Multiple Input Multiple Output system control

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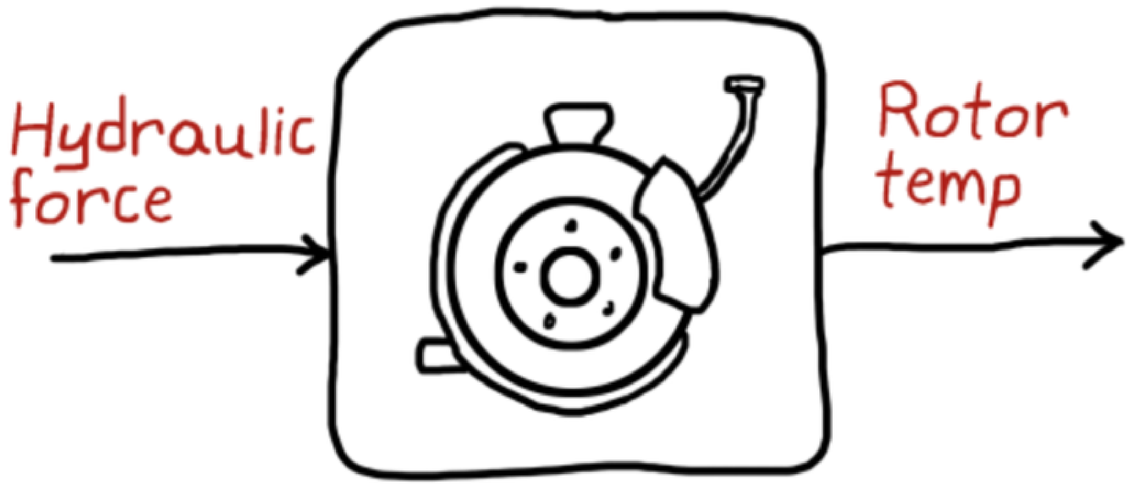
Course grade breakdowns

