Controllability and Observability

Elena VANNEAUX

elena.vanneaux@ensta-paris.fr

Course grade breakdowns

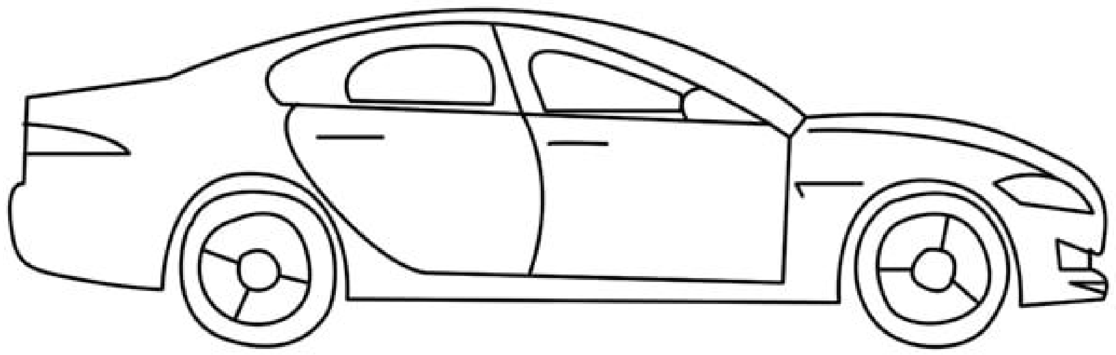
Labs - 50%

Final project - 50 %

What is a control theory?

input

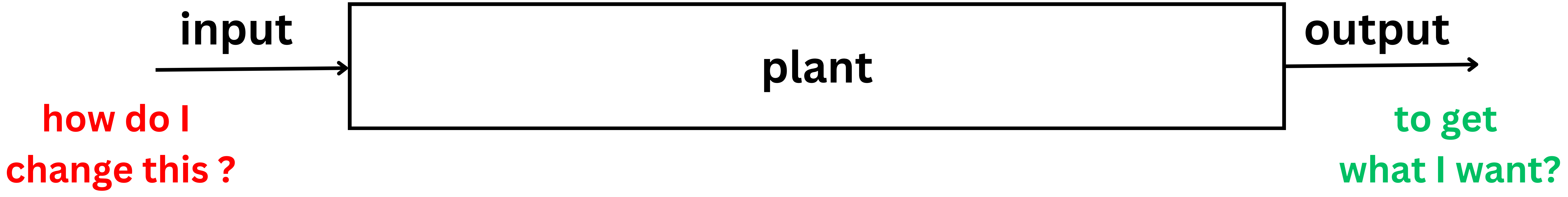
how do I
change this ?



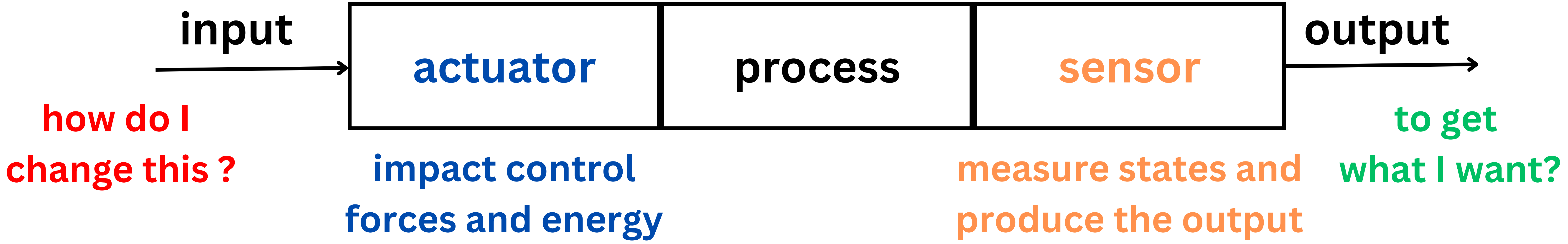
output

to get
what I want?

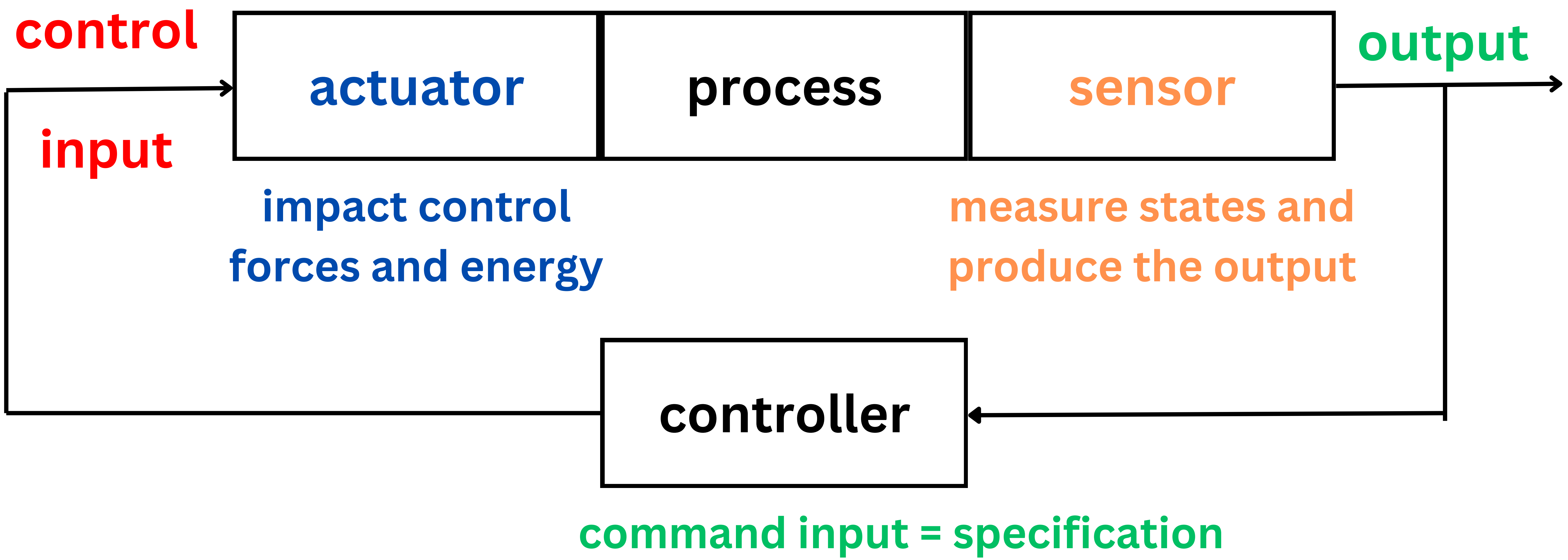
What is a control theory?



What is a control theory?

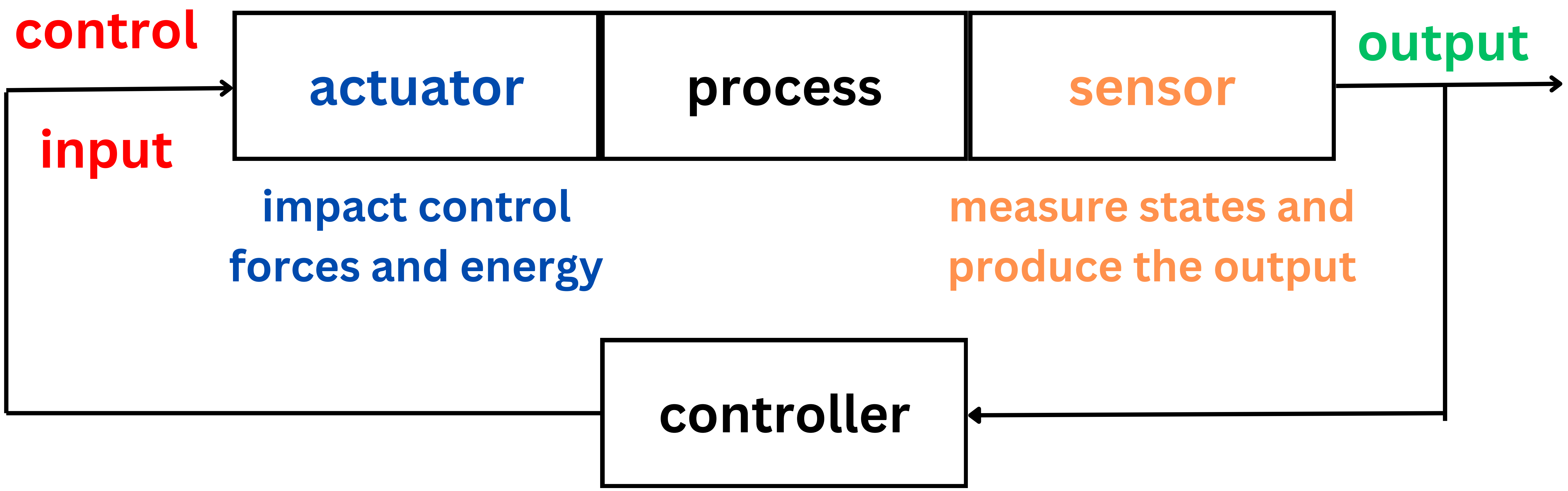


What is a control theory?



