Control of Discrete-time System

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Course grade breakdowns

Labs - 50%

Final project - 50 %

Space model of discrete-time system

Continuous-time systems

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Space model of discrete-time system

Continuous-time systems

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Dicrite-time systems

$$x_k = Ax_{k-1} + Bu_{k-1}$$

$$y_k = Cx_k + Du_k$$

Space model of discrete-time system

Continuous-time systems

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Dicrite-time systems

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

Discrete-time systems are either
inherently discrete (e.g. models of bank accounts, national economy growth models, population growth models, digital words)

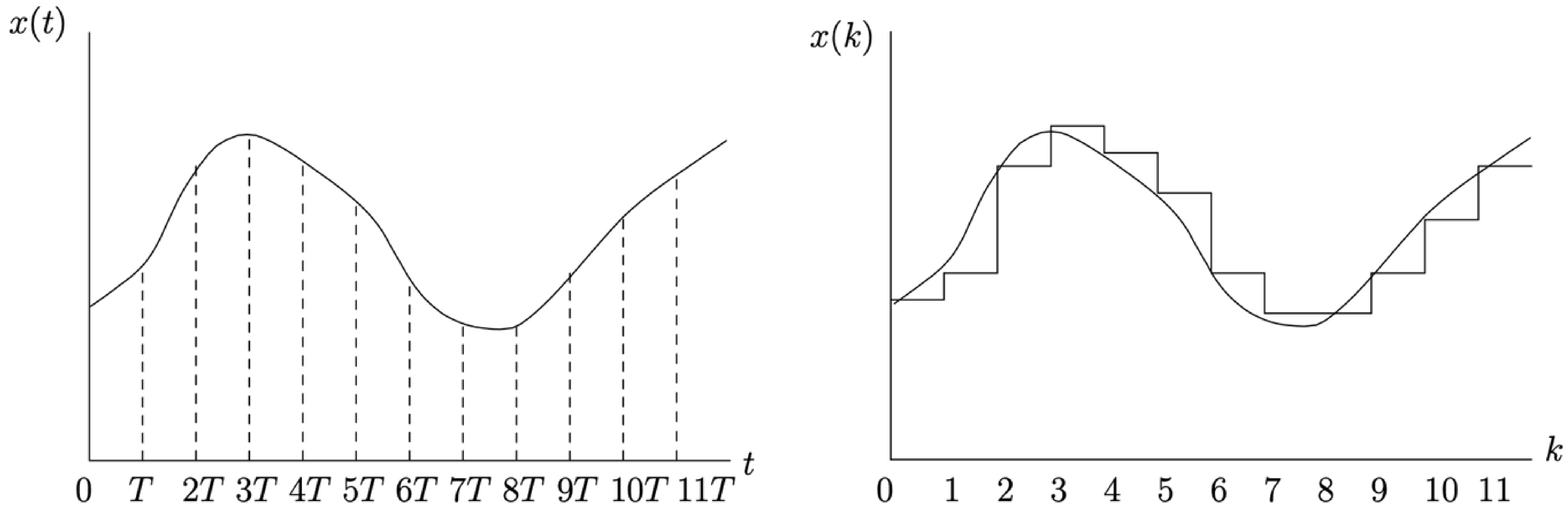
Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

with sampling-time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$



or they are obtained as a result of sampling (discretization) of continuous-time systems.

Controllability of discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

Definition of Controllability

A discrete-time linear system $x_{k+1} = Ax_k + Bu_k$ is called controllable at $k = 0$ if there exists a finite time k_N such that for any initial state x_0 and target state x_t, there exists a control sequence $\{u_k; k = 0, 1, , k_N\}$ that will transfer the system from x_0 at $k = 0$ to x_t at $k = k_N$

Observability of discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

Definition of Observability

A discrete-time linear system is called observable at $k = 0$ if there exists a finite time k_N such that for any initial state x_0, the knowledge of input $\{u_k; k = 0, 1, \ldots, k_N\}$ and $\{y_k; k = 0, 1, ..., k_N\}$ suffice to determine the state x_0.