Labs - 50%

Final project - 50 %

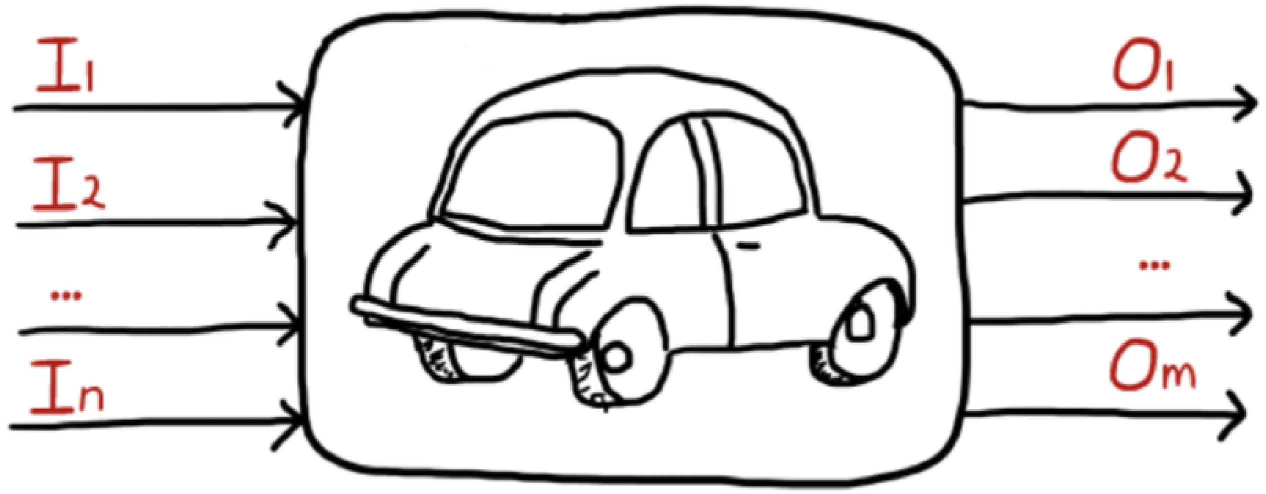
SISO system VS MIMO system

SISO



Single Input Single Output

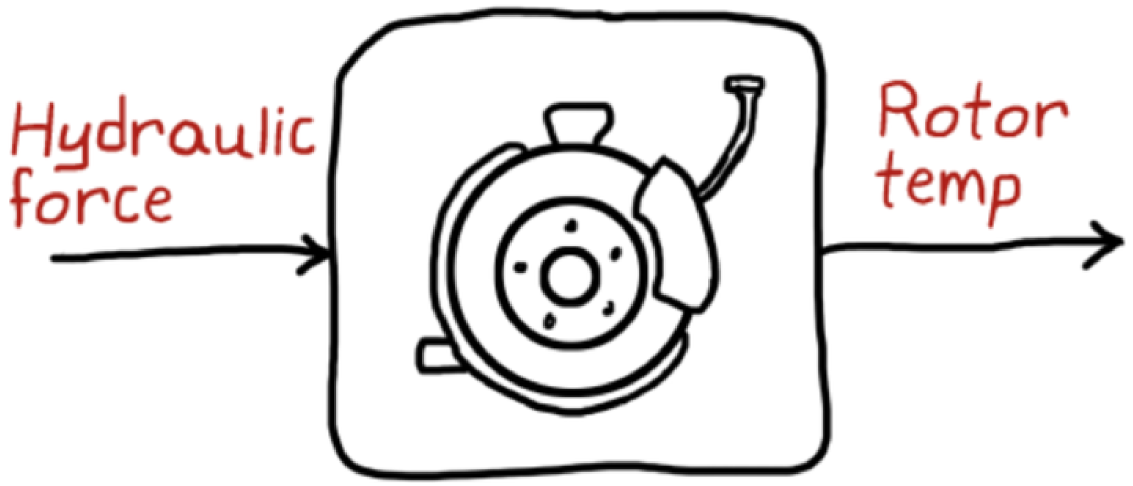
MIMO



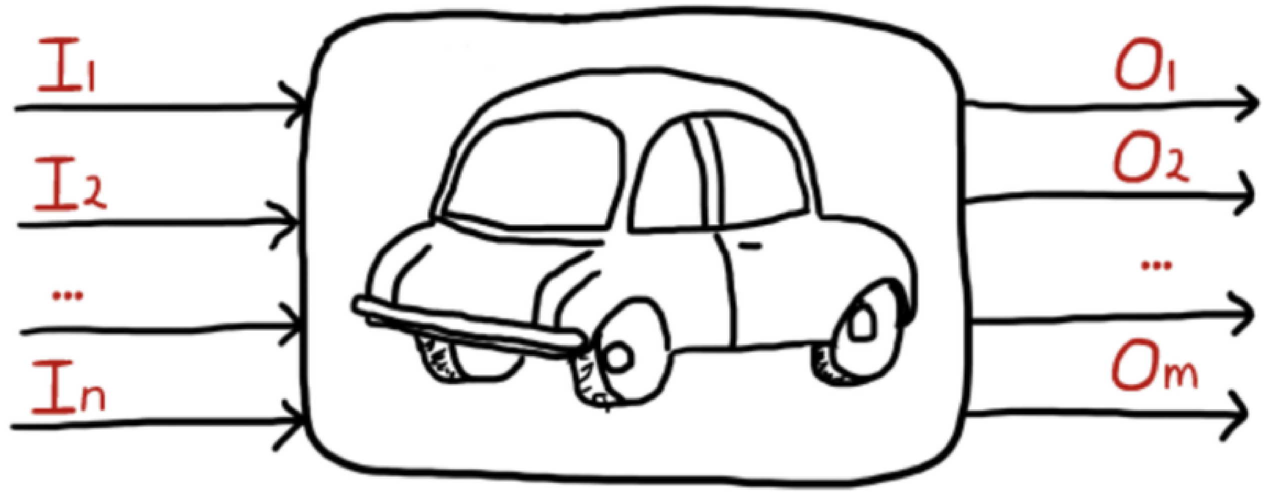
Multiple Inputs Multiple Outputs

SISO system VS MIMO system

SISO



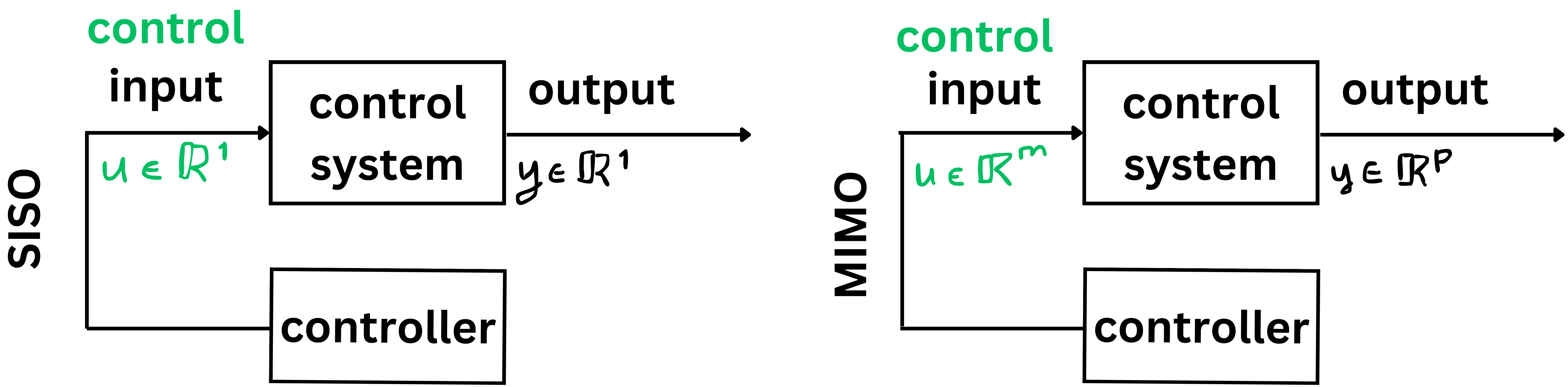
MIMO



Single Input Single Output

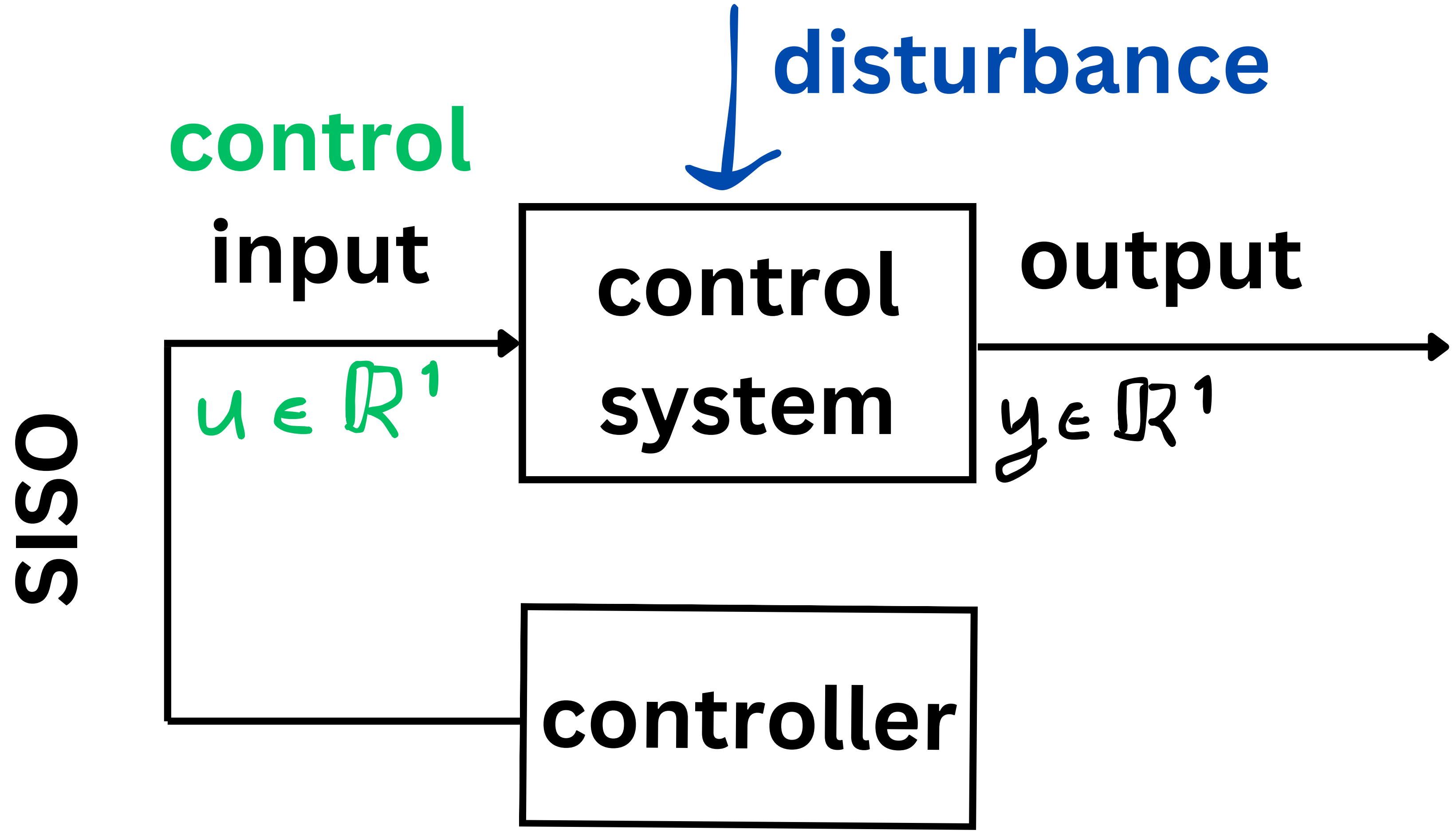
Multiple Inputs Multiple Outputs

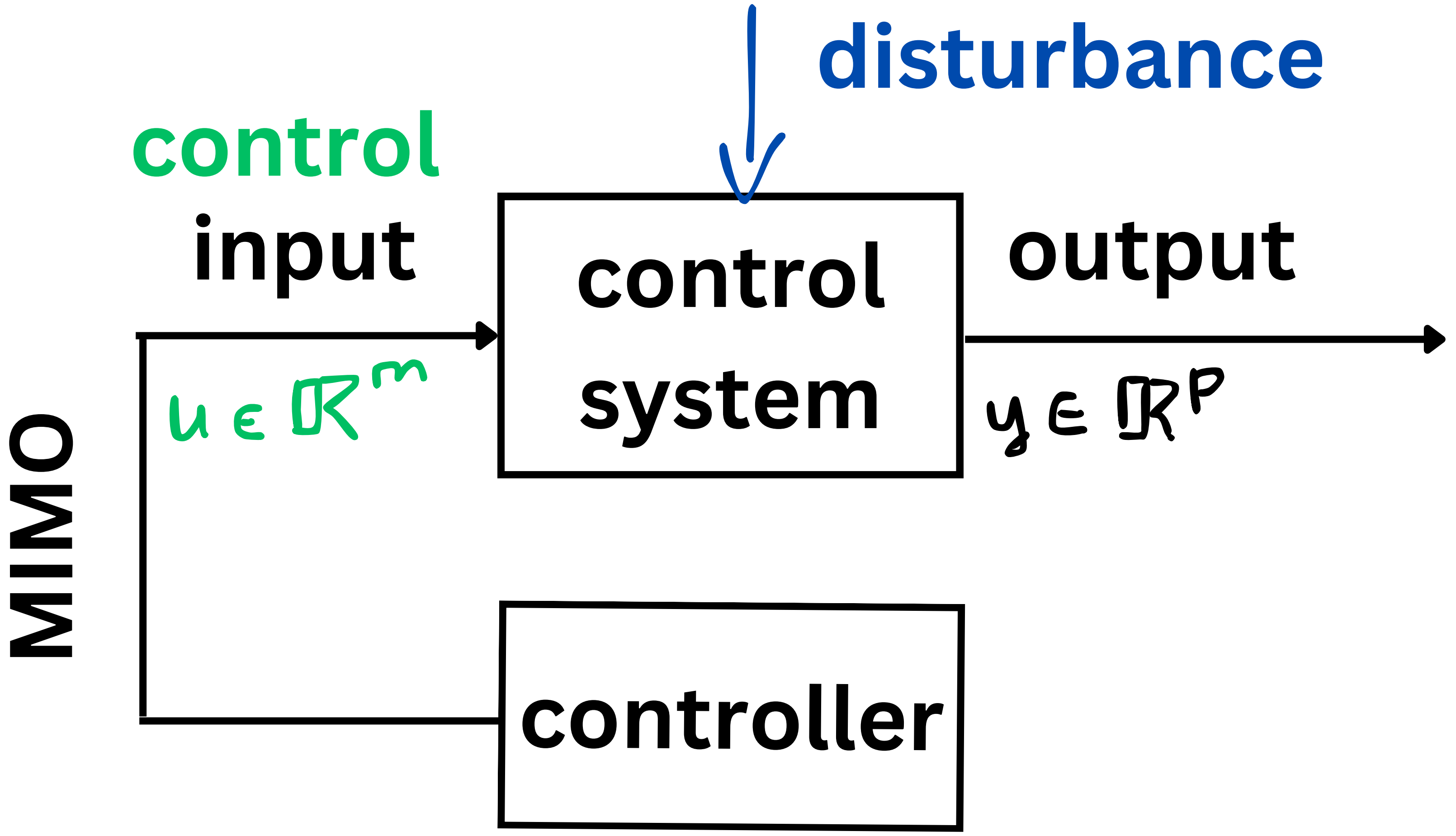
SISO system VS MIMO system



It is assumed that we count control inputs....

SISO system VS MIMO system



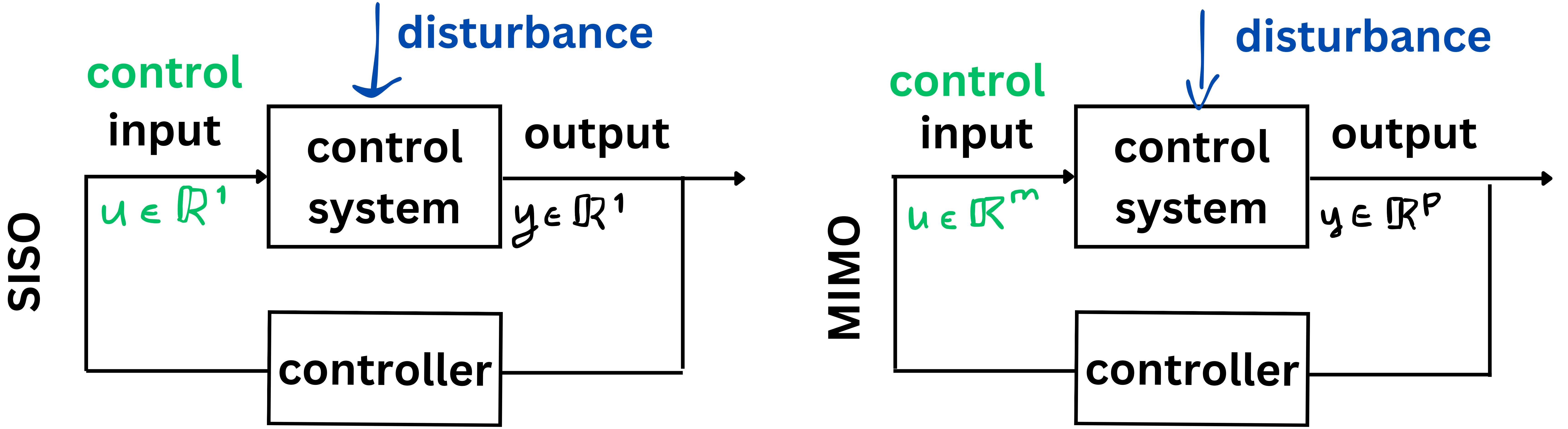


It is assumed that we count control inputs....

because there may be uncontrolled inputs (disturbances)

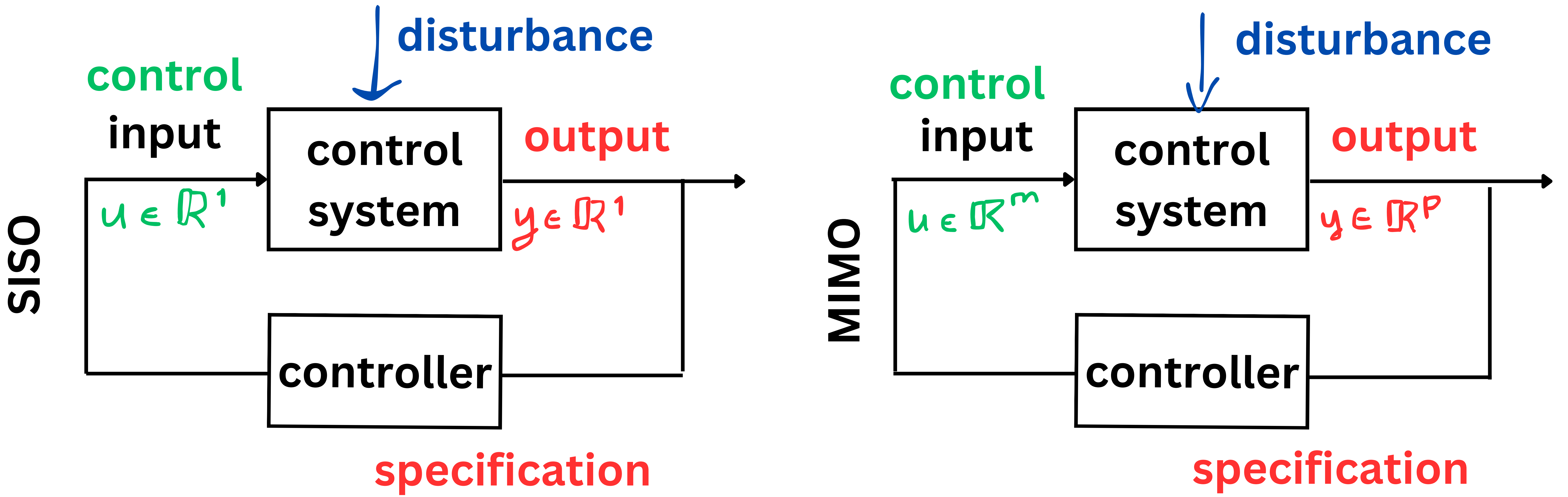
due to external factors, modeling errors, sensor noises, etc....:

SISO system VS MIMO system



Feedback controller design

SISO system VS MIMO system



Feedback controller design

how to use the sensor data (output) to generate the correct actuator commands (control input) to ensure that the output of the system satisfies the specification

What is specification?

Specification refers to a set of desired or required characteristics, behaviors, or performance metrics that a control system must satisfy or achieve.

Examples:

- Reachability: go from point A to point B in finite time t.
- Tracking: ensure that the output of the system $y(t)$ tracks the reference signal $y_r(t)$
  - Regulator: ensure that $\lim_{t \to +\infty} (y_r(t) - y(t)) = 0$, where $r(t) = const$

Specifications can include criteria such as transient response time, steady-state accuracy, disturbance rejection, robustness, notion of optimality state and actuator constraints....

Regulator for LTI systems

$$\dot{x} = Ax + Bu + w$$

state control disturbance

$$y = Cx + Du$$

output

- Regulator: ensure that $\lim_{t \to +\infty} (r(t) - y(t)) = 0$, where $r(t) = const$

Regulator for LTI systems

$$\dot{x} = Ax + Bu + w$$

state control disturbance

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output

Let us assume D = 0

- Regulator: ensure that $\lim_{t \to +\infty} (r(t) - y(t)) = 0$, where $r(t) = const$

Regulator for LTI systems

$$\dot{x} = Ax + Bu + w$$

state control disturbance

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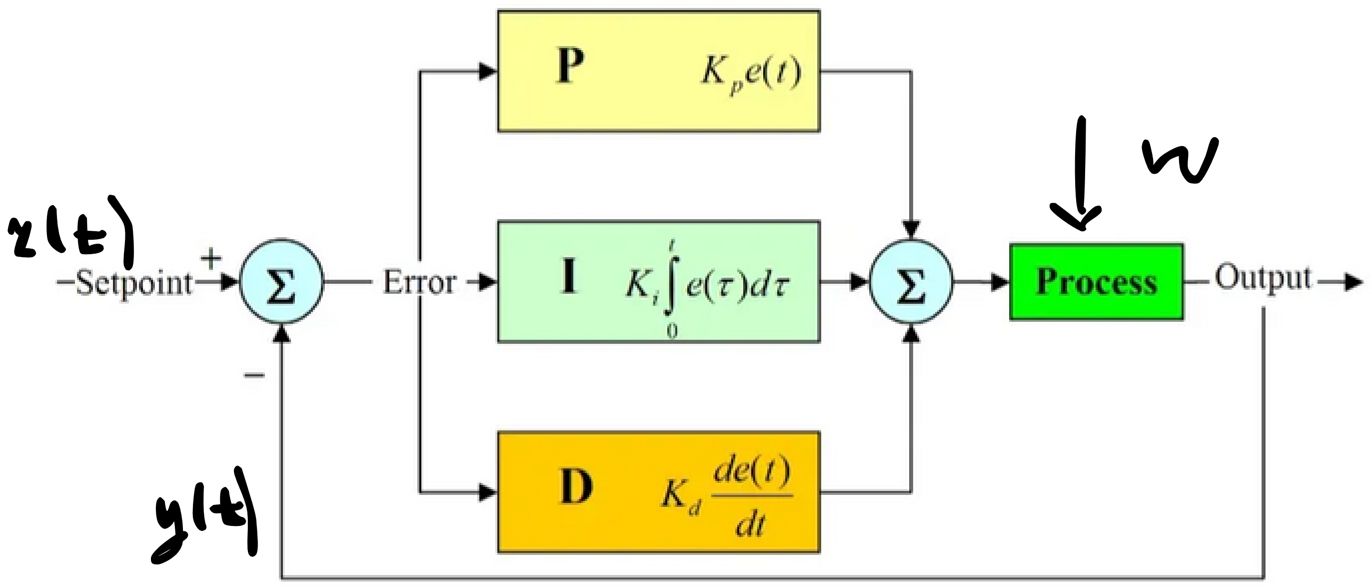
output



e(t)

- Regulator: ensure that $\lim_{t \to +\infty} (r(t) - y(t)) = 0$, where $r(t) = const$

Last time we assume that the system is SISO and we were trying to solve regulator problem by using PID feedback controller

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \dot{e}(t)$$

PID: Pros

Stability

PID controllers are capable of providing stable and accurate control over systems, ensuring that they reach and maintain the desired setpoint efficiently.

Tuning Flexibility

PID controllers offer flexibility in tuning parameters (Proportional, Integral, and Derivative gains) to achieve optimal performance for different systems and operating conditions.

Simple Implementation

Compared to more complex control algorithms, PID controllers are relatively simple to implement, making them suitable for a wide range of applications and accessible to engineers and technicians with basic control theory knowledge.

Real-Time Control

PID controllers are well-suited for real-time control applications due to their simplicity and efficiency, making them suitable for controlling systems with fast response

PID: Pros

Tuning Complexity:

Tuning PID controllers can be complex, especially for systems with nonlinear dynamics or time-varying parameters. Finding the right balance between stability and performance often requires iterative tuning processes.

Limited Robustness:

PID controllers may lack robustness compared to more advanced control algorithms, particularly in systems with uncertain parameters or external disturbances. Robust PID tuning methods exist but may require additional effort and expertise.