Controller design = how to use the sensor data (output) to generate the correct actuator commands (control input) to ensure that the output of the system satisfies the specification

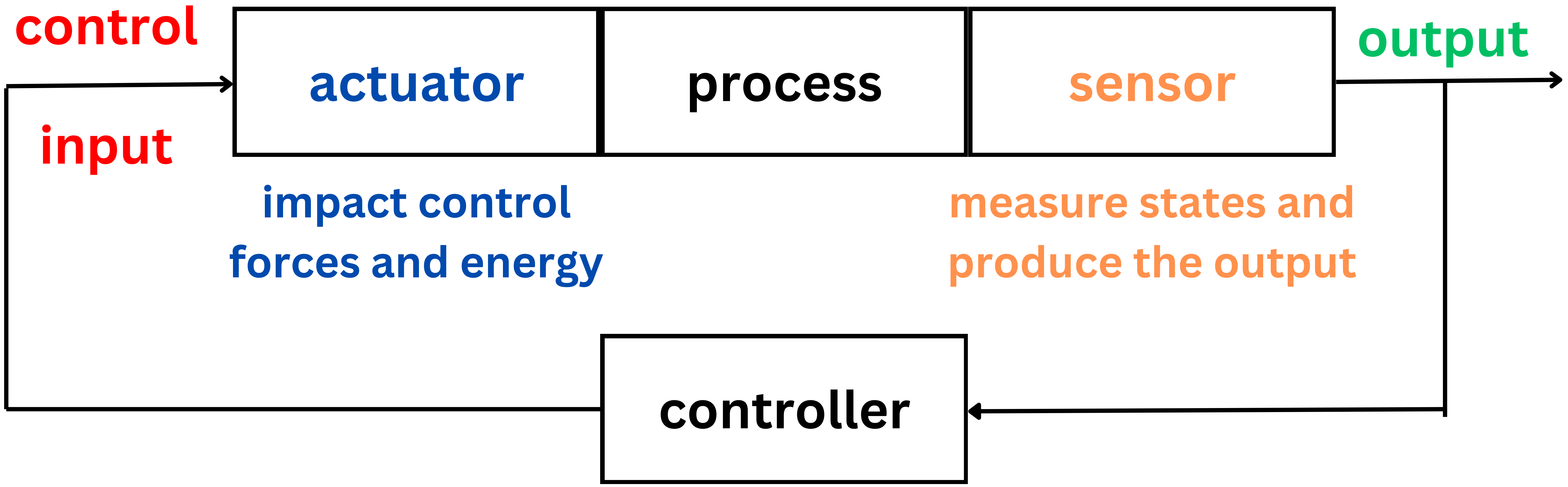
Controllability & Observability



command input = specification

To design controller

Controllability & Observability

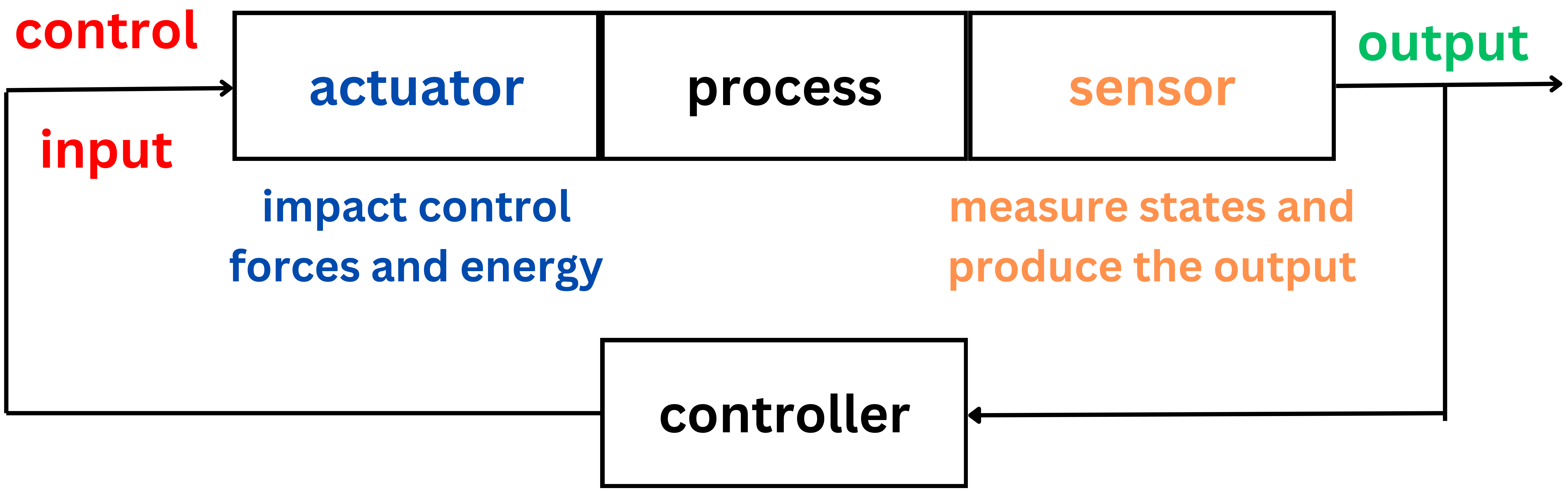


command input = specification

To design controller

you need to be able to
influence the system

Controllability & Observability



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Controllable

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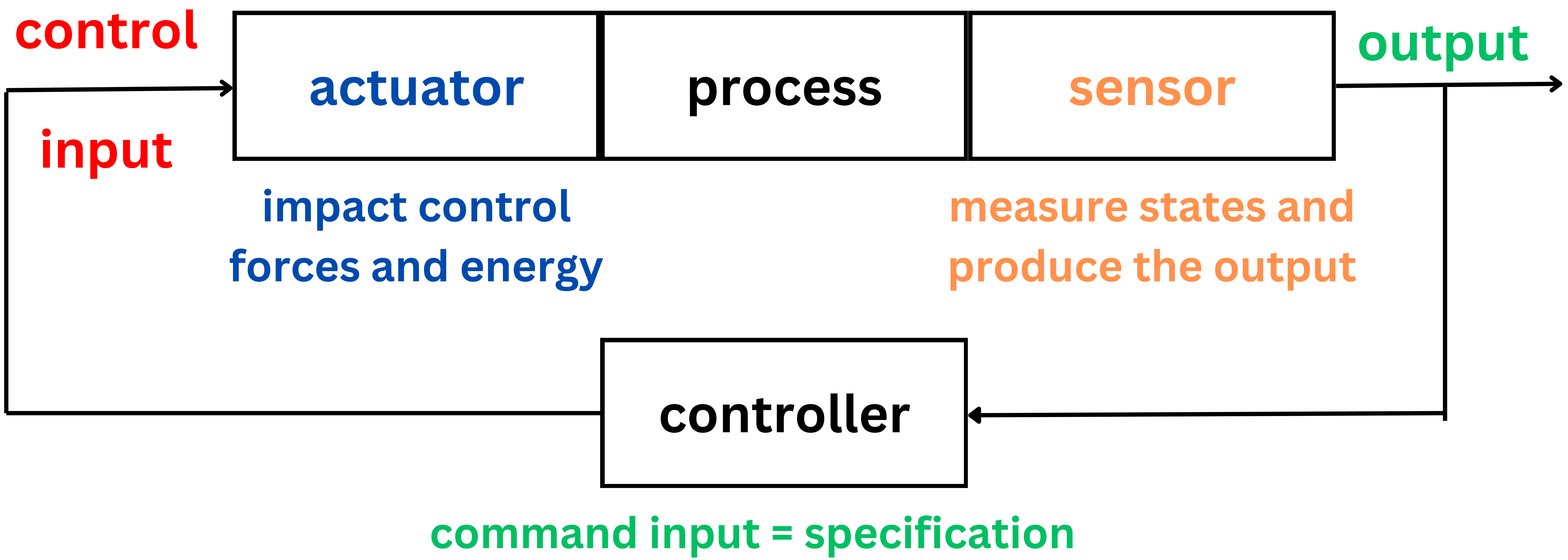
Controllability & Observability