Internal stability of discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

Definition of internal stability

A discrete-time system is stable if and only if when the input $u_k = 0$ for all $k \geq 0$, the state x_k is bounded for all $k \geq 0$ for any initial state $x_0 \in \mathbb{R}^n$

A discrete-time system is asymptotically stable if and only if it is stable and $\lim_{k \to +\infty} \|x_k\| = 0$ for any initial state $x_0 \in R^n$ n.

Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$

Continuous-time system

It's sampled version

Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

$\longrightarrow$

with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$

Attention!

Continuous-time system

Controllable

$\not\Longrightarrow$

It’s sampled version

Controllable

Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method $\longrightarrow$ with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$

Attention!

Continuous-time system		It's sampled version
Controllable	$\not\Longrightarrow$	Controllable
Observable	$\not\Longrightarrow$	Observable

Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
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Attention!

Continuous-time system		It's sampled version
Controllable	$\not\Rightarrow$	Controllable
Observable	$\not\Rightarrow$	Observable
Stable	$\not\Rightarrow$	Stable

Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
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euler method

with sampling time T

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Attention!

Continuous-time system		It's sampled version
Controllable	$\not\Longrightarrow$	Controllable ?
Observable	$\not\Longrightarrow$	Observable ?
Stable	$\not\Longrightarrow$	Stable ?

Criterion of controllability for discrete-time system

$$x_{k+1} = Ax_k + Bu_k \quad \textbf{(1)}$$

$$y_k = Cx_k + Du_k$$

Controllability matrix

$$[B, AB, \ldots, A^{n-1}B]$$

Kalman's Criterion

The linear discrete-time system (1) is controllable if and only if the controllability matrix has rank equal to n, where n is a number of state variables.

Criterion of observability for discrete-time system

$$x_{k+1} = Ax_k + Bu_k \quad \textbf{(1)}$$

$$y_k = Cx_k + Du_k \quad \textbf{(2)}$$

Observability matrix

$$\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}$$

Kalman's Criterion

The linear discrete-time system (1) with measurements (2) is observable if and only if the observability matrix has rank equal to n, where n is a number of state variables.

Criterion of Stability for discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

Criterion of stability

A discrete-time LTI system is asymptotically (internally) stable if and only if $|\lambda_j| < 1$ for all $j \in 1, \ldots, s$ where $\lambda_1, \ldots, \lambda_s$ is the set of distinct eigenvalues of A.

PID controller

Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

PID controller

SISO Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$u \in \mathbb{R}$, $y \in \mathbb{R}$

PID controller

SISO Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$u \in \mathbb{R}, \ y \in \mathbb{R}$

Specification

output of closed-loop system
should track the given reference trajectory:

$$\lim_{t \to +\infty} (y_{ref}(t) - y(t)) = 0$$

PID controller

SISO Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Specification

output of closed-loop system should track the given reference trajectory:

$$\lim_{t \to +\infty} (y_{ref}(t) - y(t)) = 0$$

PID controller

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) \mathrm{d}\tau + K_d \frac{\mathrm{d}}{\mathrm{d}t} e(t).$$

$e(t) = y_{ref}(t) - y(t)$

PID controller

SISO Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

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$y_{ref}(t) - y(t)$

Digital PID controller

SISO Control system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k$$

Specification

output of closed-loop system should track the given reference trajectory:

$$\lim_{k \to +\infty} \underbrace{(y_{ref,k} - y_k)}_{e_k} = 0$$

PID controller

$$u_k = K_\text{p} e_k + K_\text{i} \sum_{n=1}^{k} e_n + K_\text{d} \left[e_k - e_{k-1} \right]$$

Digital PID controller

SISO Control system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k$$

$y_{ref,k}$

Specification

output of closed-loop system should track the given reference trajectory:

$$\lim_{k \to +\infty} (y_{ref,k} - y_k) = 0$$

e_k

PID controller

$$u_k = K_{\mathrm{p}} e_k + K_{\mathrm{i}} \sum_{n=1}^{k} e_n + K_{\mathrm{d}} \left[e_k - e_{k-1} \right]$$