Potential for Oscillations and Instability:

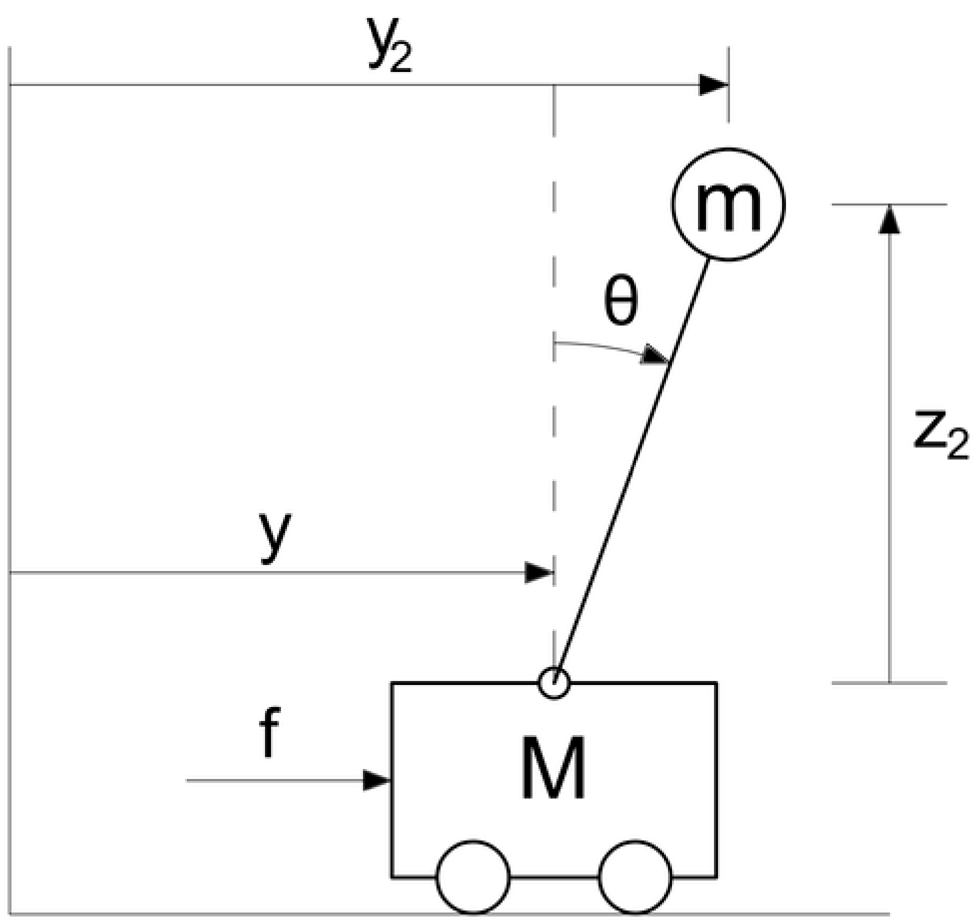
Improper tuning of PID parameters can lead to oscillations or instability in the controlled system, resulting in erratic behavior or even system damage if not addressed promptly.

Integral windup & Derivative term sensitive to measurement errors

PID: Pros

it is not clear how to design a PID controller
when system is not SISO...

Cart-pole control

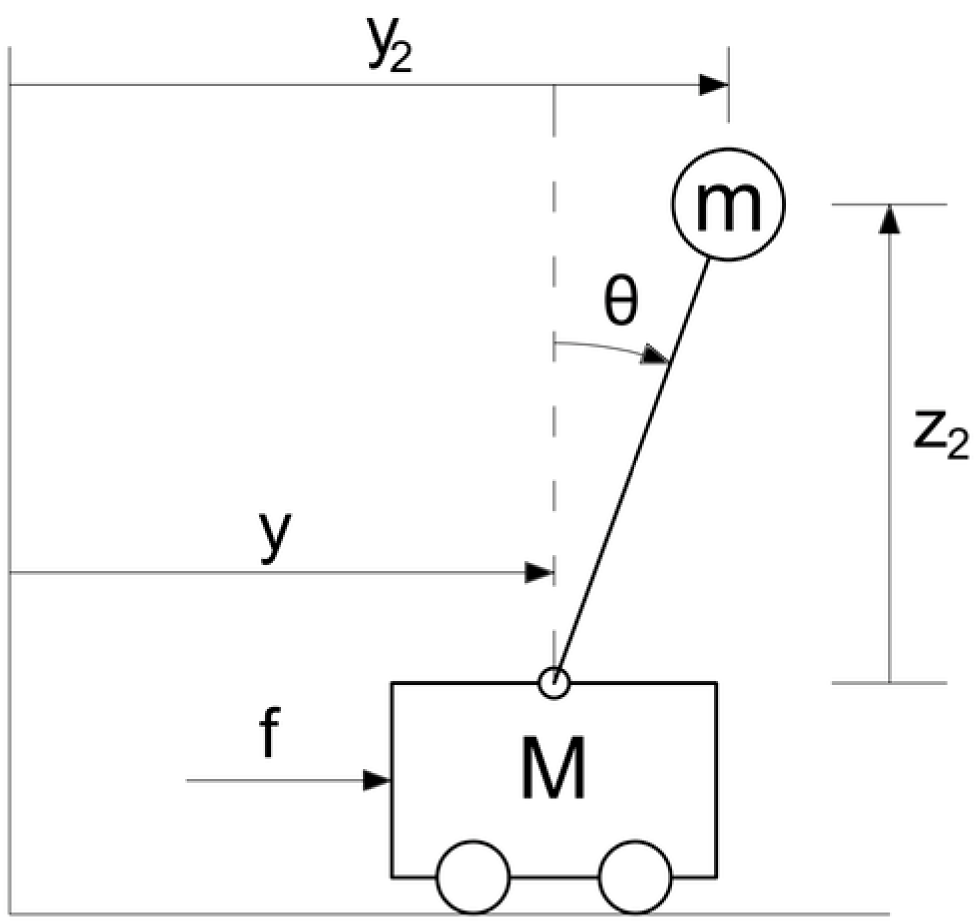


Inverted pendulum on the cart can be modeled as follows

$$(M+m)\ddot{y} + b\dot{y} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin(\theta) = F$$

$$ml\cos(\theta)\ddot{y} + (I + ml^2)\ddot{\theta} - mgl\sin\theta = 0$$

Cart-pole control



Specification

Angular velocity tracks reference trajectory equal to 1 rad

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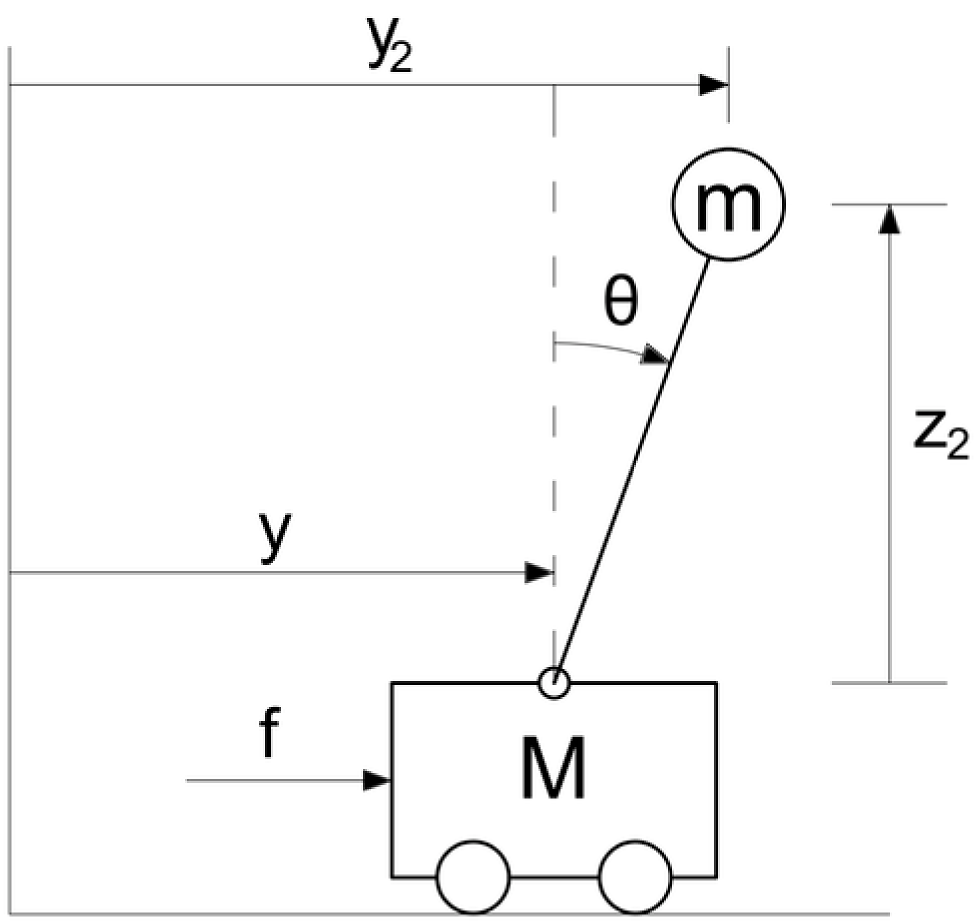
Cart-pole control



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Or in canonical state space ODE form

$$\begin{cases} \dot{y} = y_1 \\ \dot{y}_1 = \dfrac{-m^2l^2g\cos\theta\sin\theta + \left(I+ml^2\right)\left(ml\theta_1^2\sin\theta + F - by_1\right)}{(I+ml^2)(M+m) - m^2l^2\cos^2\theta} \\ \dot{\theta} = \theta_1 \\ \dot{\theta}_1 = \dfrac{(M+m)mgl\sin\theta + by_1 ml\cos\theta - m^2l^2\theta_1^2\cos\theta\sin\theta - mlF\cos\theta}{(M+m)(I+ml^2) - m^2l^2\cos^2\theta} \end{cases}$$

Specification

Angular velocity tracks reference trajectory equal to 1 rad

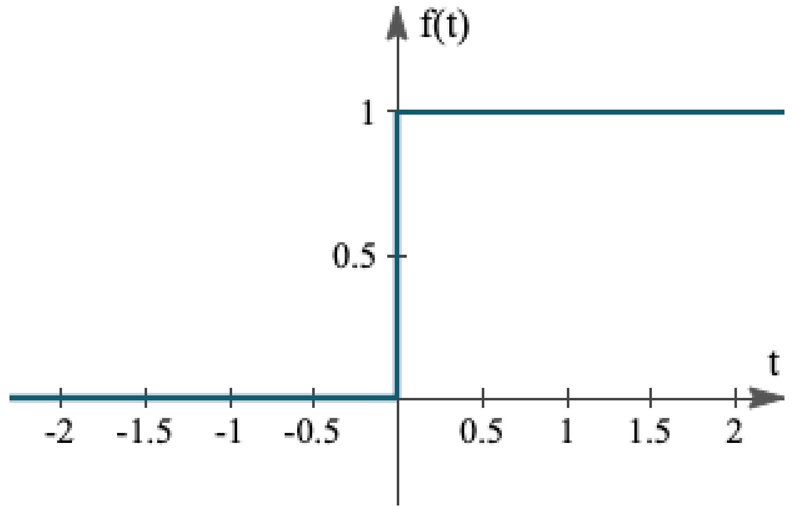
Cart-pole control

Lineralized model

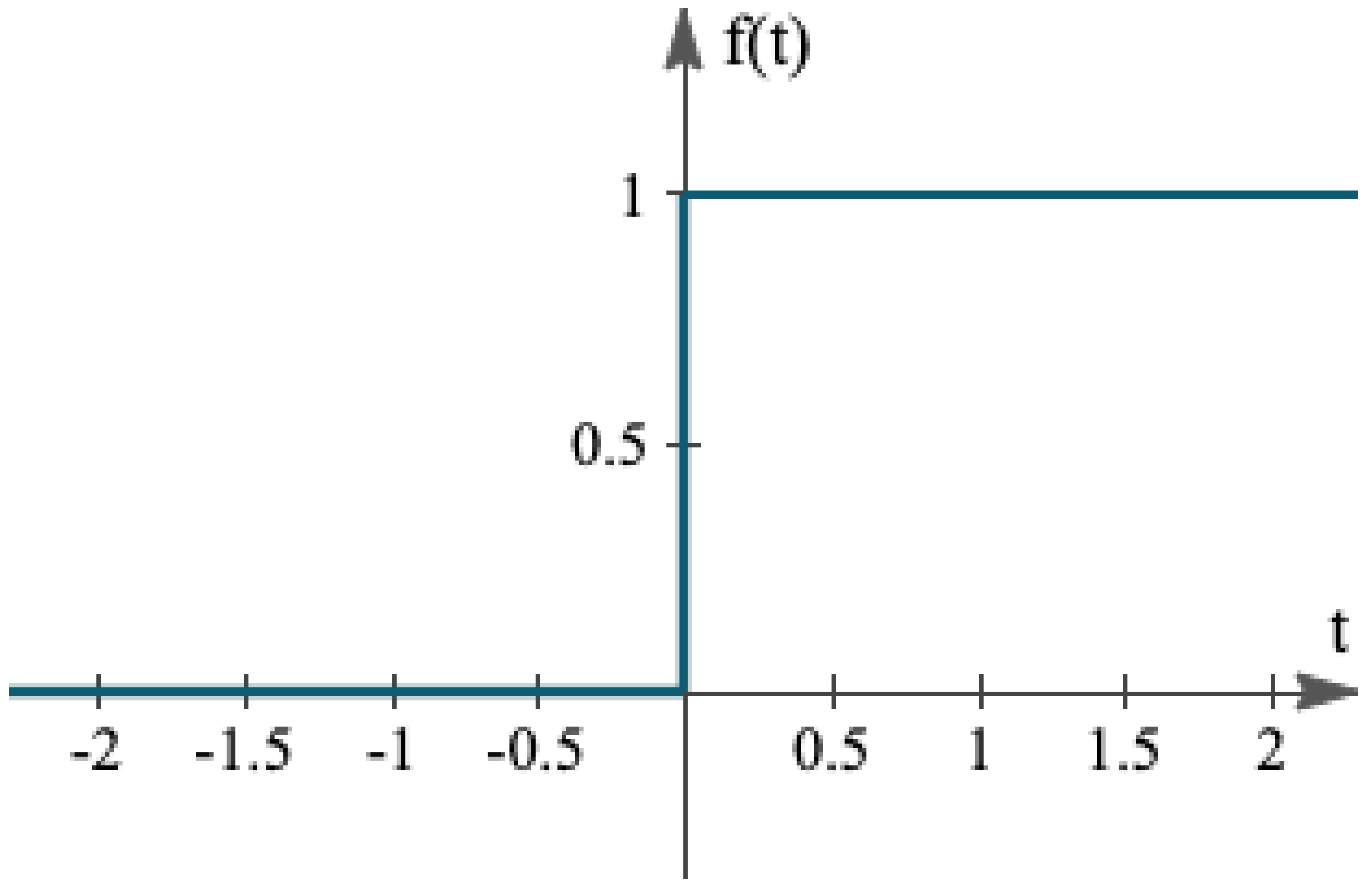
$$\begin{bmatrix} \dot{y} \\ \dot{y}1 \\ \dot{\theta} \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{-gm^2 l^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} y \\ y_1 \\ \theta \\ \theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{-ml}{I(M+m)+Mml^2} \end{bmatrix} u$$