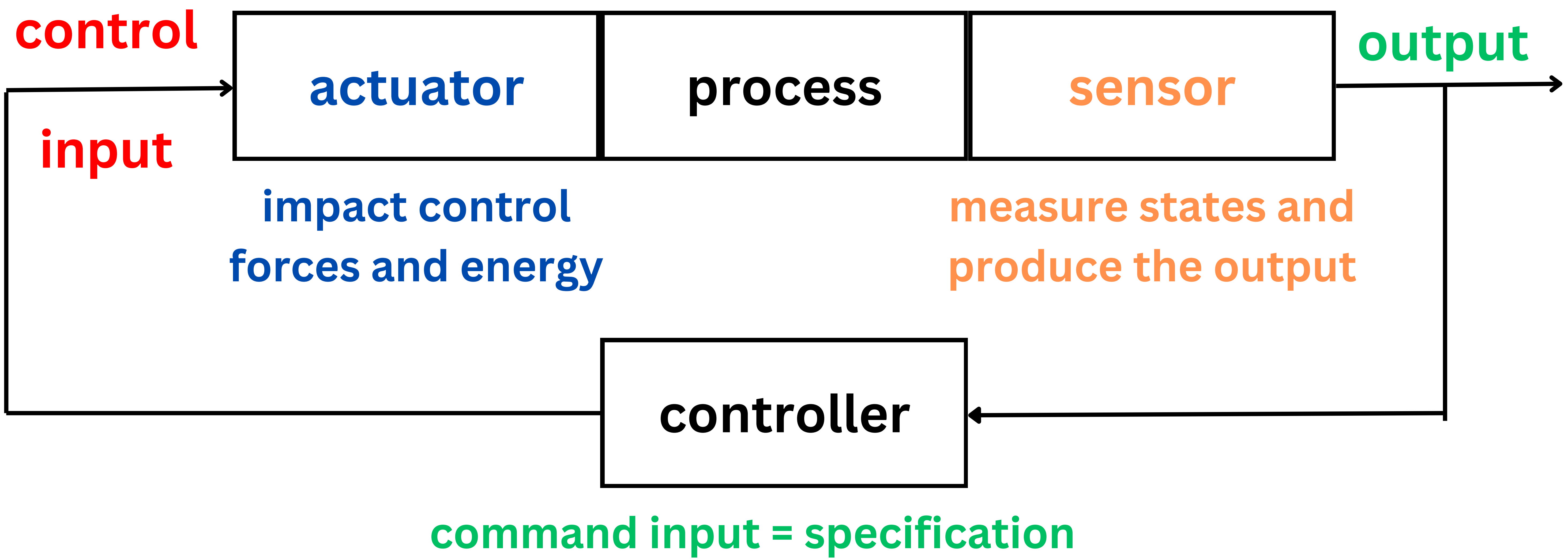
Controllable

To design controller

you need to be able to
influence the system

and know it’s changing

Controllability & Observability



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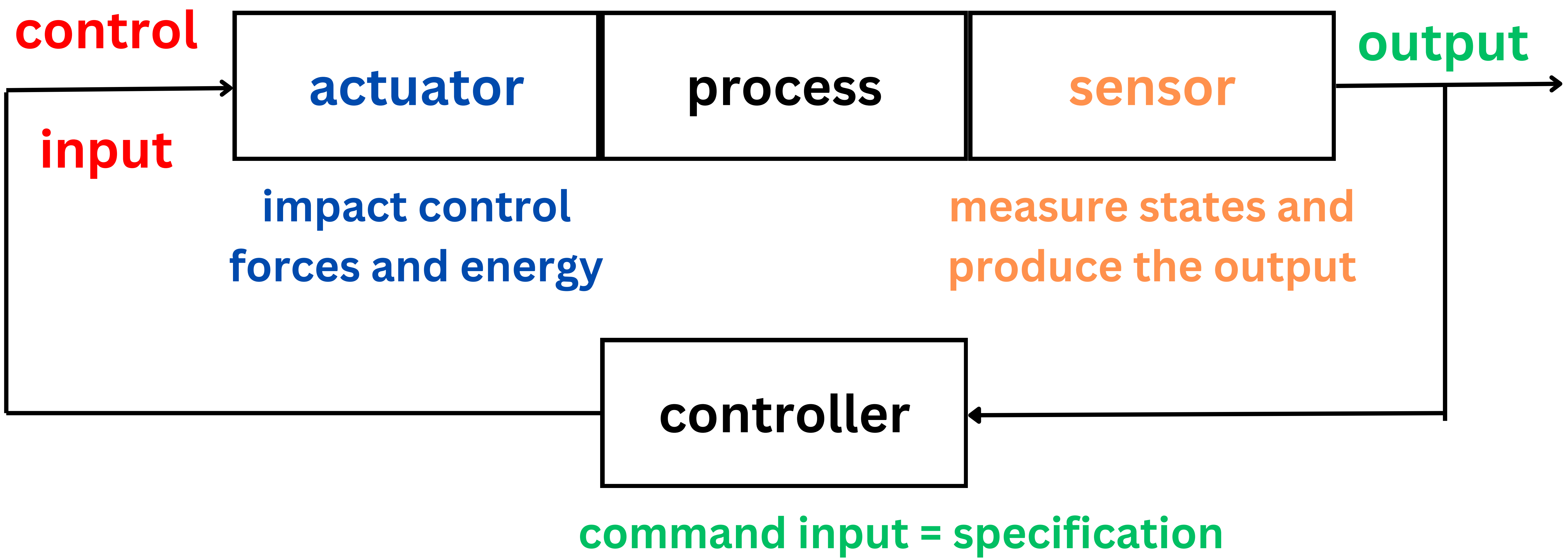
Controllability & Observability



Controllability and observability are conditions of how the system works with the actuators and sensors, and it's not tied to a specific control technique

Controllability

Controllability (null reachability) means that there exists control signal which allows the system to move from any any initial state to any final state in a finite time interval

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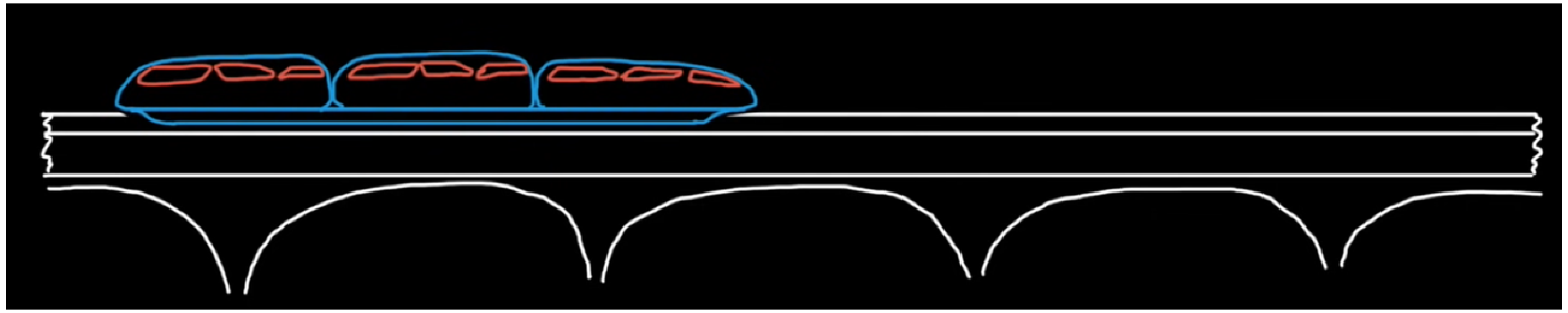
Monorail

$$\dot{v} = u$$

$$\dot{p} = v$$



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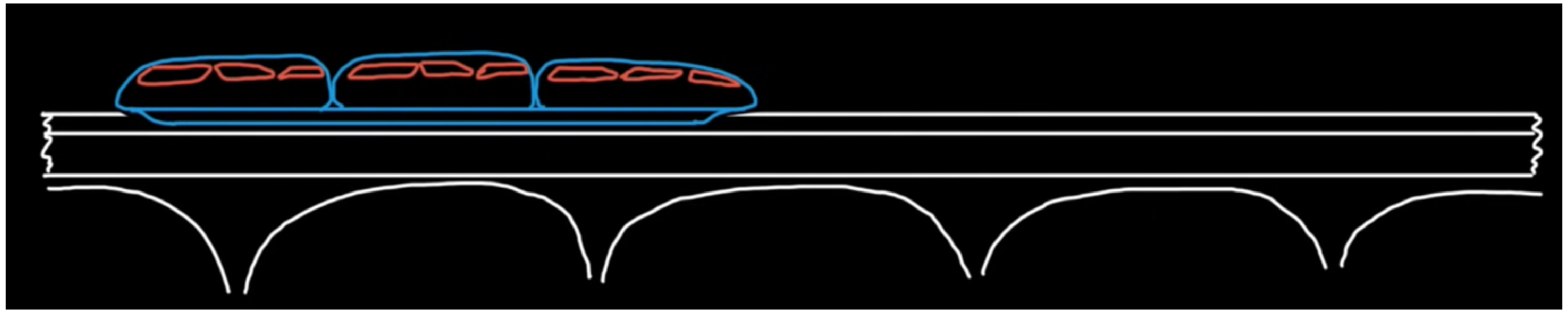