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output of closed-loop system should track the given reference trajectory:

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e_k

PID controller

The digital PID-controller is usually implemented using the so-called velocity form

$$u_k = u_{k-1} + K_p\left[e_k - e_{k-1}\right] + K_i e_k + K_d\left[e_k - 2e_{k-1} + e_{k-2}\right]$$

to avoid keep track of the sum

PID: Summary

$$u_k = u_{k-1} + K_p \left[e_k - e_{k-1} \right] + K_i e_k + K_d \left[e_k - 2e_{k-1} + e_{k-2} \right]$$

PID: Pros

- Real-Time Control
- Simple Implementation
- Tuning flexibilty

PID: Cons

- Requires Tuning
- Wrongly tuned might be unstable
- Not Ideal for Complex Processes
- Don’t take into account state and input constraints

Stabilisation by full feedback

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MIMO Control system

$$\dot{x} = Ax + Bu$$

$$y = x$$

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Specification

The closed-loop system should be asymptotically stable

$$\lim_{t->+\infty} \|x(t)\| = 0$$

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Linear Full-State Feedback Controller:

$$u = -Kx$$

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Closed-loop system

$$\dot{x} = (A - BK)x$$

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Specification

The closed-loop system should be asymptotically stable

$$\lim_{t->+\infty} \|x(t)\| = 0$$

Linear Full-State Feedback Controller: $u = -Kx$

Closed-loop system

$$\dot{x} = (A - BK)x$$

Theorem (Eigenvalue assignment — MIMO). All eigenvalues of $(\mathbf{A} - \mathbf{BK})$ can be assigned arbitrarily (provided complex eigenvalues are assigned in conjugated pairs) by selecting a real constant $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

To make closed-loop system stable assign eigenvalues with negative real part

Digital full feedback regulator

MIMO Control system

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k$$

Specification

The closed-loop system should be asymptotically stable

$$\lim_{k->+\infty} \|x_k\| = 0$$

Linear Full-State Feedback Controller: $u_k = -Kx_k$

Closed-loop system

$$x_{k+1} = (A - BK)x_k$$

Theorem (Eigenvalue assignment — MIMO). All eigenvalues of $(\mathbf{A}-\mathbf{BK})$ can be assigned arbitrarily (provided complex eigenvalues are assigned in conjugated pairs) by selecting a real constant $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

To make closed-loop system stable assign eigenvalues, s.t. $|\lambda_i| \leq 1, i = 1, \ldots n$

Stabilisation by dynamic feedback

MIMO Control system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The closed-loop system should be asymptotically stable

$$\lim_{t->+\infty} \|x(t)\| = 0$$

Stabilisation by dynamic feedback

MIMO Control system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The closed-loop system should be asymptotically stable

$$\lim_{t->+\infty} \|x(t)\| = 0$$

Luenberger Observer: $\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu$

Feedback controller: $u = K\hat{x}$

Stabilisation by dynamic feedback

MIMO Control system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The closed-loop system should be asymptotically stable

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Luenberger Observer: $\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu$

Feedback controller: $u = K\hat{x}$

If pair (A, B) is controllable we can choose K, such that for all $\lambda_i \in eig(A - BK)$ we have $Re(\lambda_i) < 0$.

If pair (A,C) is observable we can choose L, such that for all $\lambda_i \in eig(A - LC)$ we have $Re(\lambda_i) < 0$.

Stabilisation by dynamic feedback

MIMO Control system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k$$

The closed-loop system should be asymptotically stable

$$\lim_{k->+\infty} \|x_k\| = 0$$

Luenberger Observer: $\hat{x}_{k+1} = (A - LC)\hat{x}_k + Ly_k + Bu_k$

Feedback controller: $u_k = K\hat{x}_k$

If pair (A, B) is controllable we can choose K, such that for all $\lambda_i \in eig(A - BK)$ we have $|\lambda_i| < 1$.