- Regulator: ensure that $\lim_{t \to +\infty} (y_r(t) - y(t)) = 0$, where $r(t) = const$

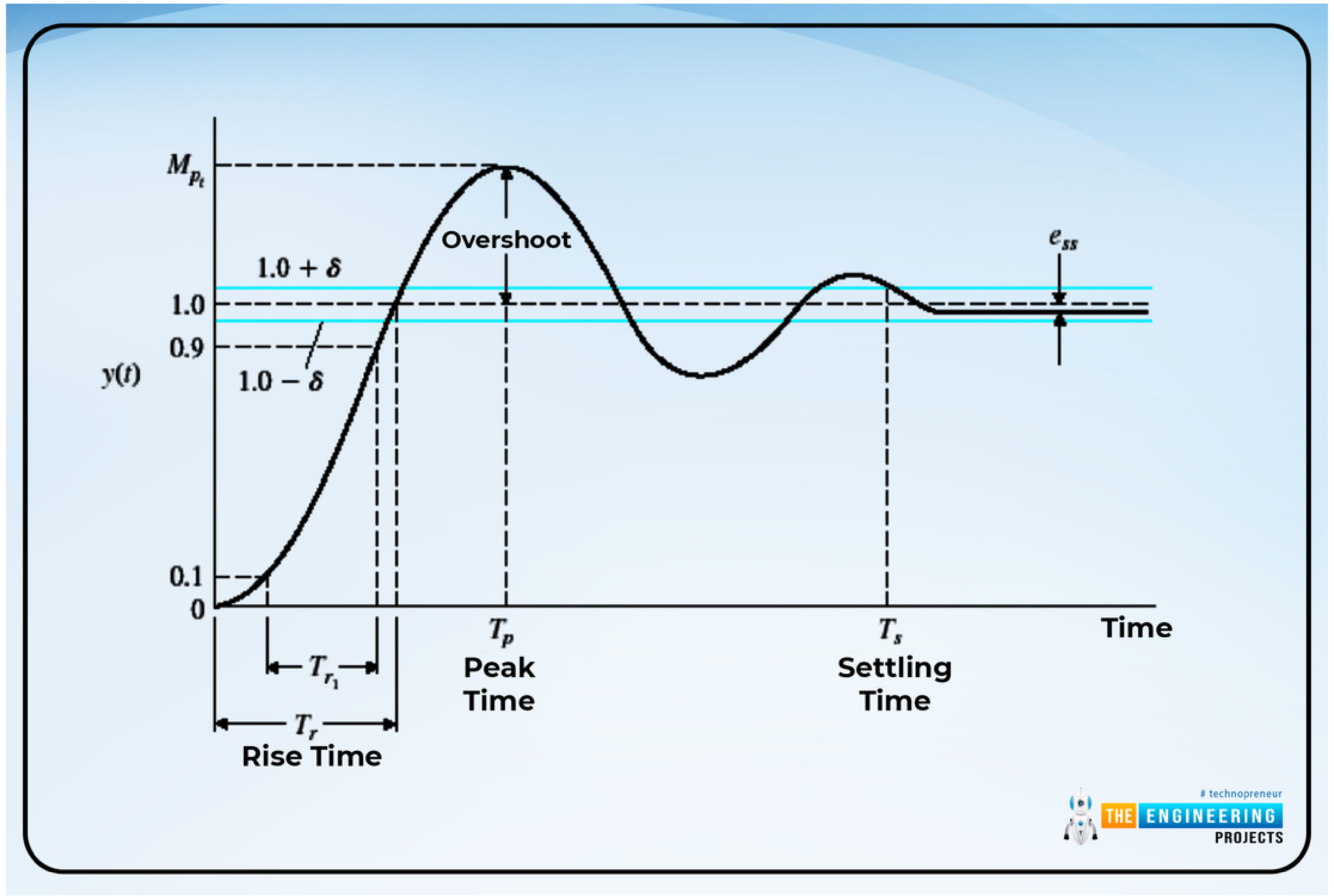
$$y_r(t) = \begin{cases} 1, t > 0 \\ 0, t <= 0 \end{cases}.$$



Cart-pole control

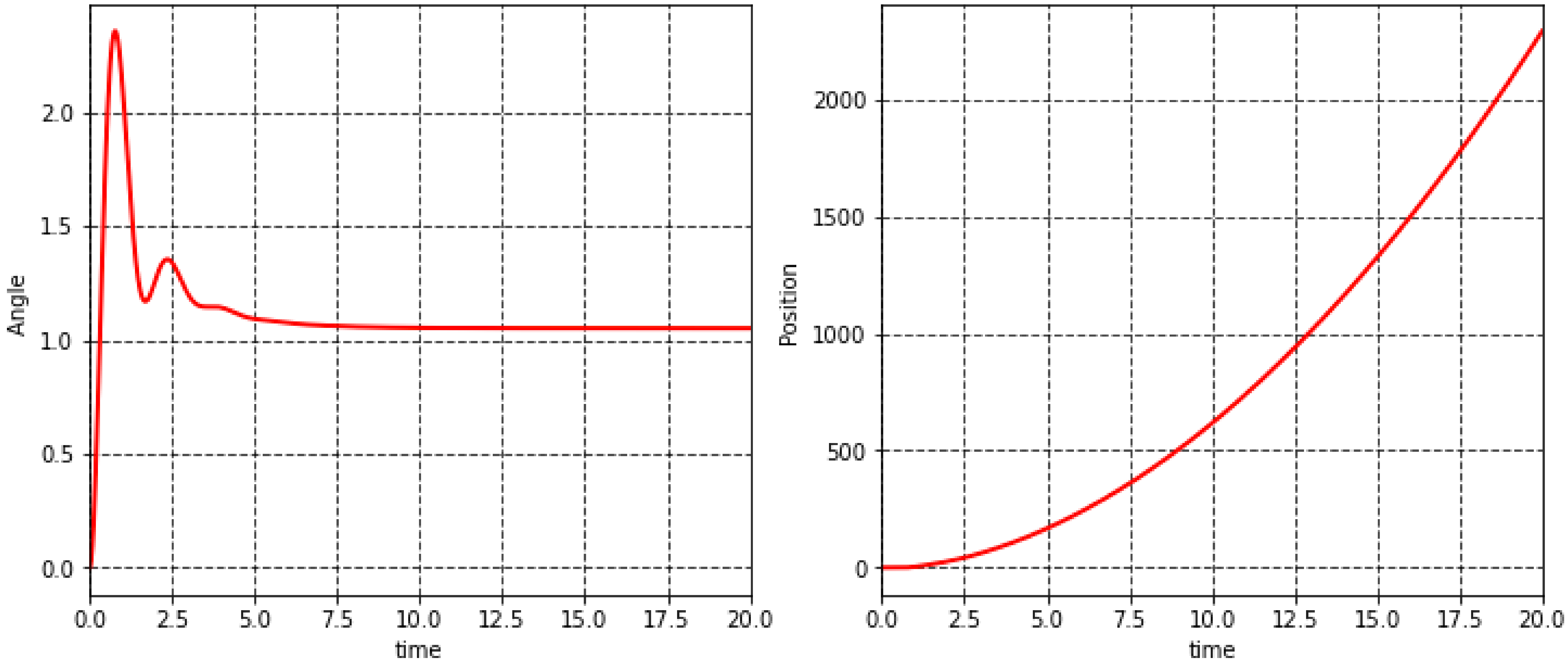


Step function



Step response

Cart-pole control



Angular velocity is stabilised, but position goes to infinity....

Regulator for LTI systems

$$\dot{x} = Ax + Bu + w$$

state control disturbance

$$y = Cx$$

output

- Regulator: ensure that $\lim_{t \to +\infty} (y_r(t) - y(t)) = 0$, where $r(t) = const$

Design a feed controller which works for MIMO systems...

Regulator for LTI systems

$$\dot{x} = \underbrace{Ax}_{\text{state}} + \underbrace{Bu}_{\text{control}} + \cancel{\underbrace{w}_{\text{disturbance}}}$$

$$\underbrace{y}_{\text{output}} = Cx$$

Let's assume that reference set point is zero and there is no external disturbance in the system

- Regulator: ensure that $\lim_{t \to +\infty} (y_r(t) - y(t)) = 0$, where $r(t) = const = 0$

Stabilsation

Design a feed controller which works for MIMO systems...

$$\lim_{t \to 0} y(t) \to 0 \Leftarrow \lim_{t \to 0} x(t) \to 0, \text{ since } y = Cx, C = const$$

Stability of LTI systems

Response of a LTI system is composed of

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

response to initial conditions

response to initial conditions

Stability of LTI systems

Response of a LTI system is composed of

$$y(t) = Ce^{At}\overset{0}{x(0)} + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

response to initial conditions

response to initial conditions

Input-Output Stability

The concept of **Input-Output Stability** refers to stability of the **response to inputs** only, assuming zero initial conditions.

BIBO Stability. A system is BIBO (**bounded-input bounded-output**) stable if *every* bounded input produces a bounded output.

Stability of LTI systems

Response of a LTI system is composed of

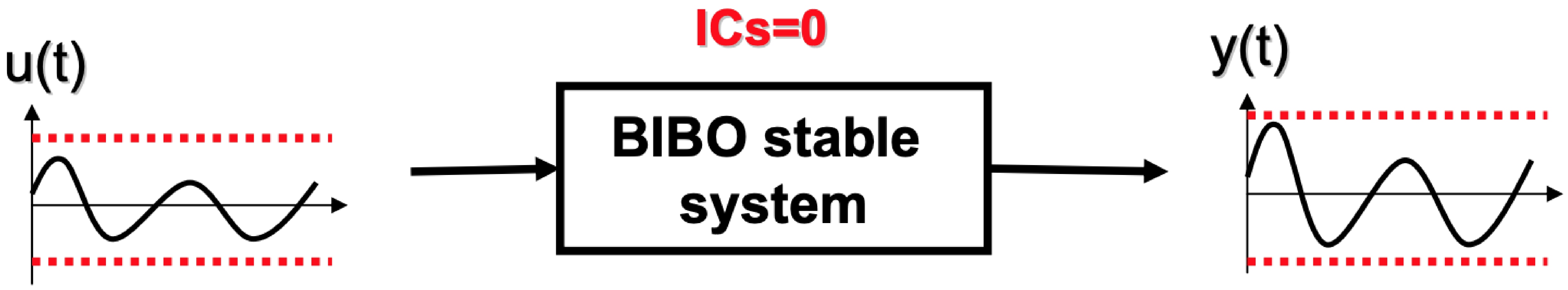
$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{response to initial conditions}} + \underbrace{C\int_0^t e^{A(t-\tau)}B\overset{=0}{u}(\tau)d\tau}_{\text{response to initial conditions}}$$

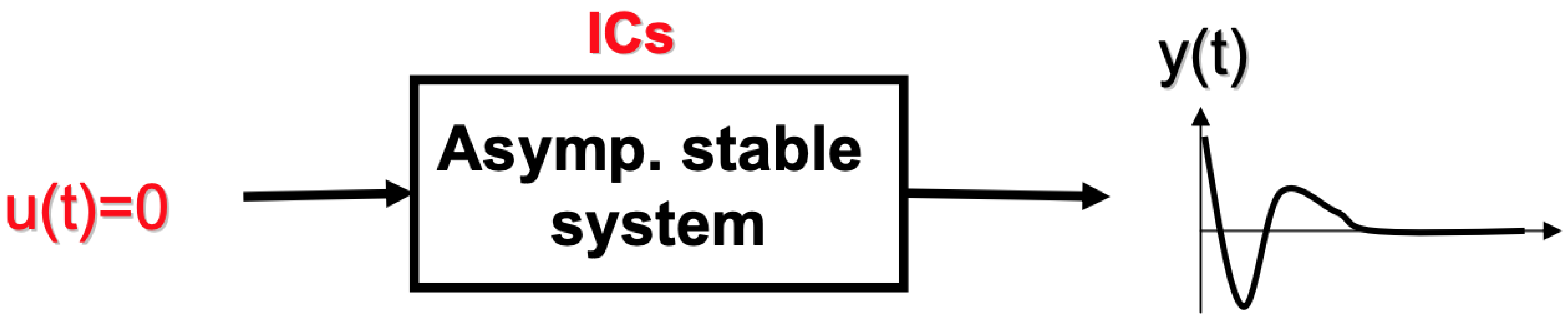
Internal Stability of LTI Systems

The concept of **Internal Stability** refers to stability of the system **response to initial conditions** only, assuming zero inputs.

Asymptotic Stability. The system $\dot{x}(t) = Ax(t)$ is asymptotically stable if *every* finite initial state x_0 excites a bounded response $x(t)$ that approaches 0 as $t \to \infty$.