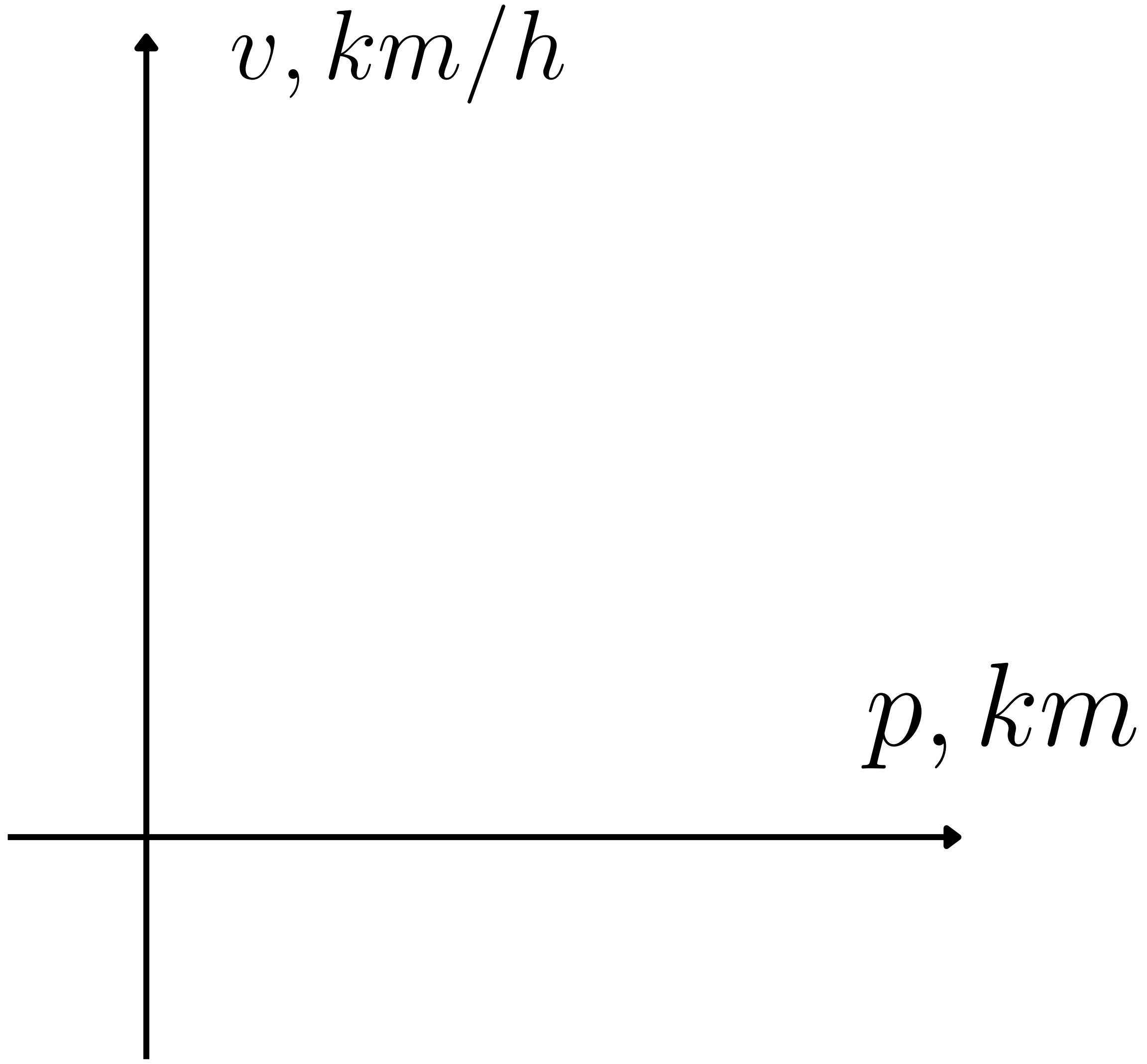

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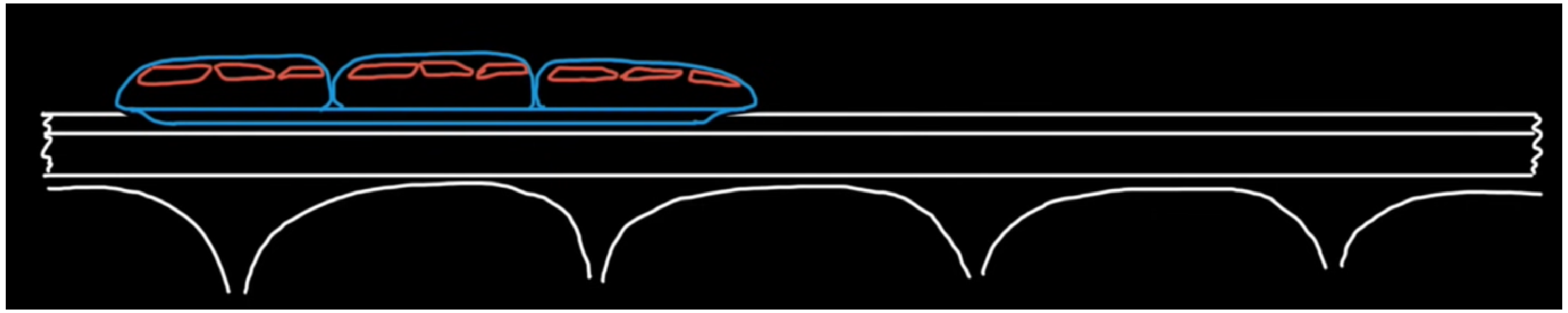

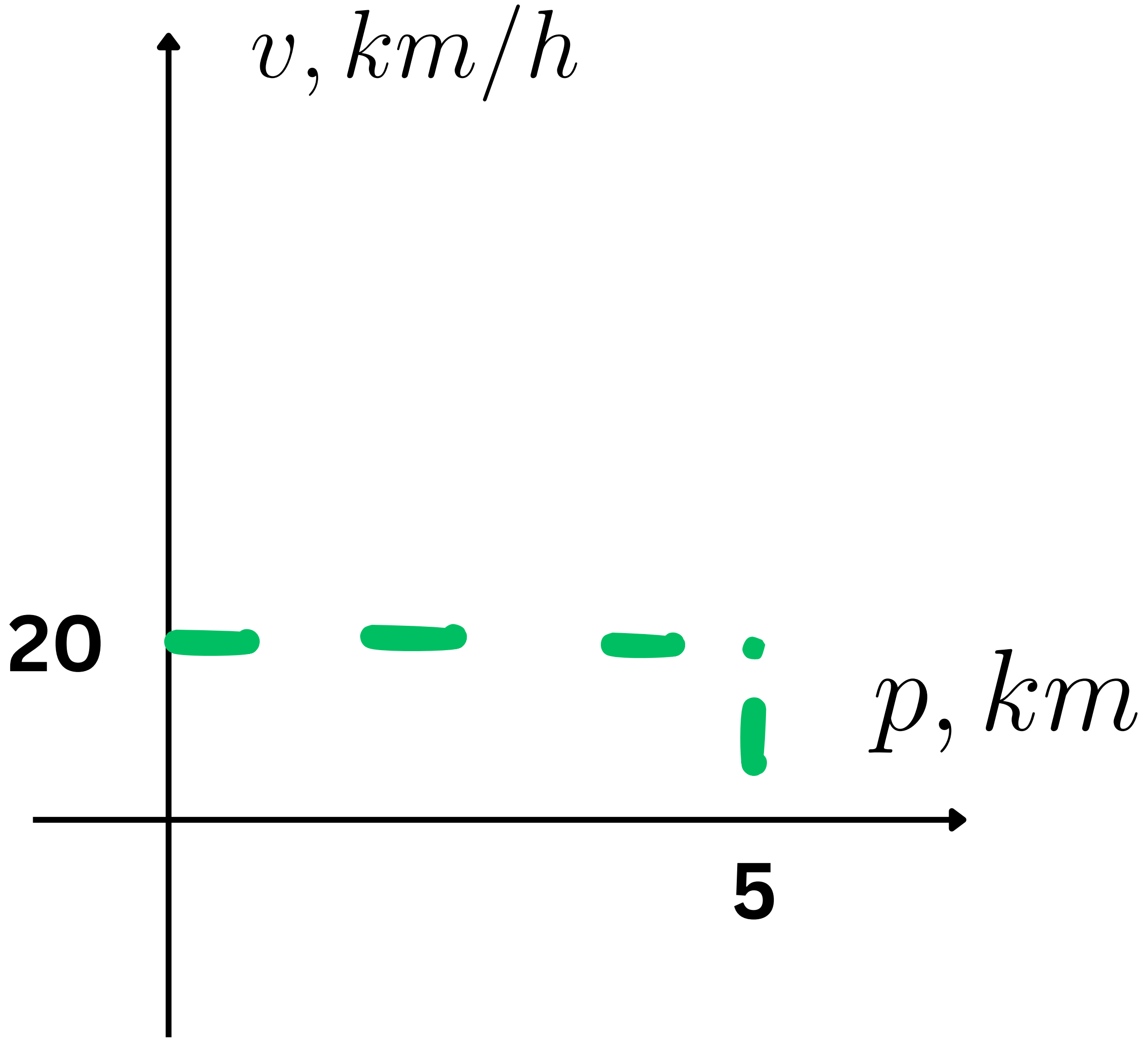

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Controllability (null reachability) means that there exists control signal which allows the system to move from any any initial state to any final state in a finite time interval

