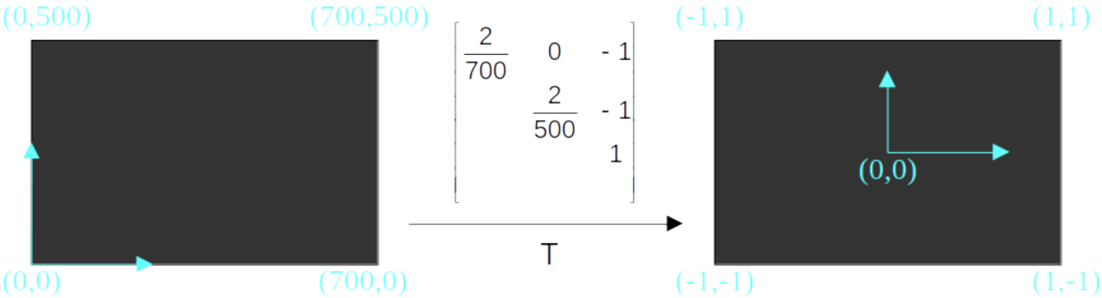
$$\mathbf{A}_i^t = \begin{array}{c} \begin{bmatrix} -x & -y & -1 & 0 & 0 & 0 & x'x & x'y & x' \\ 0 & 0 & 0 & -x & -y & -1 & y'x & y'y & y' \end{bmatrix} \\ \begin{array}{ccccccccc} 10^2 & 10^2 & 1 & 10^2 & 10^2 & 1 & 10^4 & 10^4 & 10^2 \end{array} \end{array}$$



Dependence of error distribution on the dimensions of images.

**How to transform them so that the coordinates are within $[-1, 1]$?**

# Estimation of **h**: Data normalisation



# Normalised DLT algorithm

**Objective:**
Given $N \geq 4$ 2d to 2d point correspondences $(\mathbf{x}_i, \mathbf{x}'_i)$, determine the 2d homography matrix $\mathbf{H}$ such that $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$.
**Algorithm:**

- Apply the normalisation $\tilde{\mathbf{x}}_i = \mathbf{T}_{\text{norm}}\mathbf{x}_i$ and $\tilde{\mathbf{x}}'_i = \mathbf{T}_{\text{norm}}\mathbf{x}'_i$
- apply DLT with $(\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}'_i)$
- Denormalise the homography: $\mathbf{H} = \mathbf{T}_{\text{norm}}^{-1}\tilde{\mathbf{H}}\mathbf{T}_{\text{norm}}$