Stability of LTI systems



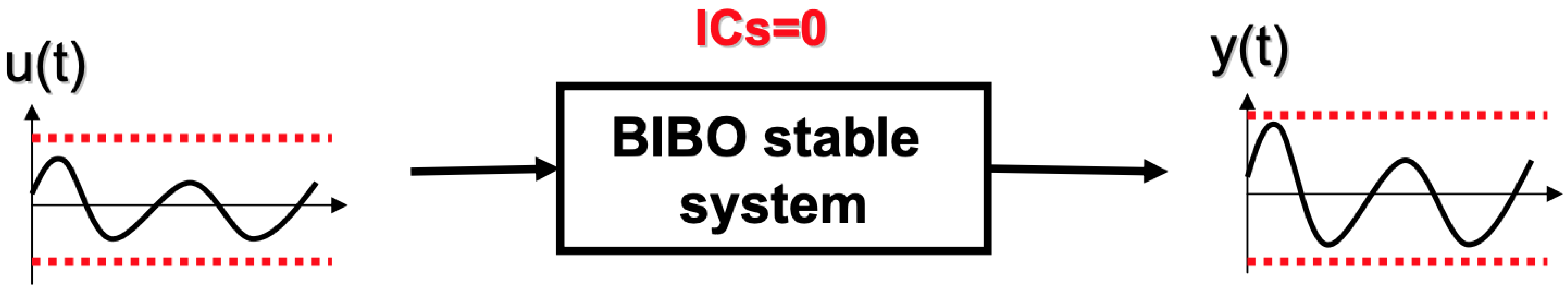


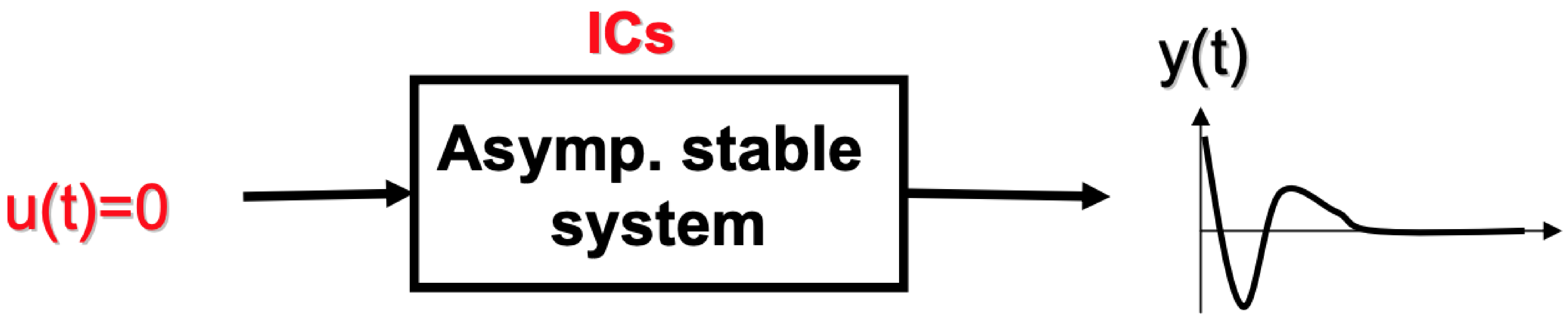
It is known

asymptotic stability $\Rightarrow$ BIBO stability

asymptotic stability $\not\Leftarrow$ BIBO stability

Stability of LTI systems





It is known

asymptotic stability $\Rightarrow$ **BIBO stability**

asymptotic stability $\not\Leftarrow$ **BIBO stability**

And since we focus on asymptotic stability

Stability of LTI systems

Internal Stability of LTI Systems

Theorem (Internal Stability). The equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is Asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ have negative real parts.

State feedback design

Linear state space control theory involves modifying the behaviour of an m-input, p-output, n-state system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{OL}$$

which we call **the plant**, or **open loop state equation**, by application of a control law of the form

$$\mathbf{u}(t) = \mathbf{N}\mathbf{r}(t) - \mathbf{K}\mathbf{x}(t), \tag{U}$$

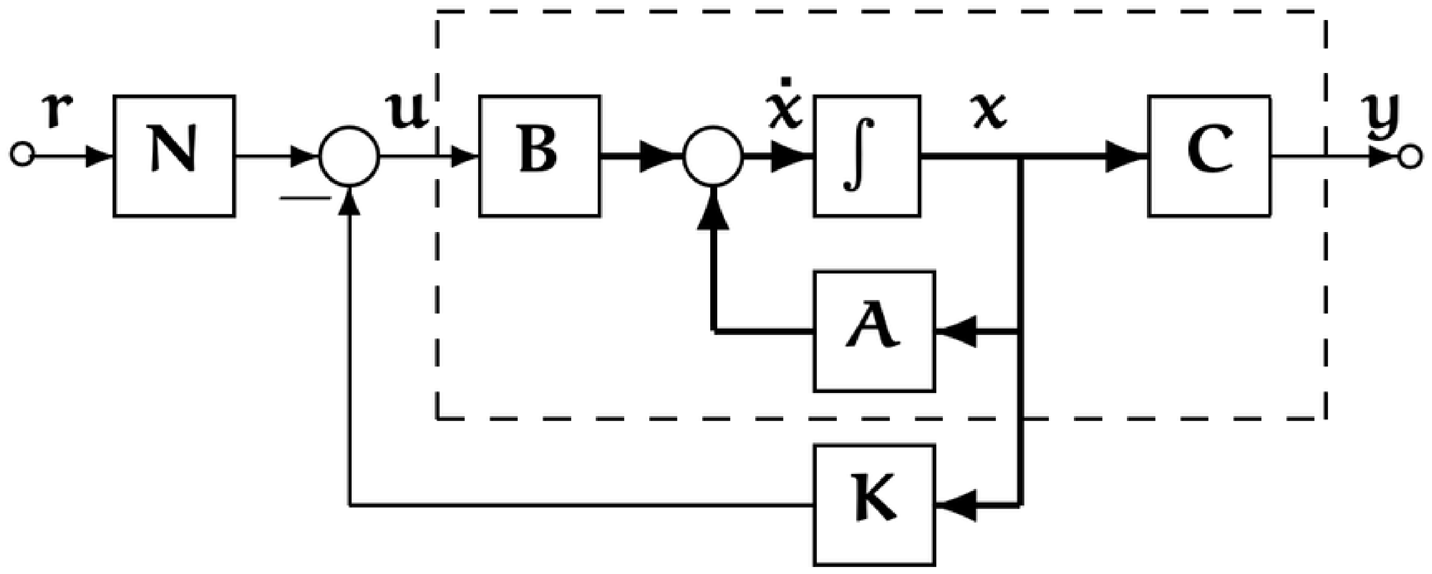
in which $\mathbf{r}(t)$ is the new (reference) input signal. The matrix $\mathbf{K}$ is the state feedback gain and $\mathbf{N}$ the feedforward gain.

State feedback design

Substitution of (U) into (OL) gives the closed-loop state equation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{N}\mathbf{r}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{aligned} \qquad \text{(CL)}$$

Obviously, the closed-loop system is also LTI.

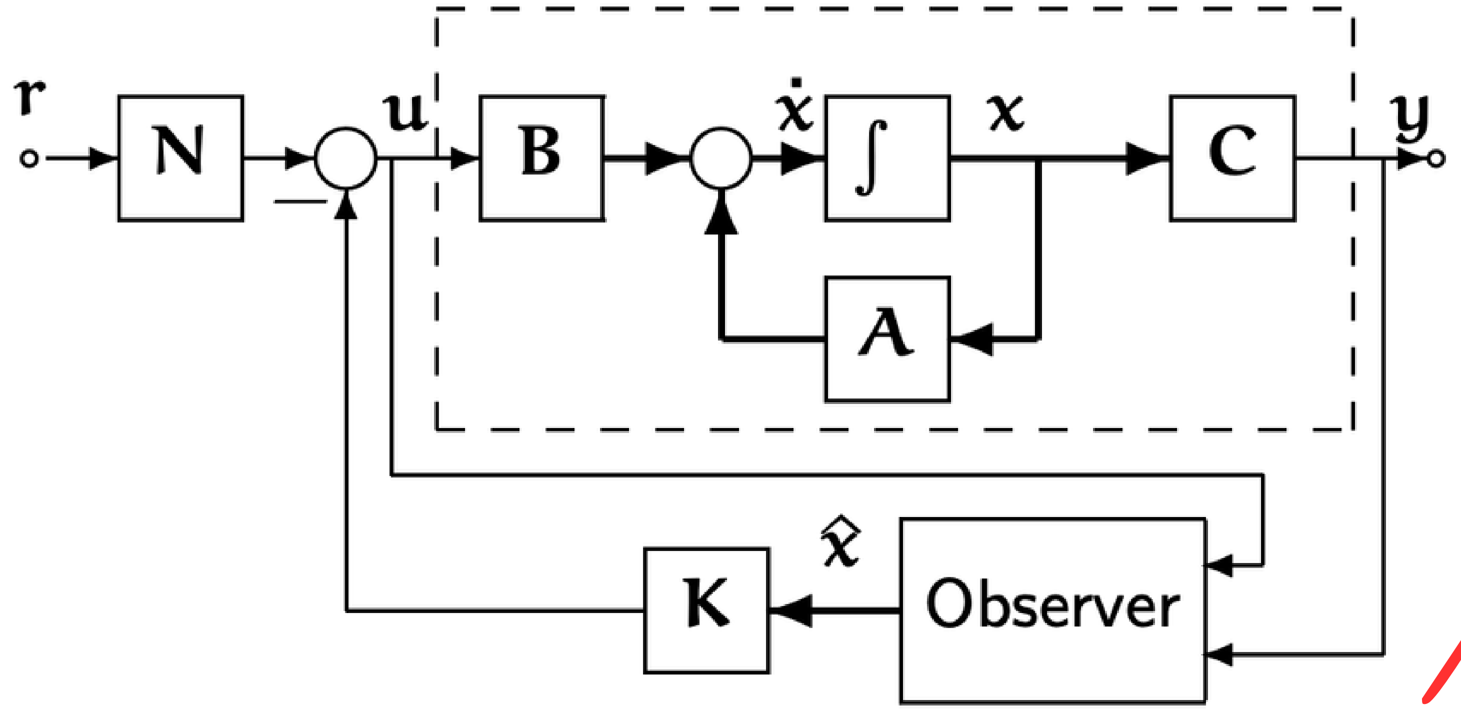


State feedback with feedforward precompensation

This type of control is said to be **static**, because $\mathbf{u}$ only depends on the present values of the state $\mathbf{x}$ and the reference $\mathbf{r}$. Note that *it requires that all states of the system be measured*.

What if full state is not available?

When not all the states of the system are measurable, we resource to their *estimation* by means of an *observer*, or *state estimator*, which reconstructs the state from measurements of the inputs $\mathbf{u}(\mathbf{t})$ and outputs $\mathbf{y}(\mathbf{t})$.



Output feedback by estimated state feedback

We will address observer design next lecture

The combination of state feedback and state estimation yields a **dynamic** output feedback controller.

Eiginvalues assigntement

Theorem (Controllability and Feedback — MIMO). The pair $(\mathbf{A} - \mathbf{BK}, \mathbf{B})$, for any $p \times n$ real matrix $\mathbf{K}$ is controllable if and only if the pair $(\mathbf{A}, \mathbf{B})$ is controllable.

Theorem (Eigenvalue assignment — MIMO). All eigenvalues of $(\mathbf{A} - \mathbf{BK})$ can be assigned arbitrarily (provided complex eigenvalues are assigned in conjugated pairs) by selecting a real constant $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

Let us proof for SISO system

Recall: Control canonical form

If the system (A,B) is controllable, we can take it to its CCF

$$\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{P}\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\bar{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1} = [\, \beta_1 \;\; \beta_2 \;\; \ldots \;\; \beta_{n-1} \;\; \beta_n \,].$$

These matrices arise from the change of coordinates $\bar{x} = \mathbf{P}x$ where

$$\mathbf{P}^{-1} = \mathcal{C}\bar{\mathcal{C}}^{-1} \quad \text{with} \quad \begin{aligned} \mathcal{C} &= [\, \mathbf{B} \;\; \mathbf{A}\mathbf{B} \;\; \ldots \;\; \mathbf{A}^{n-1}\mathbf{B} \,] \\ \bar{\mathcal{C}} &= [\, \bar{\mathbf{B}} \;\; \bar{\mathbf{A}}\bar{\mathbf{B}} \;\; \ldots \;\; \bar{\mathbf{A}}^{n-1}\bar{\mathbf{B}} \,] \end{aligned}$$

Eiginvalues assigntement for SISO system

Theorem (Eigenvalue assignment by state feedback). if the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

is controllable, then the state feedback control law

$$\boxed{\mathbf{u} = \mathbf{r} - \mathbf{K}\mathbf{x}}, \qquad \text{where } \mathbf{K} \in \mathbb{R}^{1\times n},$$

assigns the eigenvalues of the closed-loop state equation

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{r}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

to any desired, arbitrary locations, provided that complex eigenvalues are assigned in conjugate pairs.

Eiginvalues assintement for SISO system

Proof: If the system is controllable, we can take it to its CCF by the change of coordinates $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$, which yields $\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ and $\bar{\mathbf{B}} = \mathbf{B}\mathbf{P}$. It is not difficult to verify that

$$\bar{\mathcal{C}} \triangleq [\bar{\mathbf{B}}, \bar{\mathbf{A}}\bar{\mathbf{B}}, \ldots, \bar{\mathbf{A}}^{n-1}\bar{\mathbf{B}}] = \mathbf{P}[\mathbf{B}, \mathbf{A}\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{P}\mathcal{C},$$

and thus $\mathbf{P}^{-1} = \mathcal{C}\bar{\mathcal{C}}^{-1}$.

On substituting $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ in the state feedback law

$$u = r - \mathbf{K}\mathbf{x} = r - \mathbf{K}\mathbf{P}^{-1}\bar{\mathbf{x}} \triangleq r - \bar{\mathbf{K}}\bar{\mathbf{x}},$$

Since $\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}} = \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{P}^{-1}$, we see that $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}}$ are similar, and thus **have the same eigenvalues**.

Now, say that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the desired closed-loop eigenvalue locations. We can then generate the desired characteristic polynomial

$$\begin{aligned}\Delta_K(s) &= (s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_n) \\ &= s^n + \bar{\alpha}_1 s^{n-1} + \cdots + \bar{\alpha}_n.\end{aligned}$$

Eiginvalues assintement for SISO system

If we choose

$$\bar{\mathbf{K}} = [(\bar{\alpha}_1 - \alpha_1), (\bar{\alpha}_2 - \alpha_2), \ldots, (\bar{\alpha}_n - \alpha_n)],$$

the closed-loop state equation becomes (in the $\bar{\mathbf{x}}$ coordinates)

$$\dot{\bar{\mathbf{x}}}(\mathbf{t}) = (\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}})\bar{\mathbf{x}}(\mathbf{t}) + \bar{\mathbf{B}}\mathbf{r}(\mathbf{t})$$

$$= \left(\begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} - \begin{bmatrix} (\bar{\alpha}_1 - \alpha_1) & (\bar{\alpha}_2 - \alpha_2) & \cdots & (\bar{\alpha}_n - \alpha_n) \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right) \bar{\mathbf{x}}(\mathbf{t}) + \bar{\mathbf{B}}\mathbf{r}$$

$$= \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & \cdots & -\bar{\alpha}_{n-1} & -\bar{\alpha}_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \bar{\mathbf{x}}(\mathbf{t}) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mathbf{r}(\mathbf{t}).$$

Because the closed-loop evolution matrix $(\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}})$ is still in companion form, we see from the last expression that its characteristic polynomial is the desired one $\Delta_{\mathbf{K}}(\mathbf{s})$. Finally, from $\bar{\mathbf{K}} = \mathbf{K}\mathbf{P}^{-1}$, we get that $\mathbf{K} = \bar{\mathbf{K}}\mathbf{P}$. □

Eiginvalues assintement for SISO system

Procedure for pole placement by state feedback (Bass-Gura Formula)

1. Obtain the coefficients of the open loop characteristic polynomial $\Delta(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$.
2. Form the controllability matrices $\mathcal{C} = [\, B \;\; AB \;\; \cdots \;\; A^{n-1}B \,]$ and

$$\bar{\mathcal{C}} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 1 & \alpha_1 & \cdots & \alpha_{n-3} & \alpha_{n-2} \\ 0 & 0 & 1 & \cdots & \alpha_{n-4} & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^{-1}$$

(note the inverse!)