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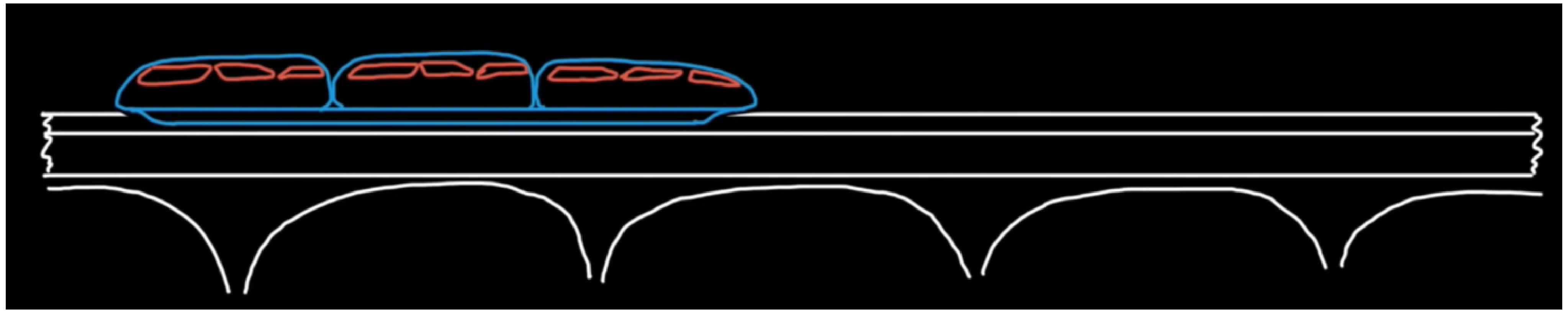
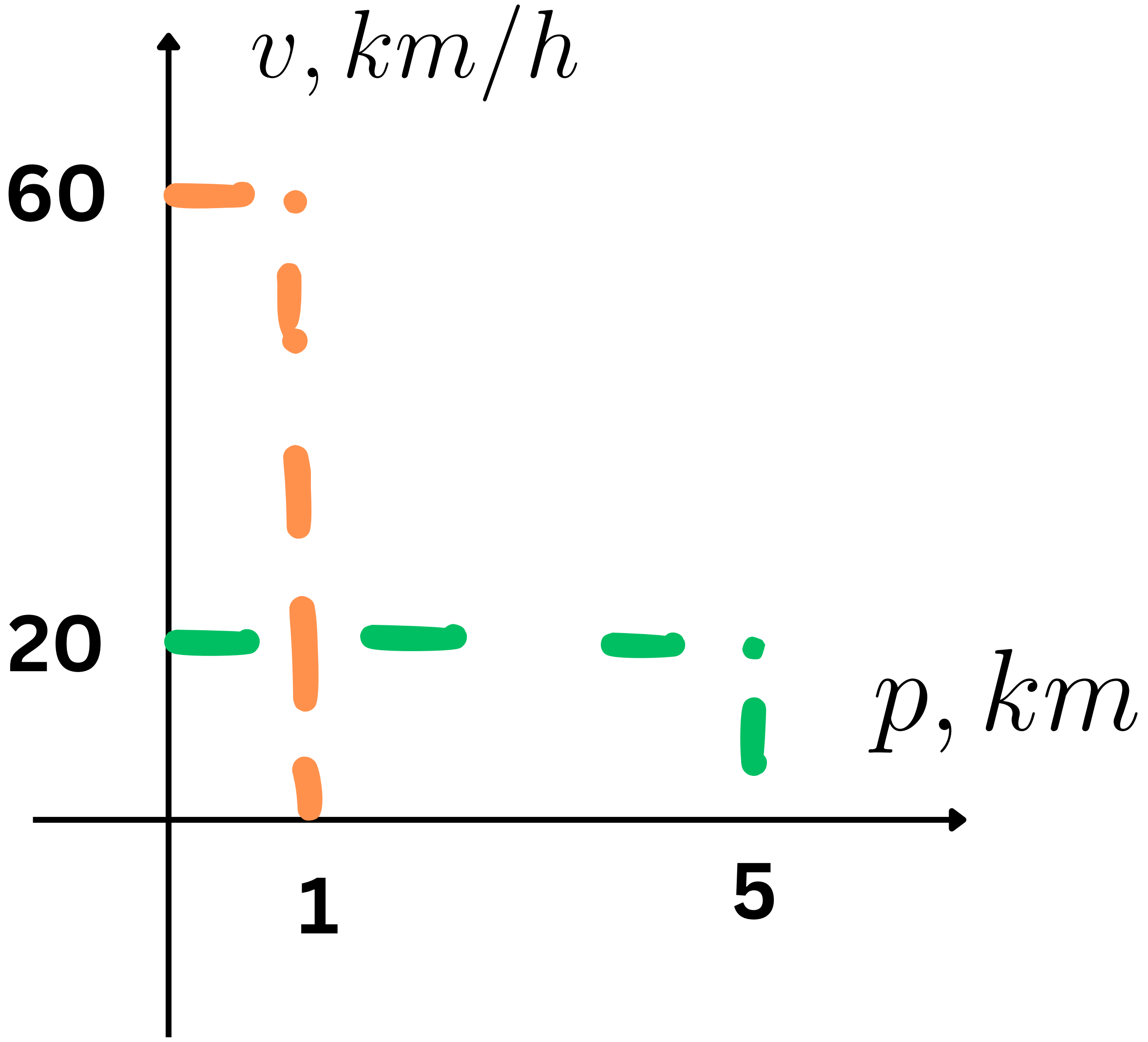
controllability does not mean that the state must be maintained, only that it can be reached...

even if infinite amount of energy is required for that....





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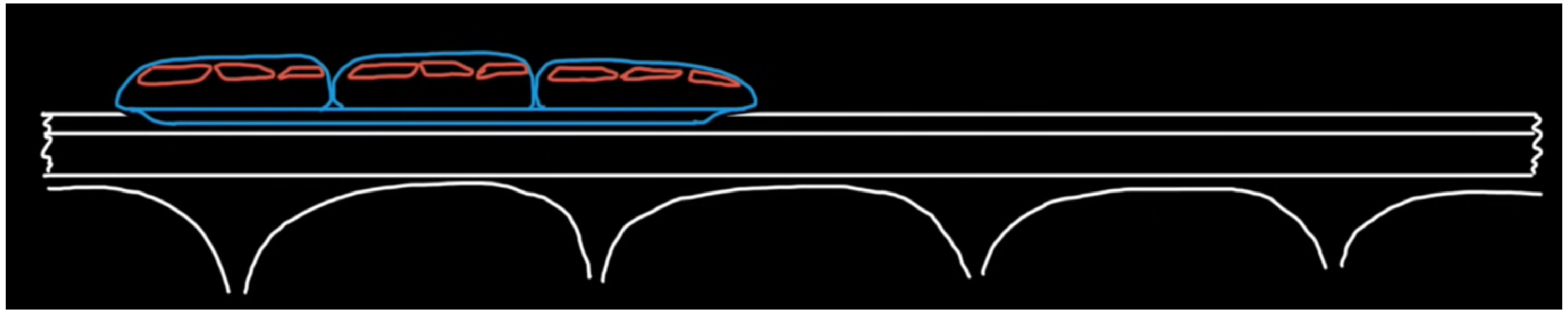


Example of uncontrollable system

imagine we lost control of gaz pedal

$$\dot{v} = 0\,u$$

$$\dot{p} = v$$

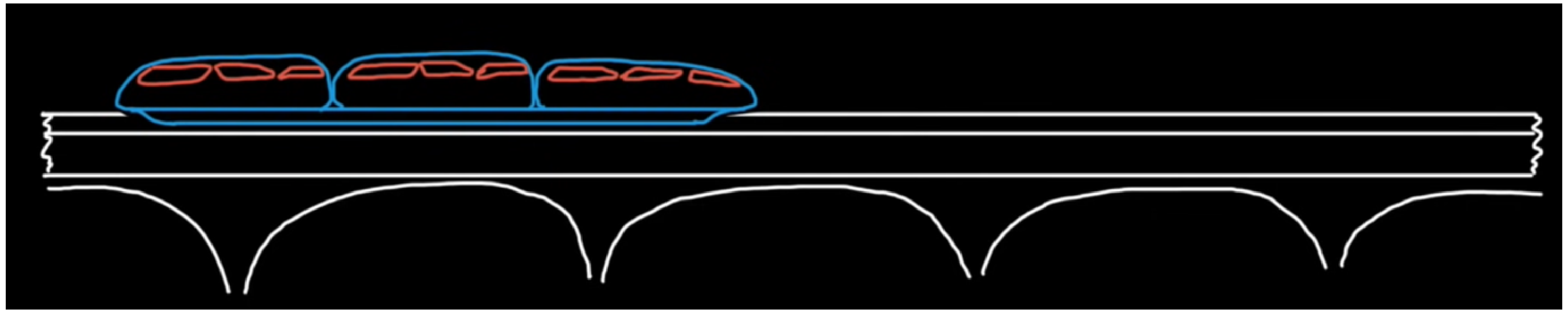
Observability

Observability means that all states can be known from the outputs of the system

Monorail

$$\dot{v} = u$$

$$\dot{p} = v$$



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impractical to know every state of the system

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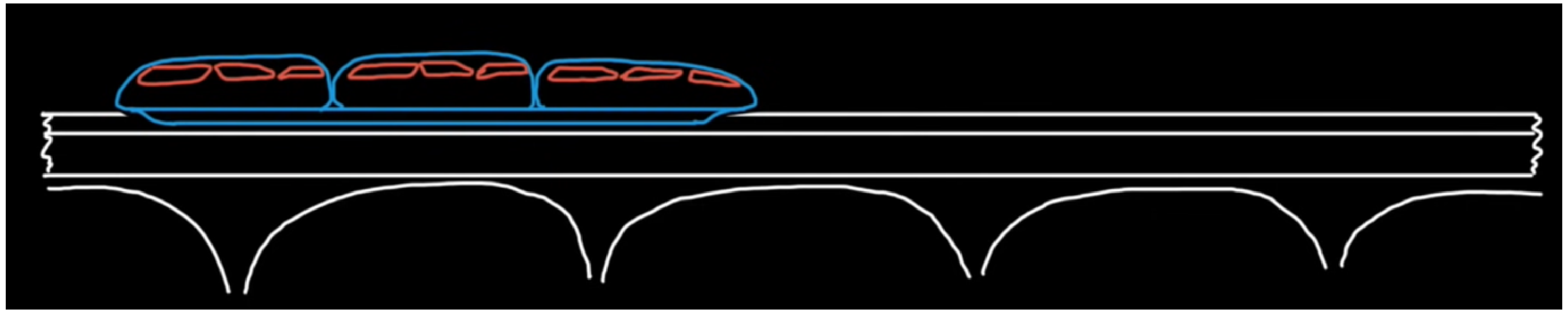

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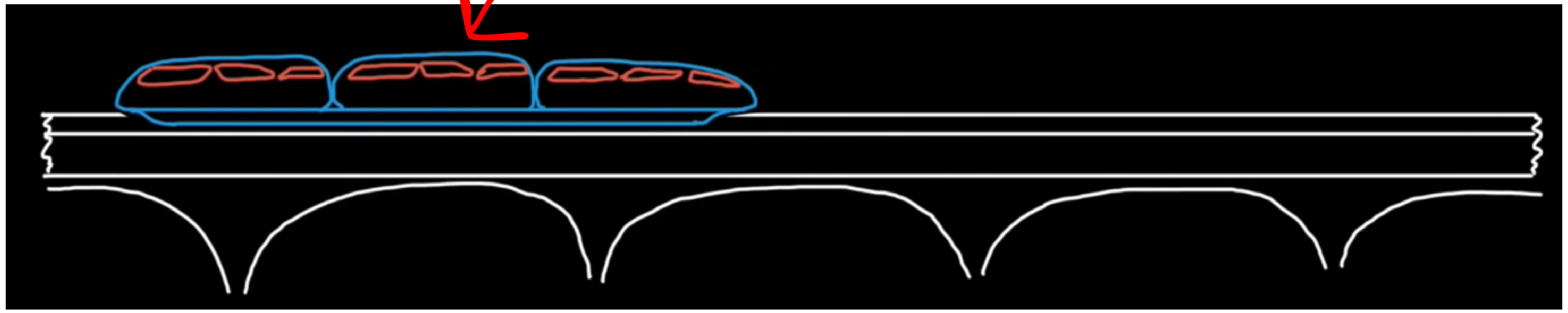
$t = 22°C$