If pair (A,C) is observable we can choose L, such that for all $\lambda_i \in eig(A - LC)$ we have $|\lambda_i| < 1$.

LQR: continuous system

For a continuous-time linear system described by:

$$\dot{x} = Ax + Bu$$

with a cost function defined as:

$$J = \int_0^\infty \left(x^T Q x + u^T R u + 2 x^T N u \right) dt$$

the feedback control law that minimizes the value of the cost is:

$$u = -Kx$$

where K is given by:

$$K = R^{-1}(B^T P + N^T)$$

and P is found by solving the continuous time algebraic Riccati equation:

$$A^T P + PA - (PB + N)R^{-1}(B^T P + N^T) + Q = 0$$

LQR: discrete system

For a discrete-time linear system described by:

$$x_{k+1} = Ax_k + Bu_k$$

with a performance index defined as:

$$J = \sum_{k=0}^{\infty} \left(x_k^T Q x_k + u_k^T R u_k + 2x_k^T N u_k\right)$$

the optimal control sequence minimizing the performance index is given by:

$$u_k = -Fx_k$$

where:

$$F = (R + B^T PB)^{-1}(B^T PA + N^T)$$

and P is the unique positive definite solution to the discrete time algebraic Riccati equation (DARE):

$$P = A^T PA - (A^T PB + N)\left(R + B^T PB\right)^{-1}(B^T PA + N^T) + Q.$$

Why LQR is “better” than PID?

- It can handle multiple-input multiple-output (MIMO) systems.
- It is an optimal control, taking into account the system dynamics and control effort. This can lead to better performance and efficiency compared to PID, which focuses on reducing error but doesn't optimize a specific criterion.
- LQR more robust than PID in uncertain environments.

What are the limitations?

Don’t take into account state and input constraints!

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Ideally, we want

MIMO Control system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The closed-loop system should be asymptotically stable

$$\lim_{t->+\infty} \|x(t)\| = 0$$

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but, also ensure

$$x \in \mathcal{X}, \quad u \in \mathcal{U}$$

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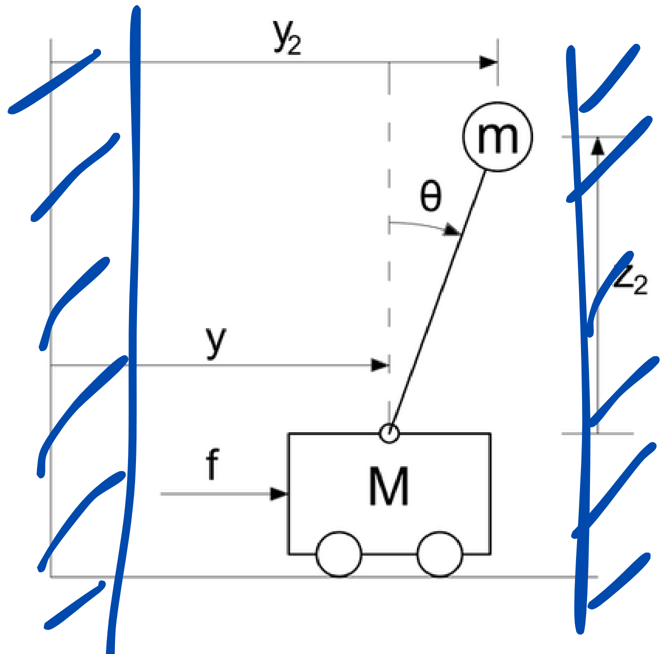