3. Select the coefficients of the desired **closed-loop** characteristic polynomial $\Delta_K(s) = s^n + \bar{\alpha}_1 s^{n-1} + \cdots + \bar{\alpha}_n$ and build the state-feedback gain in $\bar{x}$ coordinates,

$$\bar{K} = [\, (\bar{\alpha}_1 - \alpha_1) \;\; (\bar{\alpha}_2 - \alpha_2) \;\; \cdots \;\; (\bar{\alpha}_n - \alpha_n) \,]$$

4. Compute the state-feedback gain in the original x coordinates

$$K = \bar{K}\bar{\mathcal{C}}\mathcal{C}^{-1}$$

Eiginvalues assintement for MIMO system

Theorem (Controllability and Feedback — MIMO). The pair $(\mathbf{A} - \mathbf{BK}, \mathbf{B})$, for any $\mathbf{p} \times \mathbf{n}$ real matrix $\mathbf{K}$ is controllable if and only if the pair $(\mathbf{A}, \mathbf{B})$ is controllable.

Theorem (Eigenvalue assignment — MIMO). All eigenvalues of $(\mathbf{A}-\mathbf{BK})$ can be assigned arbitrarily (provided complex eigenvalues are assigned in conjugated pairs) by selecting a real constant $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

Eiginvalues assintement for MIMO system

A MIMO system in state space is described with the same formalism we have been using for SISO systems, i.e.,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad && \mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^p \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) && \mathbf{y} \in \mathbb{R}^q \end{aligned}$$

When the system has $\mathbf{p}$ inputs, the state feedback gain $\mathbf{K}$ in a feedback law

$$\mathbf{u} = -\mathbf{K}\mathbf{x} = - \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ k_{p1} & k_{p2} & \cdots & k_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will have $\mathbf{p} \times \mathbf{n}$ parameters. That is, $\mathbf{K} \in \mathbb{R}^{p \times n}$.

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Eiginvalues assintement for MIMO system

Example (Nonuniqueness of K in MIMO state feedback). As a simple MIMO system consider the second order system with two inputs

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

The system has two eigenvalues at $s = 0$, and it is controllable, since $\mathbf{B} = \mathbf{I}$, so $\mathcal{C} = [\,\mathbf{B}\ \ A\mathbf{B}\,]$ is full rank.

Let's consider the state feedback

$$\mathbf{u}(t) = -\mathbf{K}x(t) = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} x(t)$$

Then the closed loop evolution matrix is

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} -k_{11} & -k_{12} \\ 1-k_{21} & -k_{22} \end{bmatrix}$$

Eiginvalues assintement for MIMO system

Example (Continuation). Suppose that we would like to place both closed-loop eigenvalues at $s = -1$, i.e., the roots of the characteristic polynomial $s^2 + 2s + 1$. Then, one possibility would be to select

$$\begin{cases} k_{11} = 2 \\ k_{12} = 1 \\ k_{21} = 0 \\ k_{22} = 0 \end{cases} \Rightarrow \mathbf{A} - \mathbf{BK} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \text{eigenvalues at } s = -1$$

But the alternative selection

$$\begin{cases} k_{11} = \ 1 \\ k_{12} = \text{free} \\ k_{21} = \ 1 \\ k_{22} = -1 \end{cases} \Rightarrow \mathbf{A} - \mathbf{BK} = \begin{bmatrix} -1 & k_{12} \\ 0 & -1 \end{bmatrix} \Rightarrow \text{also eigenvalues at } s = -1$$

As we see, **there are infinitely many** possible selections of $\mathbf{K}$ that will give the same eigenvalues of $(\mathbf{A} - \mathbf{BK})$! □

Eiginvalues assintement for MIMO system

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Eiginvalues assintement for MIMO system

The "excess of freedom" in MIMO state feedback design could be a problem if we don't know how to best use it...

There are several ways to tackle the problem of selecting $\mathbf{K}$ from an infinite number of possibilities, among them

- Cyclic Design. Reduces the problem to one of a **single input**, so we can apply the known rules.
- Controller Canonical Form Design. Extends the Bass-Gura formula to MIMO.
- Optimal Design. Computes the **best** $\mathbf{K}$ by optimising a suitable cost function.

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Let us focus on Optimal design

Optimal design for state feedback control

Theorem (LQR). Consider the state space system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^p$$
$$y = Cx, \quad y \in \mathbb{R}^q$$

and the performance criterion

$$J = \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt, \rightarrow \min \qquad \text{(J)}$$

where Q is non negative definite and R is positive definite. Then the optimal control minimising (J) is given by the **linear** state feedback law

$$\boxed{u(t) = -Kx(t)} \quad \text{with} \quad \boxed{K = R^{-1}B^TP}$$

and where P is the unique positive definite solution to the matrix **Algebraic Riccati Equation** (ARE)

$$\boxed{A^TP + PA - PBR^{-1}B^TP + Q = 0}$$

Eiginvalues assigntement

Theorem (Controllability and Feedback — MIMO). The pair $(A - BK, B)$, for any $p \times n$ real matrix K is controllable if and only if the pair (A, B) is controllable.

Theorem (Eigenvalue assignment — MIMO). All eigenvalues of $(A - BK)$ can be assigned arbitrarily (provided complex eigenvalues are assigned in conjugated pairs) by selecting a real constant K if and only if (A, B) is controllable.

What happens if system is not controllable?

We have seen that if a state equation is controllable, then we can assign its eigenvalues arbitrarily by state feedback. But, what happens when the state equation is **not** controllable?

We know that we can take any state equation to the Controllable/Uncontrollable Canonical Form

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{\mathcal{C}} \\ \dot{\bar{\mathbf{x}}}_{\bar{\mathcal{C}}} \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{\mathbf{A}}_{\mathcal{C}} & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{\bar{\mathcal{C}}} \end{bmatrix}}_{\bar{\mathbf{A}}} \begin{bmatrix} \bar{\mathbf{x}}_{\mathcal{C}} \\ \bar{\mathbf{x}}_{\bar{\mathcal{C}}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{\mathcal{C}} \\ 0 \end{bmatrix} u$$

Because the evolution matrix $\bar{\mathbf{A}}$ is **block-triangular**, its eigenvalues are the union of the eigenvalues of the diagonal blocks: $\bar{\mathbf{A}}_{\mathcal{C}}$ and $\bar{\mathbf{A}}_{\bar{\mathcal{C}}}$.

What happens if system is not controllable?

The state feedback law

$$
\begin{aligned}
\mathbf{u} &= \mathbf{r} - \mathbf{K}\mathbf{x} \\
&= \mathbf{r} - \bar{\mathbf{K}}\bar{\mathbf{x}} \\
&= \mathbf{r} - \begin{bmatrix} \bar{\mathbf{K}}_{\mathcal{C}} & \bar{\mathbf{K}}_{\tilde{\mathcal{C}}} \end{bmatrix} \begin{bmatrix} \dot{\bar{\mathbf{x}}}_{\mathcal{C}} \\ \dot{\bar{\mathbf{x}}}_{\tilde{\mathcal{C}}} \end{bmatrix}
\end{aligned}
$$

yields the closed-loop system

$$
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{\mathcal{C}} \\ \dot{\bar{\mathbf{x}}}_{\tilde{\mathcal{C}}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{\mathcal{C}} - \bar{\mathbf{B}}_{\mathcal{C}}\bar{\mathbf{K}}_{\mathcal{C}} & \bar{\mathbf{A}}_{12} - \bar{\mathbf{B}}_{\mathcal{C}}\bar{\mathbf{K}}_{\tilde{\mathcal{C}}} \\ 0 & \bar{\mathbf{A}}_{\tilde{\mathcal{C}}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{\mathcal{C}} \\ \bar{\mathbf{x}}_{\tilde{\mathcal{C}}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{\mathcal{C}} \\ 0 \end{bmatrix} \mathbf{r}.
$$

We see that the eigenvalues of $\bar{\mathbf{A}}_{\tilde{\mathcal{C}}}$ are **not** affected by the state feedback, so they remain **unchanged**.

The value of $\bar{\mathbf{K}}_{\tilde{\mathcal{C}}}$ is **irrelevant** — the uncontrollable states cannot be affected.

What happens if system is not controllable?

We conclude that the condition of **Controllability** is not only sufficient, but also necessary to place **all** eigenvalues of $\mathbf{A} - \mathbf{BK}$ in desired locations.

A notion of interest in control that is weaker than that of **Controllability** is that of Stabilisability.

Stabilisability. The system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

is said to be **stabilisable** if all its **uncontrollable states are asymptotically stable**.

This condition is equivalent to asking that the matrix $\bar{\mathbf{A}}_{\tilde{c}}$ be **Hurwitz**.

Stabilisation VS tracking a constant reference

$$\dot{x} = \underbrace{Ax}_{\text{state}} + \underbrace{Bu}_{\text{control}} + \cancel{\underbrace{w}_{\text{disturbance}}}$$

$$\underbrace{y}_{\text{output}} = Cx$$

- Regulator: ensure that $\lim_{t \to +\infty} (y_r(t) - y(t)) = 0$, where $r(t) = \mathit{const}$ $\neq 0$

Stabilisation VS tracking a constant reference

Linear state space control theory involves modifying the behaviour of an m-input, p-output, n-state system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned} \qquad \text{(OL)}$$

we apply the control law $\mathbf{u}(t) = \mathbf{N}\mathbf{r}(t) - \mathbf{K}\mathbf{x}(t)$ and obtain the **closed-loop** state equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{N}\mathbf{r}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

If the system is **controllable**, by appropriately designing $\mathbf{K}$, we are able to place the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K})$ at any desired locations.

Let’s first consider a case of SISO systems

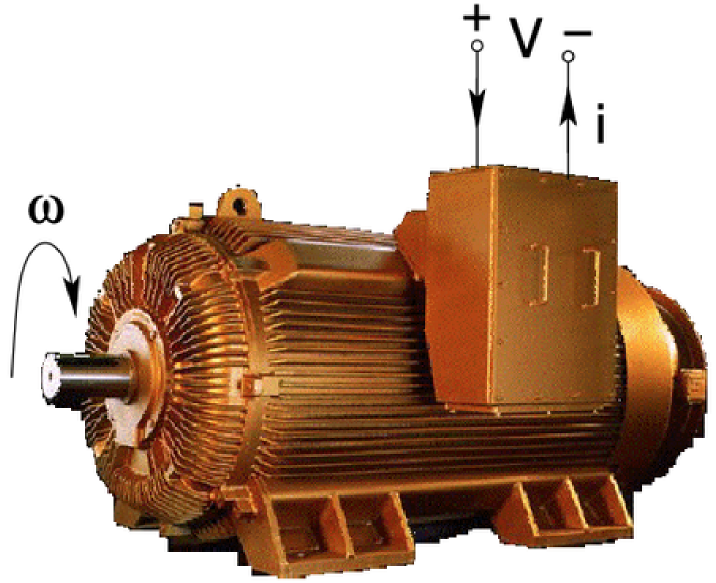
Tracking a constant reference

We review the state feedback design procedure with an example.

Example (Speed control of a DC motor). We consider a DC motor described by the state equations

$$\frac{d}{dt}\begin{bmatrix}\omega(t)\\ i(t)\end{bmatrix} = \begin{bmatrix}-10 & 1\\ -0.02 & -2\end{bmatrix}\begin{bmatrix}\omega(t)\\ i(t)\end{bmatrix} + \begin{bmatrix}0\\ 2\end{bmatrix}V(t)$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\omega(t)\\ i(t)\end{bmatrix}$$



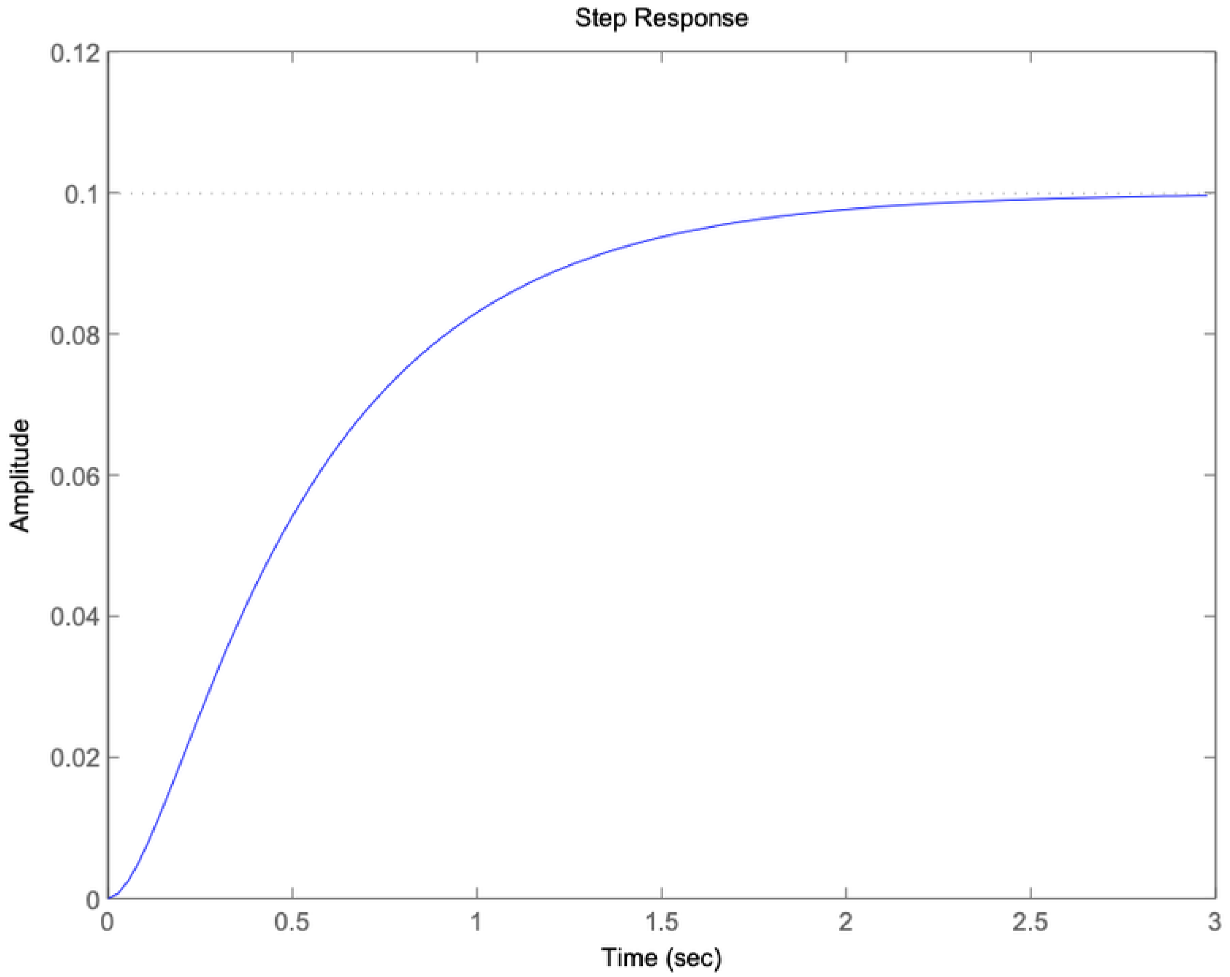
The input to the DC motor is the voltage $V(t)$, and the states are the current $i(t)$ and rotor velocity $\omega(t)$. We assume that we can measure both, and take $\omega(t)$ as the output of interest.