Observability

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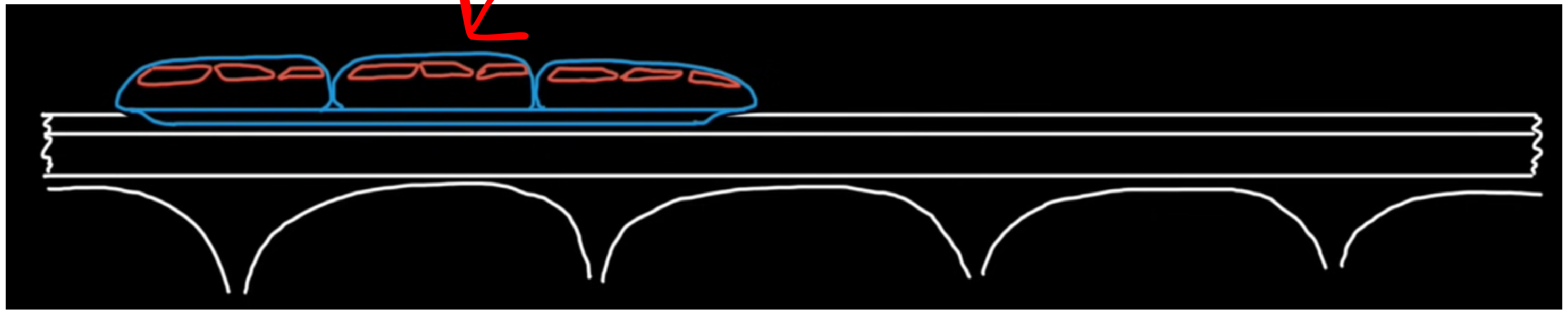
$$\dot{v} = u$$

$$\dot{p} = v$$

most don't impact the system in any meaningful way

$t = 22°C$



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$$\dot{v} = u$$

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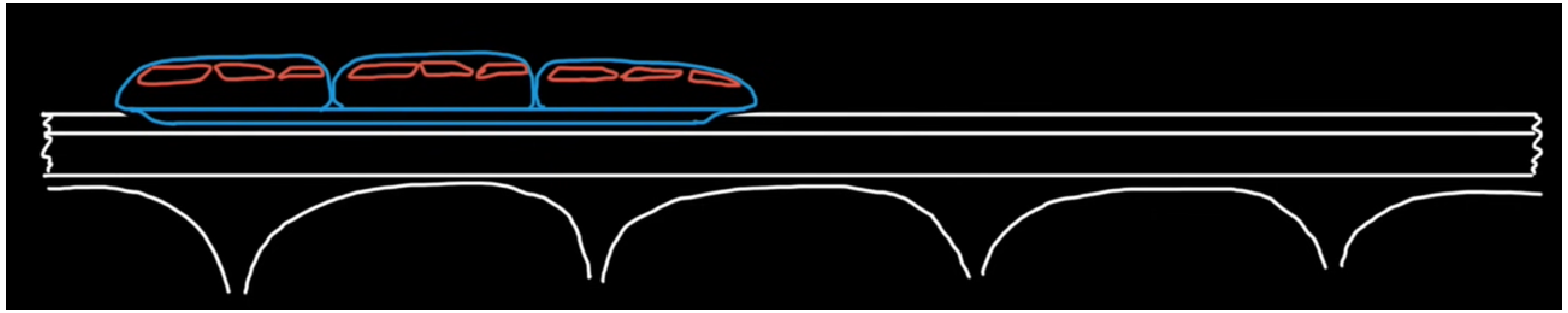
and we do not consider them in the state vector of the model

$$x = (p, v, \cancel{t})$$



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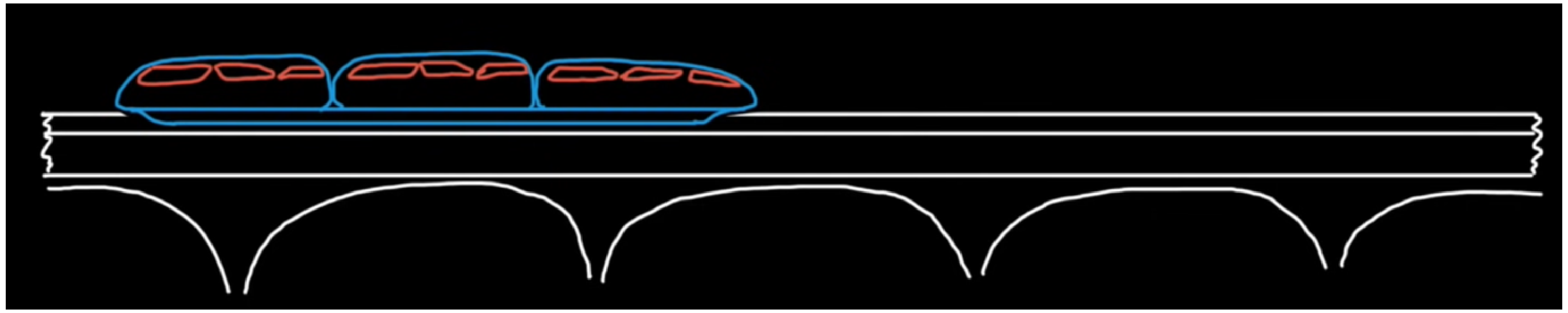


Observability

Observability means that all critical states can be known from the outputs of the system

What does it mean to observe a state?

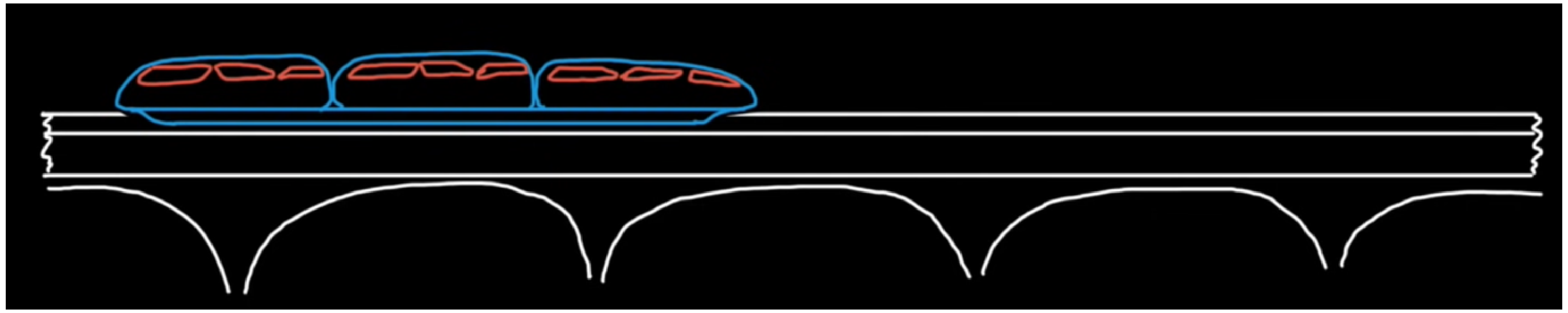
Monorail

$$\dot{v} = u$$

$$\dot{p} = v$$

we can mesure it

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}$$



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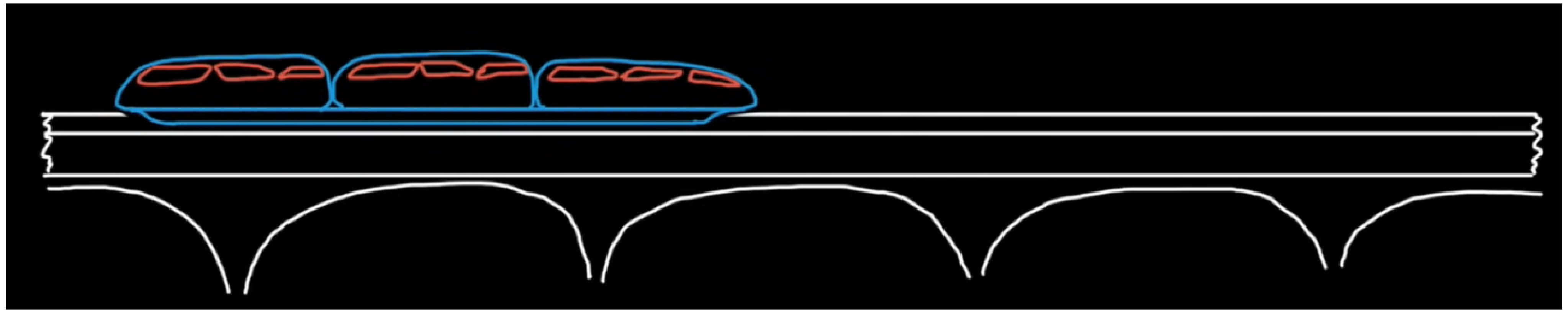

we can mesure it

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}$$

we can estimate it from available information

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} \qquad v = \dot{p}$$

measure position **estimate speed**

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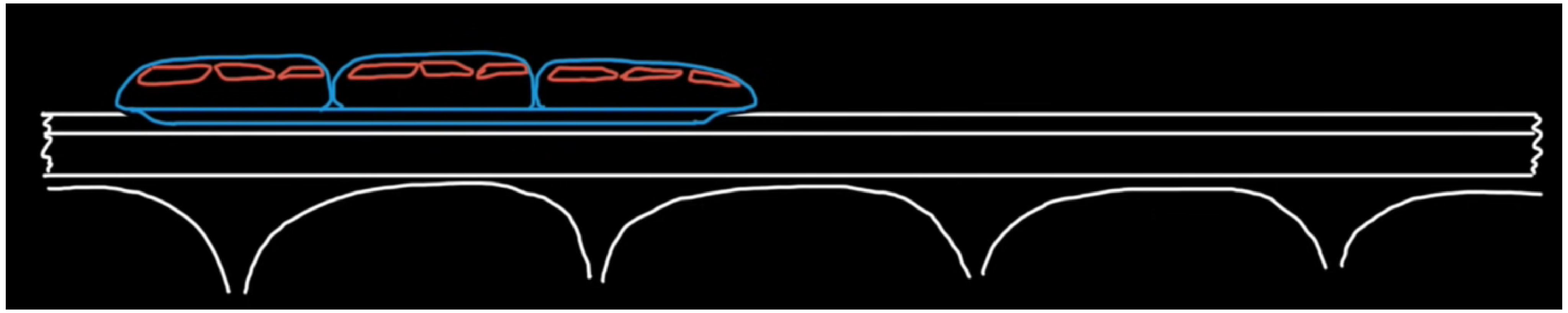