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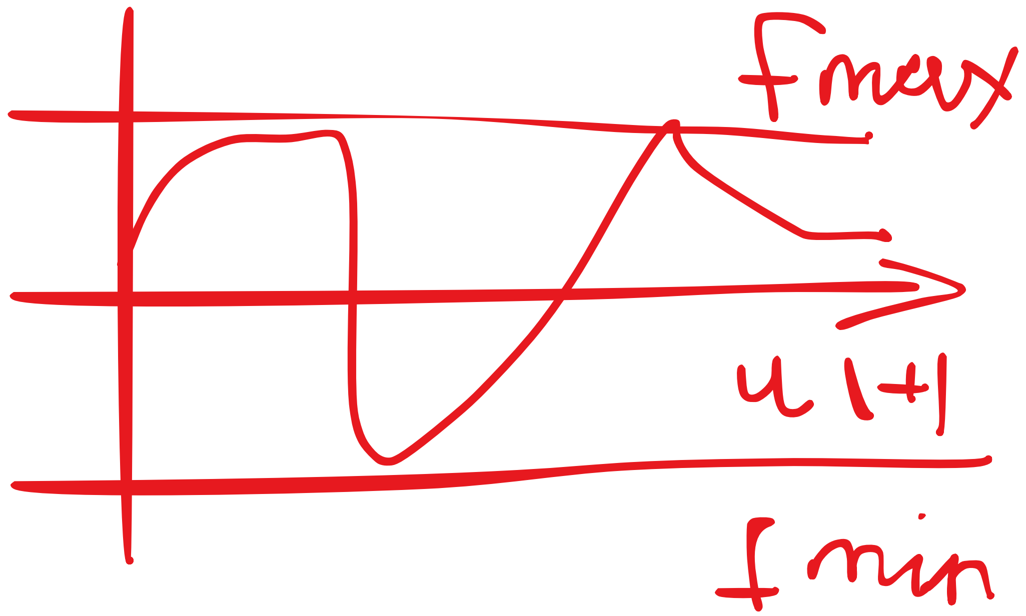


Do we know how to solve it?

- **Constrained LQR for continuous system?**

$$\dot{x} = Ax + Bu \quad x \in \mathcal{X},\ u \in \mathcal{U}$$

and

$$J = \int_0^\infty \left(x^T Q x + u^T R u + 2x^T N u\right) dt \rightarrow \min$$

- **well, not really...**

- **Constrained LQR for dicrete system?**

$$x_{k+1} = Ax_k + Bu_k \quad x \in \mathcal{X},\ u \in \mathcal{U}$$

and

$$J = \sum_{k=0}^{\infty} \left(x_k^T Q x_k + u_k^T R u_k + 2x_k^T N u_k\right) \rightarrow \min$$

- **well, not really...**

Do we know how to solve it?

- **Finite horizon version** of constrained LQR for discrete system?

$$x_{k+1} = Ax_k + Bu_k$$

$$x \in \mathcal{X}, \;\; u \in \mathcal{U}$$

min

$$J = x_{H_p}^T Q_{H_p} x_{H_p} + \sum_{k=0}^{H_p-1} \left(x_k^T Q x_k + u_k^T R u_k + 2 x_k^T N u_k \right), \text{ where } H_p \text{ is the time horizon}$$

- **yes, there are solvers capable to do so, at least when constraints are convex**

Do we know how to solve it?