Tracking a constant reference

Example (continuation). ① The open-loop characteristic polynomial is

$$\Delta(s) = \det(sI - A) = \det \begin{bmatrix} s+10 & -1 \\ 0.02 & s+2 \end{bmatrix} = s^2 + 12s + 20.02$$

which has two stable roots at $s = -9.9975$ and $s = -2.0025$. The motor **open-loop step response** is



The system takes about 3s to reach steady-state. The final speed is about $1/10$ the amplitude of the voltage step.

We would like to design a state feedback control to make the motor response faster and obtain tracking of $\omega(t)$ to constant reference inputs r.

Tracking a constant reference

Example (continuation). To design the state feedback gain, we next ② compute the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix}$$

which is full rank $\Rightarrow$ the system is **controllable**.

Also, from the open-loop characteristic polynomial we form **controllability matrix in $\bar{x}$ coordinates** is

$$\bar{\mathcal{C}} = \begin{bmatrix} 1 & \alpha_1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}$$

Tracking a constant reference

Example (continuation). We now ③ propose a desired characteristic polynomial. Suppose that we we would like the closed-loop eigenvalues to be at $s = -5 \pm j$, which yield a step response with 0.1% overshoot and about 1s settling time.

The desired (closed-loop) characteristic polynomial is then

$$\Delta_K(s) = (s + 5 - j)(s + 5 + j) = s^2 + 10s + 26$$

With $\Delta_K(s)$ and $\Delta(s)$ we determine the state feedback gain in $\bar{x}$ coordinates

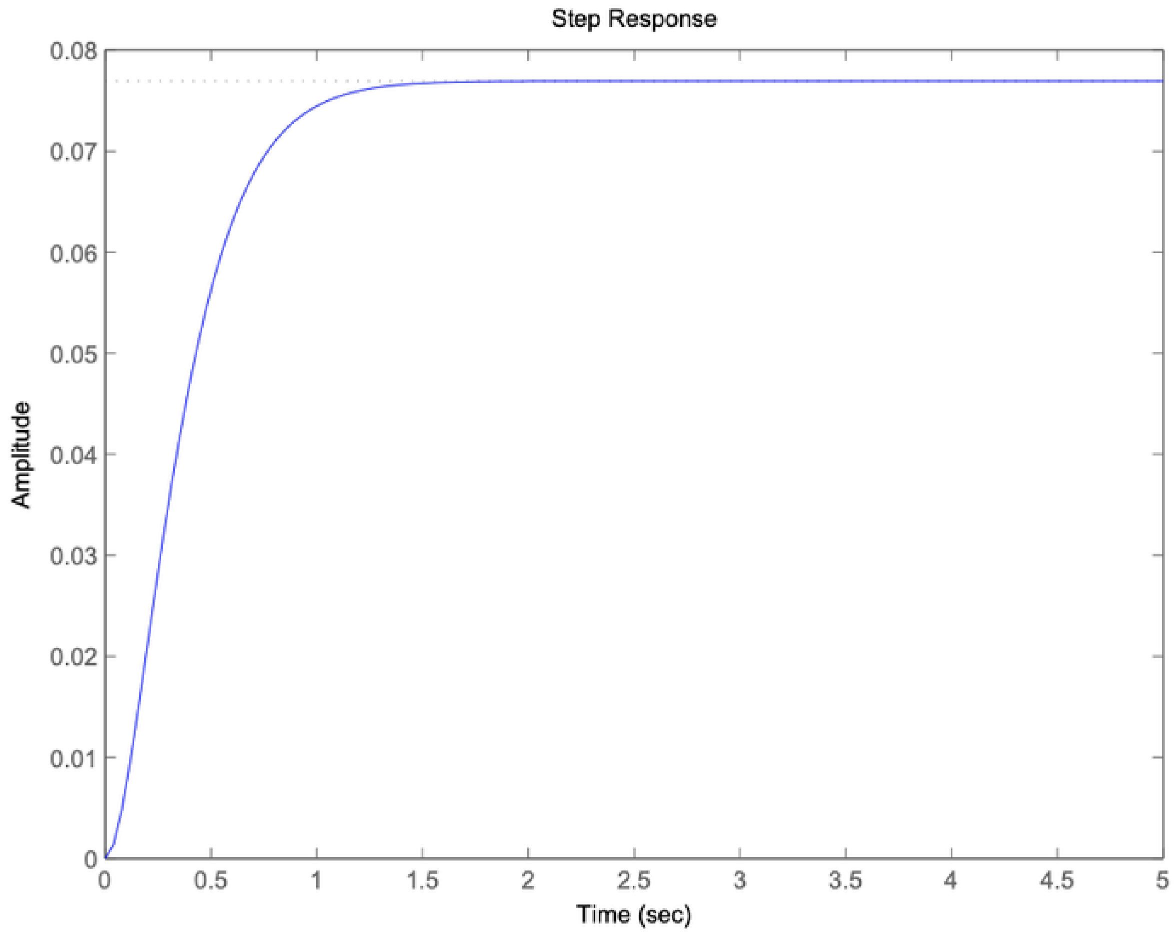
$$\bar{K} = \begin{bmatrix} (\bar{\alpha}_1 - \alpha_1) & (\bar{\alpha}_2 - \alpha_2) \end{bmatrix} = \begin{bmatrix} (10 - 12) & (26 - 20.02) \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 5.98 \end{bmatrix}$$

Tracking a constant reference

Example (continuation). Finally, ④ we obtain the state feedback gain $\mathbf{K}$ in the original coordinates using **Bass-Gura** formula,

$$\mathbf{K} = \bar{\mathbf{K}}\bar{\mathcal{C}}\mathcal{C}^{-1} = \begin{bmatrix} -2 & 5.98 \end{bmatrix} \begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 12.99 & -1 \end{bmatrix}$$

As can be verified, the eigenvalues of $(\mathbf{A} - \mathbf{BK})$ are as desired.

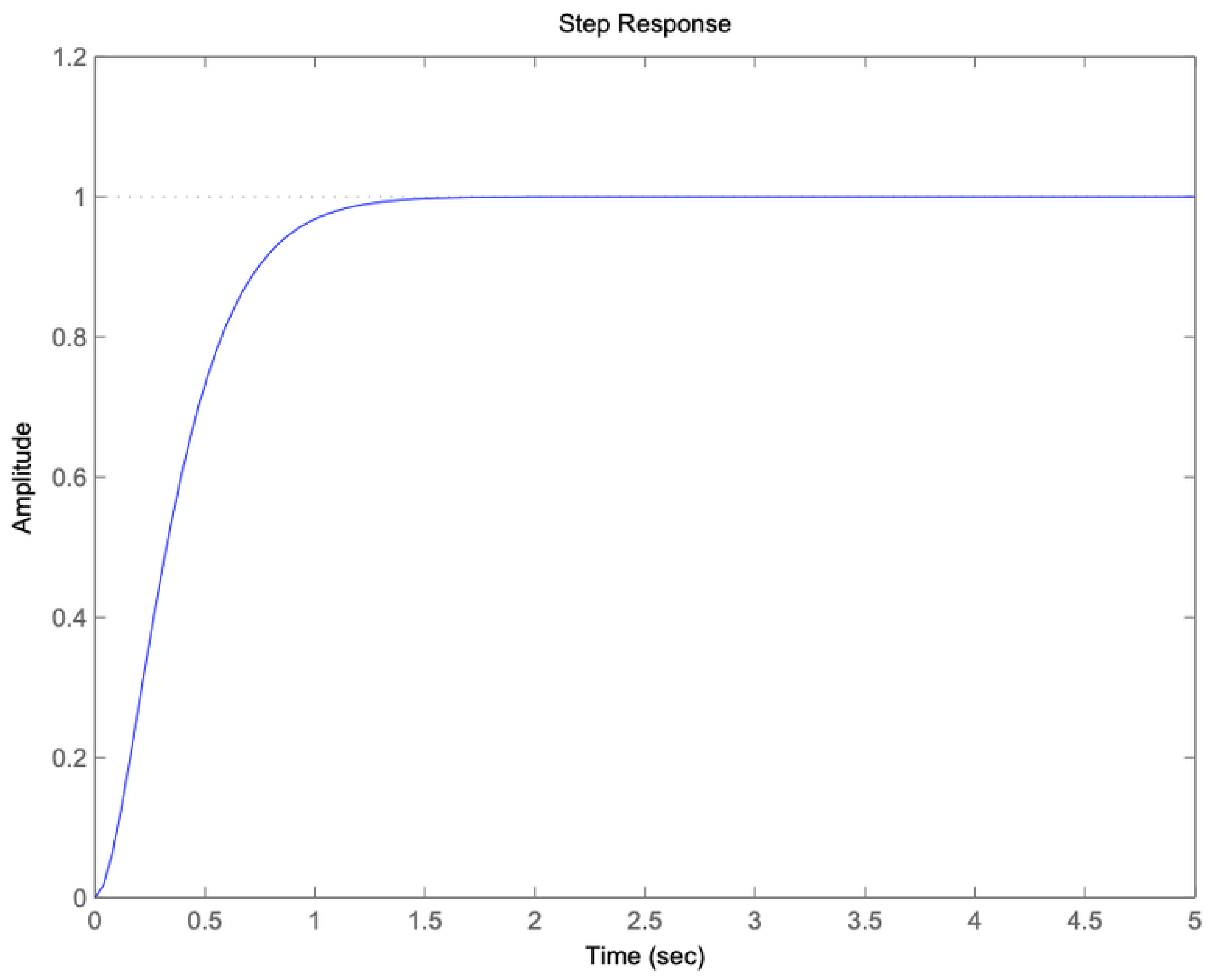


The **closed-loop step response**, as desired, settles in 1s, with no significant overshoot.

Note, however, that we still have steady-state error ($\omega(t) \to 0.0769$). To fix it, we use the feed-forward gain $\mathbf{N}$.

Tracking a constant reference

Example (continuation). The system transfer function does not have a **zero at** $s = 0$, which would prevent tracking of constant references (as we can see in the step response, which otherwise, would asymptotically go to 0).



Thus, ⑤ we determine N with the formula

$$N = -\frac{1}{G_{cl}(0)} = \frac{-1}{C(A - BK)^{-1}B} = 13$$

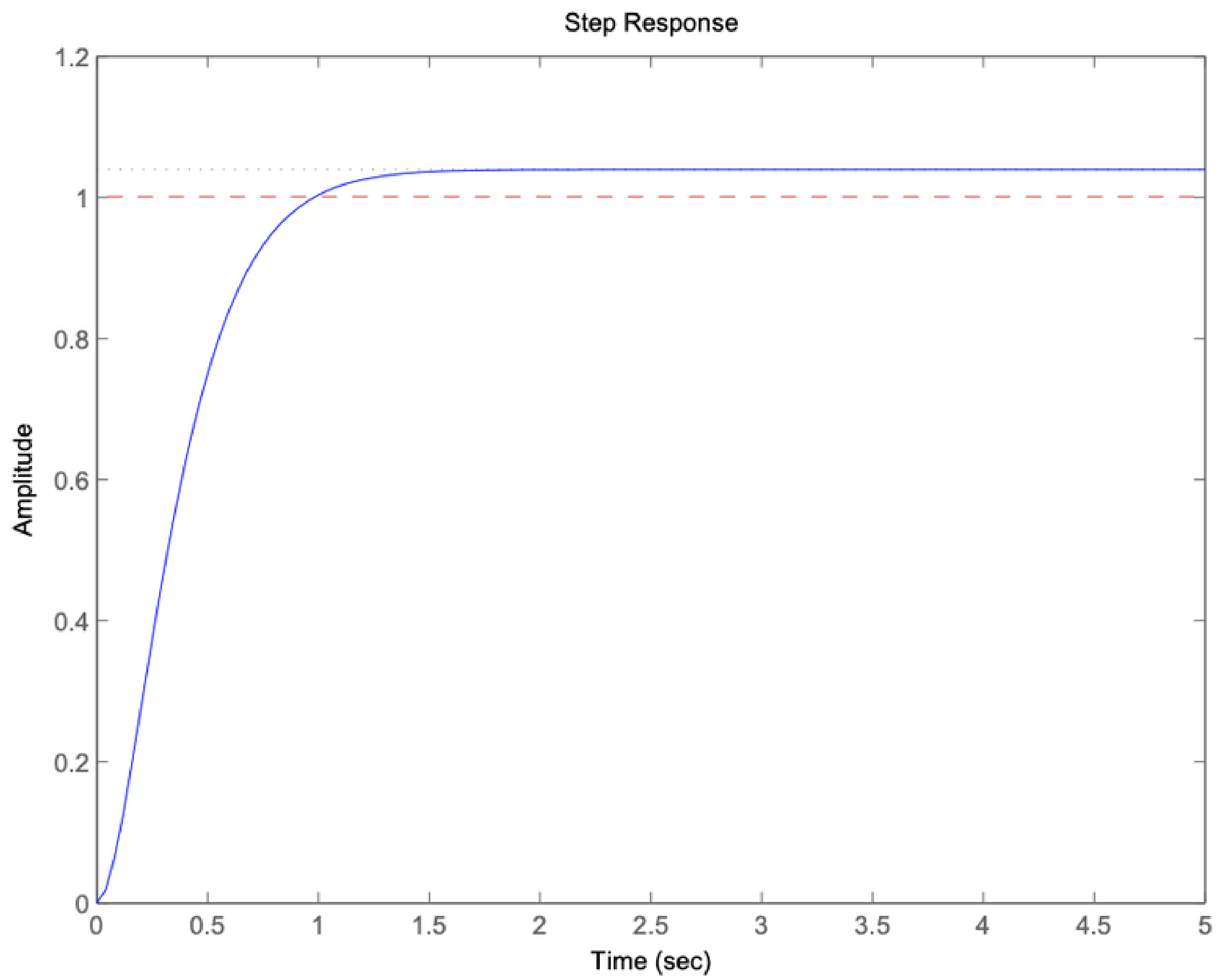
and achieve zero steady-state error in the closed-loop step response.