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measure position **estimate speed**

$$y = \underline{\begin{pmatrix}1 & 0\end{pmatrix}}\begin{pmatrix}v \\ p\end{pmatrix} \qquad \underline{p} = \int v\,dt + C$$

measure speed **estimate position**

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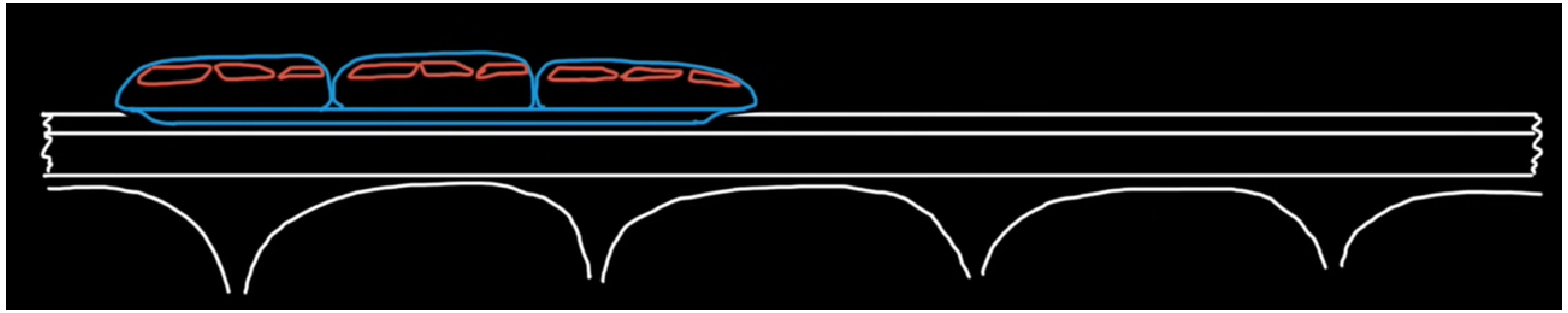
we can mesure it

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}$$

adding additional sensors can be expensive



we can estimate it from available information

$$v = \dot{p}$$

estimations are sensitive to measurement errors

estimate speed

$$p = \int v dt + C$$

estimate position

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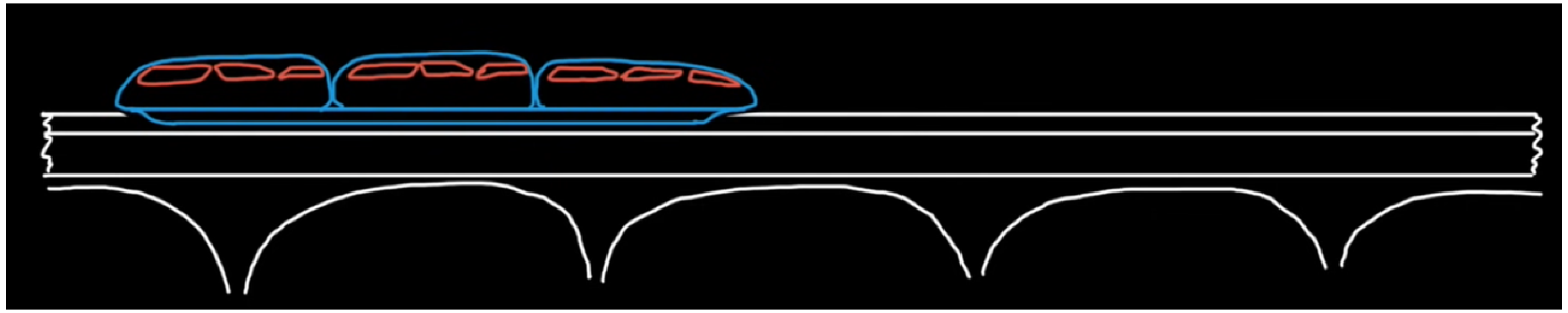

Example of unobservable system

imagine we lost all the sencors

$$y = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}$$

Controllability and observability of LTI system

Controllability & Observability of LTI system

State equation

$$\dot{x} = Ax + Bu$$

Output equation

$$y = Cx + Du$$

Dimensions

n states

p controls   m outputs

Controllability means that there exists control signal which allows the system to move from any any initial state to any final state in a finite time interval

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Controllability & Observability of LTI system

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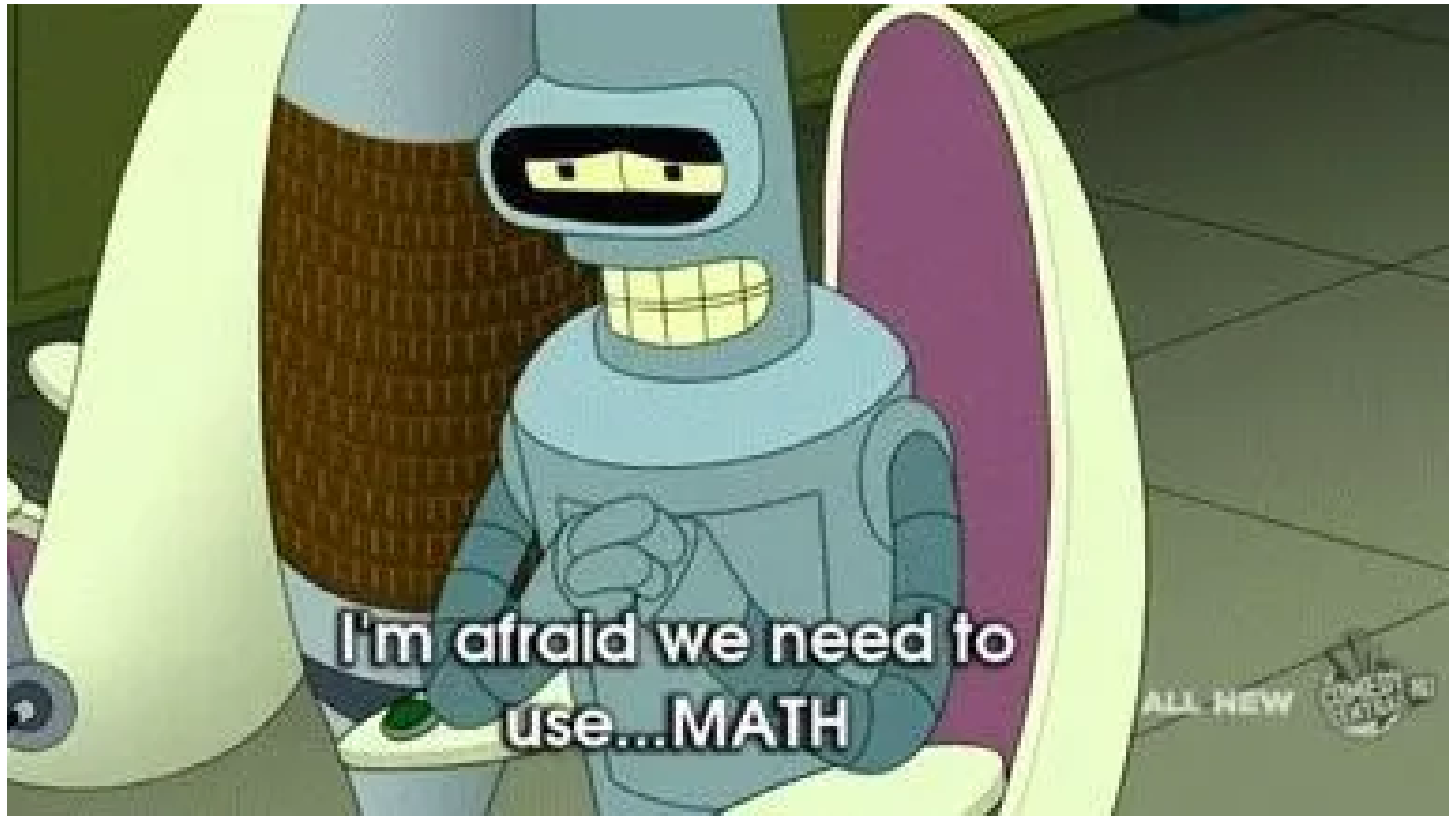

Controllability & Observability of LTI system

State equation

$$\dot{x} = Ax + Bu$$

Dimensions

n states
p controls

Solution of a state equation

$$x(t) = \underline{e^{At}}\,x(0) + \int_0^t \underline{e^{A(t-\tau)}}\,Bu(\tau)d\tau$$

<u>**matrix exponential**</u>

Let me remind...