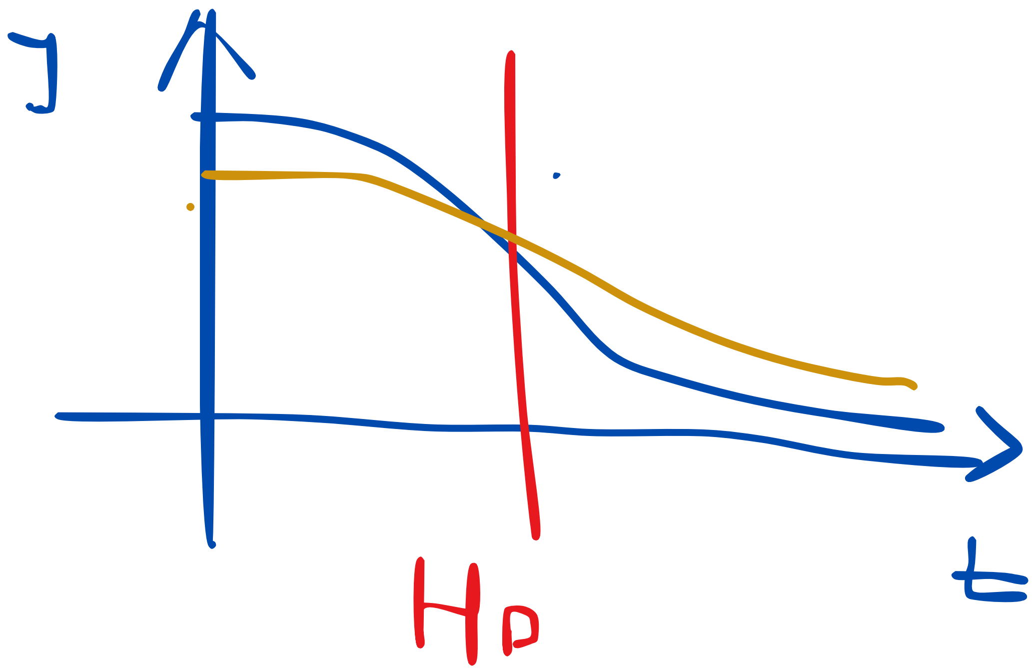
- Finite time horizon version of constrained LQR for discrete system?

$$x_{k+1} = Ax_k + Bu_k$$

$$x \in \mathcal{X},\ u \in \mathcal{U}$$

min

$$J = x_{H_p}^T Q_{H_p} x_{H_p} + \sum_{k=0}^{H_p-1} \left(x_k^T Q x_k + u_k^T R u_k + 2x_k^T N u_k\right),$$ where H_p is the time horizon



- yes, there are solvers capable to do so, at least when constraints are convex
  - Suboptimal solution w.r.t. original infinite time horizon problem