Tracking a constant reference

Example (continuation). We have designed a state feedback controller for the speed of the DC motor. However, the tracking achieved by **feedforward precompensation** would not tolerate (it is not **robust** to) to uncertainties in the plant model.

To see this, suppose the **real** matrix A in the system is slightly different from the one we used to compute $\mathbf{K}$,

$$\tilde{\mathbf{A}} = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$



The closed-loop step response given by the designed gains $\mathbf{N}, \mathbf{K}$ (based on a **different** A-matrix) doesn't yield tracking. □

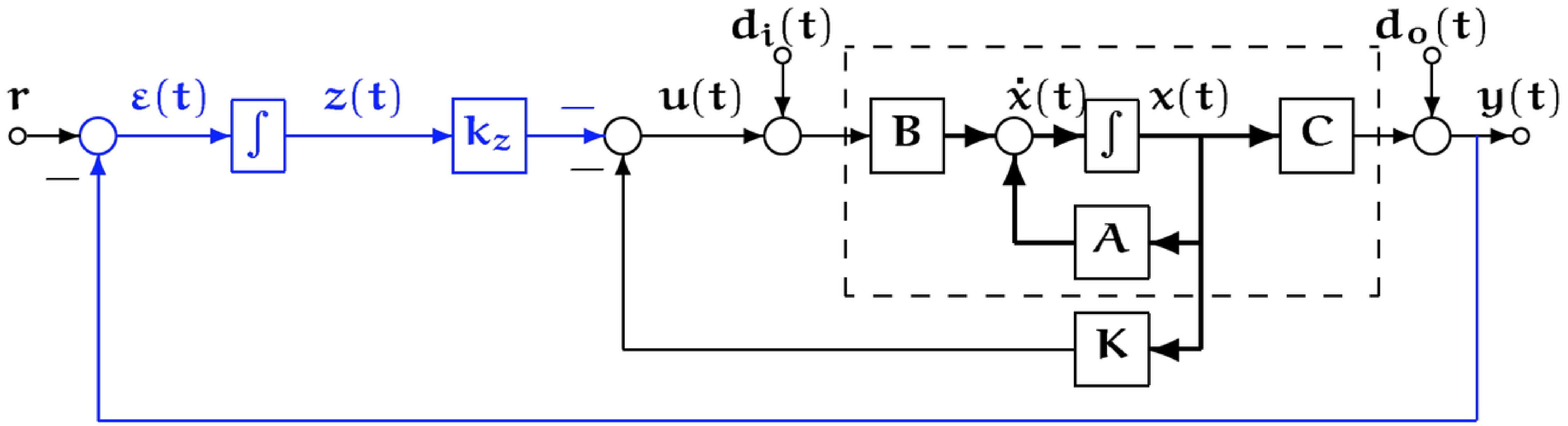
Robust tracking: integral action

We now introduce a **robust** approach to achieve constant reference tracking by state feedback. This approach consists in the **addition of integral action** to the state feedback, so that

- the error $\boldsymbol{\varepsilon}(\mathbf{t}) = \mathbf{r} - \mathbf{y}(\mathbf{t})$ will approach $\mathbf{0}$ as $\mathbf{t} \to \infty$, and this property will be preserved
  - under moderate uncertainties in the plant model
  - under constant input or output disturbance signals.

Robust tracking: integral action

The State Feedback with Integral Action scheme:



The main idea in the addition of integral action is to **augment the plant** with an extra state: the integral of the tracking error $\varepsilon(t)$,

$$\dot{z}(t) = r - y(t) = r - Cx(t) \qquad \text{(IA1)}$$

The control law for the **augmented plant** is then

$$u(t) = -\begin{bmatrix} K & k_z \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \qquad \text{(IA2)}$$

Robust tracking: integral action

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{A_a} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} - \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_a} \underbrace{\begin{bmatrix} K & k_z \end{bmatrix}}_{K_a} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$= (A_a - B_a K_a) \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

The state feedback design with integral action can be done as a normal state feedback design for the **augmented plant**

If K_a is designed such that the closed-loop augmented matrix $(A_a - B_a K_a)$ is rendered Hurwitz, then necessarily in steady-state

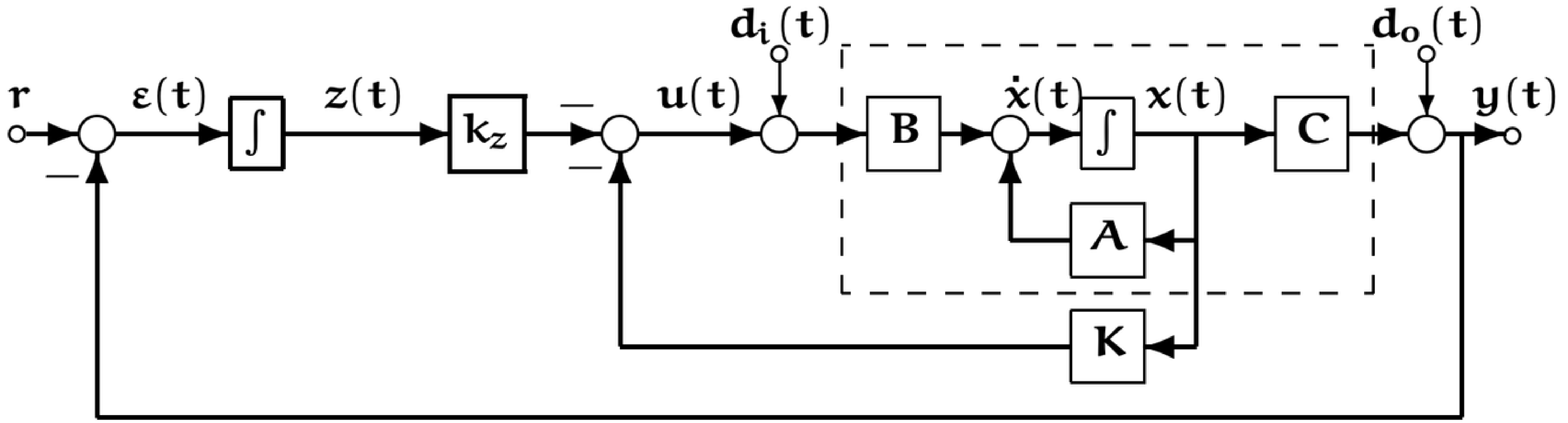
$$\lim_{t \to \infty} \dot{z}(t) = 0 \quad \Rightarrow \quad \lim_{t \to \infty} y(t) = r, \quad \text{achieving tracking.}$$

Robust tracking: integral action for MIMO system

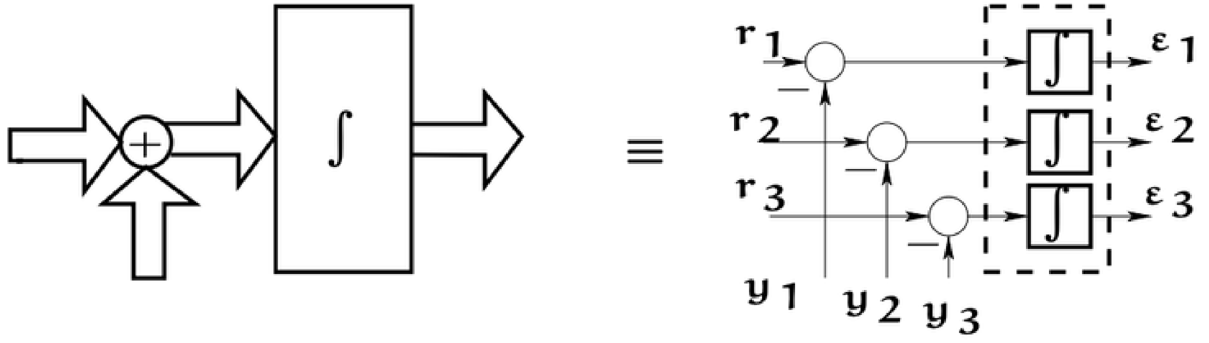
Robust tracking for MIMO system

Tracking with **Integral Action** is subject to the same restrictions: **we can only achieve asymptotic tracking of a maximum of as many outputs as control inputs are available.**

The scheme and computation procedure is the same as in SISO



Note that now the **integral action** is applied to each of the q reference input channels.



Robust tracking for MIMO system

The procedure to compute $\mathbf{K}$ and $\mathbf{k}_z$ for the state feedback control with integral action is exactly as in the SISO case,

$$\dot{z}(t) = \mathbf{r} - \mathbf{y}(t) = \mathbf{r} - \mathbf{C}\mathbf{x}(t)$$

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{K} & \mathbf{k}_z \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z(t) \end{bmatrix}$$

where $\mathbf{K}_a = [\,\mathbf{K}\ \mathbf{k}_z\,]$ is computed to place the eigenvalues of the **augmented plant** $(\mathbf{A}_a, \mathbf{B}_a)$ at desired locations, where

$$\mathbf{A}_a = \begin{bmatrix} \mathbf{A} & 0_{n\times q} \\ -\mathbf{C} & 0_{q\times q} \end{bmatrix}, \quad \mathbf{B}_a = \begin{bmatrix} \mathbf{B} \\ 0_{q\times p} \end{bmatrix}$$

Robust tracking for MIMO system

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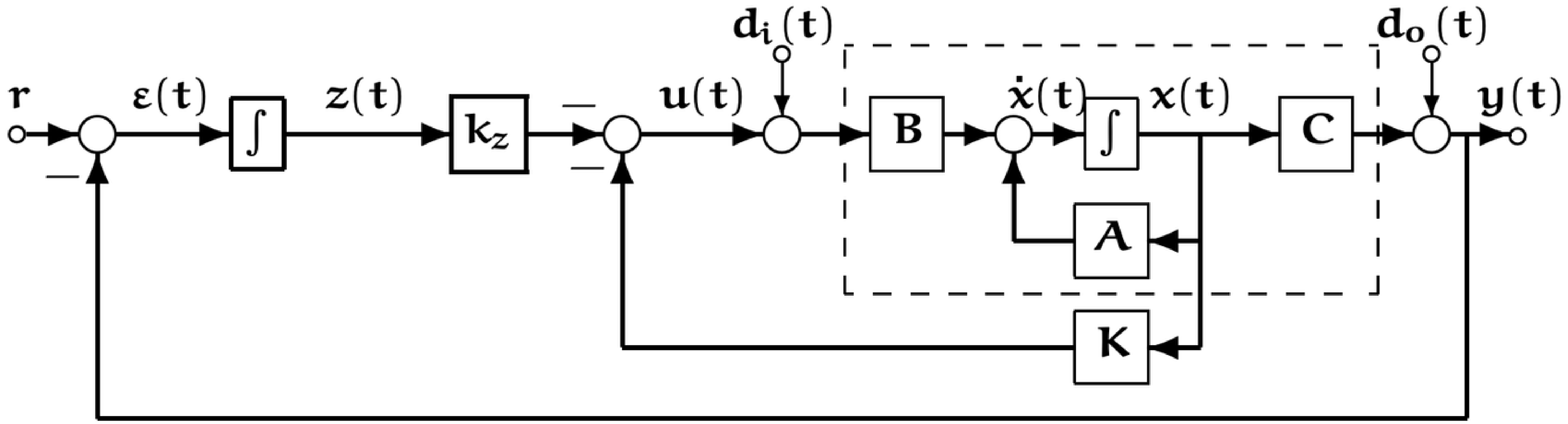

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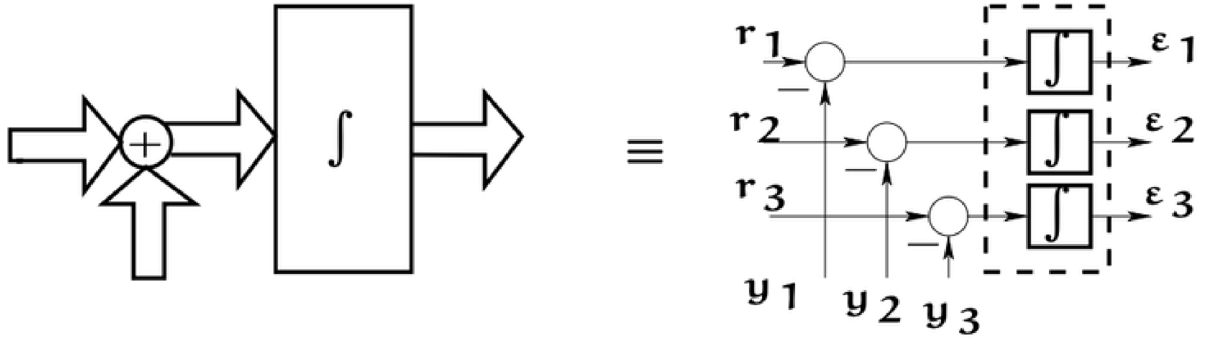

Summary

We have discussed how to design a linear feedback controller, solving the regulator problem. This controller is robust to constant disturbance

$$\underbrace{\dot{x} = Ax}_{\text{state}} + \underbrace{Bu}_{\text{control}} + \underbrace{w}_{\text{disturbance}} \quad = \text{const}$$

$$\underbrace{y}_{\text{output}} = Cx$$

- Regulator: ensure that $\lim_{t \to +\infty} (y_r(t) - y(t)) = 0$, where $r(t) = const$