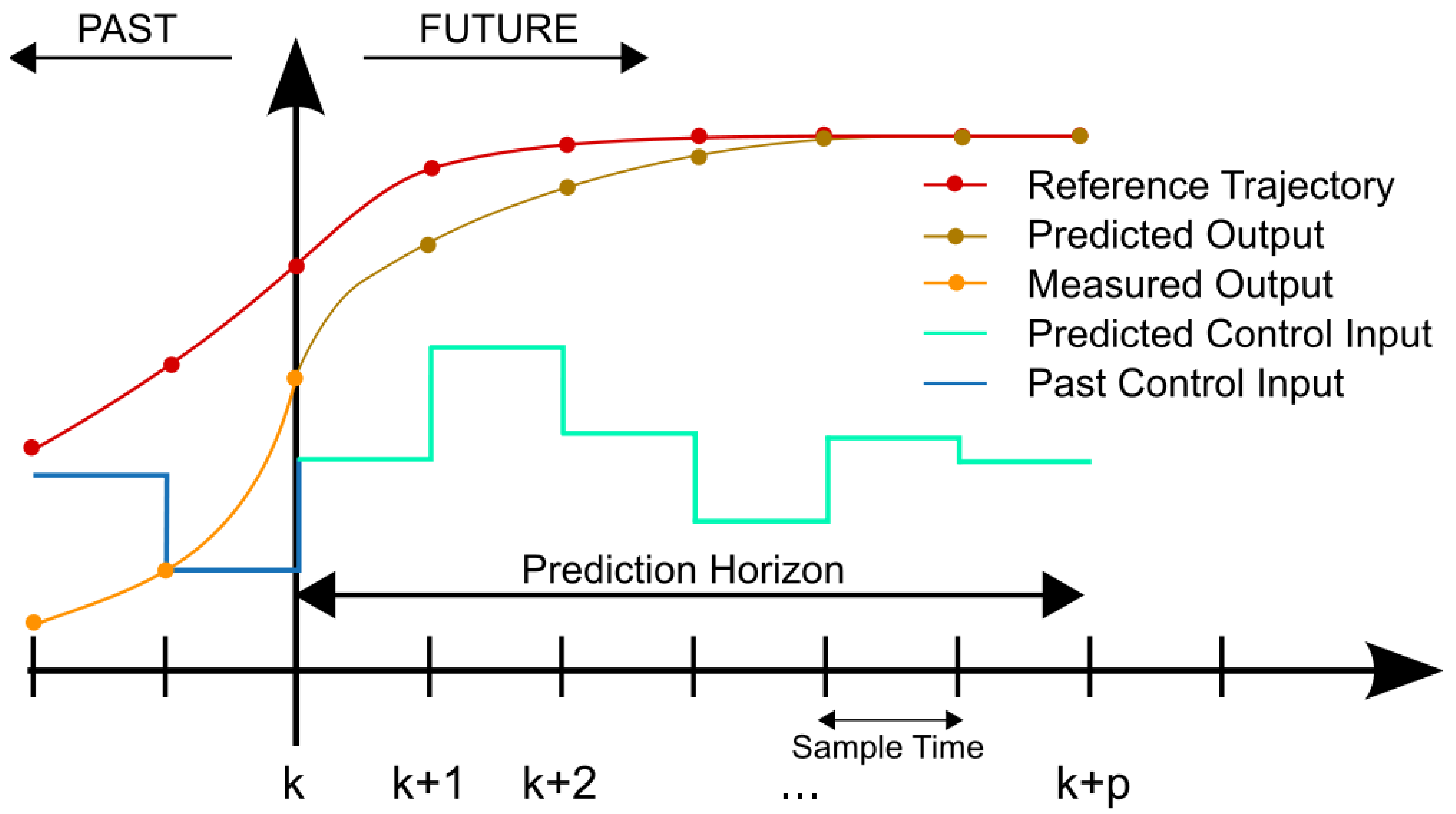
Model predictive control

- at each time t solve the (planning) problem

$$\begin{array}{ll} \text{minimize} & \sum_{\tau=t}^{t+T} \ell(x(\tau), u(\tau)) \\ \text{subject to} & u(\tau) \in \mathcal{U}, \quad x(\tau) \in \mathcal{X}, \quad \tau = t, \ldots, t+T \\ & x(\tau+1) = Ax(\tau) + Bu(\tau), \quad \tau = t, \ldots, t+T-1 \\ & x(t+T) = 0 \end{array}$$

  with variables $x(t+1), \ldots, x(t+T)$, $u(t), \ldots, u(t+T-1)$
  and data $x(t)$, A, B, ℓ, $\mathcal{X}$, $\mathcal{U}$
- call solution $\tilde{x}(t+1), \ldots, \tilde{x}(t+T)$, $\tilde{u}(t), \ldots, \tilde{u}(t+T-1)$
- we interpret these as *plan of action* for next T steps
- we take $u(t) = \tilde{u}(t)$
- this gives a complicated state feedback control $u(t) = \phi_{\text{mpc}}(x(t))$

Model predictive control



MPC for tracking

The Model Predictive Control (MPC) problem solved by pyMPC is:

$$\arg\min_{U} \overbrace{\frac{1}{2}\left(x_N - x_{\text{ref}}\right)^\top Q_{x_N}\left(x_N - x_{\text{ref}}\right)}^{=J_{Q_{x_N}}} + \overbrace{\frac{1}{2}\sum_{k=0}^{N_p-1}\left(x_k - x_{\text{ref}}\right)^\top Q_x\left(x_k - x_{\text{ref}}\right)}^{J_{Q_x}} +$$

$$+ \overbrace{\frac{1}{2}\sum_{k=0}^{N_p-1}\left(u_k - u_{\text{ref}}\right)^\top Q_u\left(u_k - u_{\text{ref}}\right)}^{J_u} + \overbrace{\frac{1}{2}\sum_{k=0}^{N_p-1}\Delta u_k^\top Q_{\Delta u}\Delta u_k}^{J_{\Delta u}} \tag{2a}$$

subject to :

$$x_{k+1} = Ax_k + Bu_k \tag{2b}$$

$$x_k \in \mathcal{X}, \; u_k \in \mathcal{U}$$

Tnank you for you attention!