- Let $A \in \mathbb{R}^{n \times n}$, the exponential of A, denoted by e^A is the $n \times n$ matrix given by the power series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

- Let $A \in \mathbb{R}^{n \times n}$ and I_n is $n \times n$ identity matrix. Then

$$p(\lambda) = det(\lambda I_n - A) = \lambda^n + \alpha_{n-1} A^{n-1} + \ldots + a_1 \lambda + \alpha_0 I_n$$

is called the characteristic polynomial of A.

Let me remind...

Theorem Caley-Hamilton

Let $A \in \mathbb{R}^{n \times n}$ then A satisfy its own characteristic polynomial equation, i.e.

$$p(A) = A^n + \alpha_{n-1}A^{n-1} + \ldots + \alpha_1 A + \alpha_0 I_n = 0.$$

- The theorem allows A^n to be expressed as a linear combination of the lower matrix powers of A

Controllability of LTI system

- The LTI system is called controllable if for any initial state x_0 and any final state x_f, there exists input signal $u(t)$ such that the system, starting from $x(0) = x_0$, reaches $x(t_f) = x_f$ in some finite time t_f.

- Starting at 0 is not a special case — if we can get to any state in finite time from the origin, then we can get from any initial condition to that state in finite time as well.

- $x(t_f) = \int_0^{t_f} e^{A(t_f - \tau)} Bu(\tau)\, d\tau$

Solution of
a state equation

Controllability of LTI system

- Change the variables $\tau_2 = \tau - t_f, d\tau = d\tau_2$ gives us a form

$$x(t_f) = \int_0^{t_f} \underline{e^{-A\tau_2}} Bu(t_f + \tau_2)\, d\tau_2$$

Controllability of LTI system

- Change the variables $\tau_2 = \tau - t_f, d\tau = d\tau_2$ gives us a form

$$x(t_f) = \int_0^{t_f} e^{-A\tau_2} Bu(t_f + \tau_2)\, d\tau_2$$

- Assume the system has p inputs. From the definition of matrix exponential and Cayley-Hamilton theorem, we have

$$e^{-A\tau_2} = \sum_{i=0}^{\infty} \frac{A^i}{i!}(-\tau_2)^i = \sum_{i=0}^{n-1} A^i \alpha_i(\tau_2)$$

for some computable scalars $\alpha_i(\tau_2)$.

Controllability of LTI system

- Hence

$$x(t_f) = \int_0^{t_f} e^{-A\tau_2} Bu(t_f + \tau_2)\, d\tau_2 =$$

$$\int_0^{t_f} \left(\sum_{i=0}^{n-1} A^i \alpha_i(\tau_2) \right) Bu(t_f + \tau_2)\, d\tau_2 =$$

$$\sum_{i=0}^{n-1} (A^i B) \int_0^{t_f} \alpha_i(\tau_2) u(t_f + \tau_2)\, d\tau_2 = \sum_{i=0}^{n-1} (A^i B) \beta_i(t_f)$$

- the coefficients $\beta_i(t_f)$ depends on the input $u(\tau_2) \in \mathbb{R}^p,\ 0 < \tau_2 \leq t_f$.

Controllability of LTI system

- In matrix form, we have $x(t_f) = [B, AB, \ldots, A^{n-1}B] \begin{bmatrix} \beta_0(t_f) \\ , \ldots \\ \beta_{n-1}(t_f) \end{bmatrix}$

Controllability of LTI system

- In matrix form, we have $x(t_f) = \overset{\text{controllability matrix}}{\underbrace{[B, AB, \ldots, A^{n-1}B]}_{\mathcal{C}(A,B)}} \begin{bmatrix} \beta_0(t_f) \\ , \ldots \\ \beta_{n-1}(t_f) \end{bmatrix}$

- A solution of this equation exists for any $x(t_f) \in \mathbb{R}^{n\times 1}$ if and only if

$$rank(\mathcal{C}(A, B)) = n.$$

Controllability of LTI system

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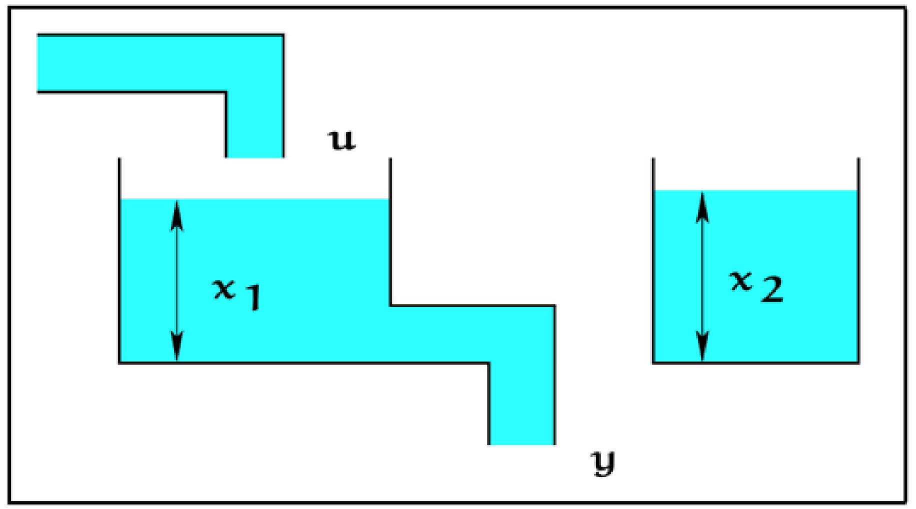
Kalman's Controllability Rank Condition

The LTI system $\dot{x} = Ax + Bu, \;\; x \in \mathbb{R}^{n\times 1}$ is controllable if and only if the controllability matrix $\mathcal{C}(A, B) = [B, AB, \ldots, A^{n-1}B]$ has full rank, i.e.

$$rank(\mathcal{C}(A, B)) = n.$$

Controllability Examples

Example.



In the hydraulic system on the left it is obvious that the input cannot affect the level x_2, so it is intuitively evident that the 2-tank system is not controllable.

A linearised model of this system with unitary parameters gives

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

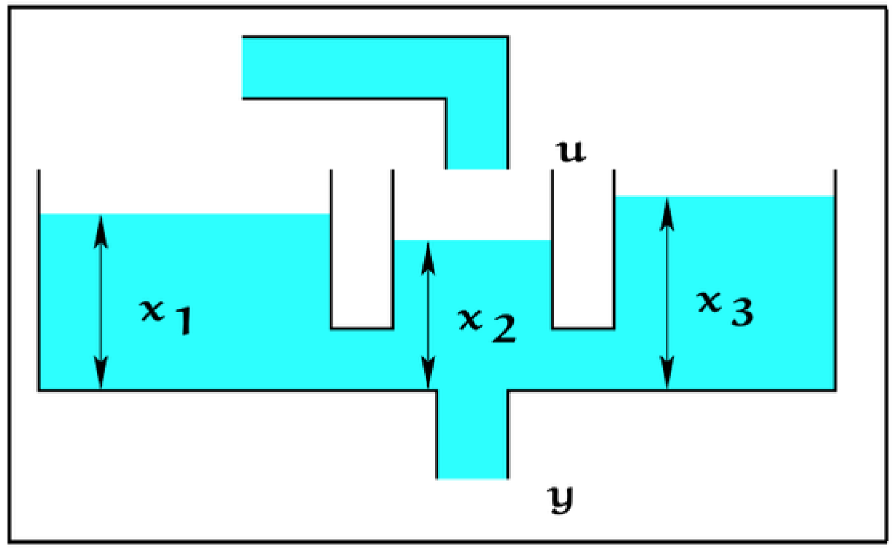
The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is not full rank, so the system is not controllable.

Controllability Examples

Example.



The controllability of the hydraulic system on the left is not so obvious, although we can see that $x_1(t)$ and $x_3(t)$ cannot be affected independently by $u(t)$.

The linearised model in this case is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

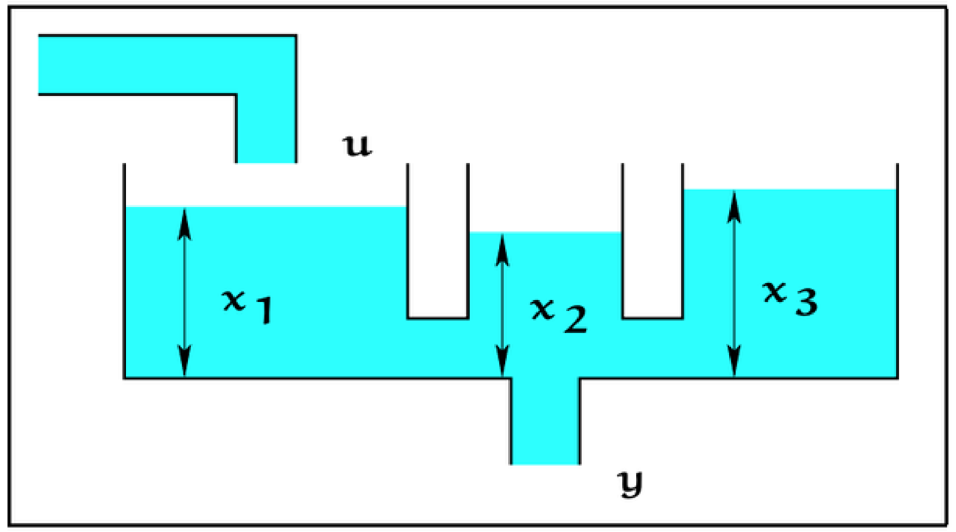
The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ 1 & -3 & 11 \\ 0 & 1 & -4 \end{bmatrix}$$

which has rank 2, showing that the system is not controllable.

Controllability Examples

Example.



Now in the previous system suppose that the input is applied in the first tank, as shown in the figure. In this case the linearised model is the same as before, except that the matrix $\mathbf{B}$ is now different

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(t)$$

The controllability matrix is now

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

which has rank 3, showing that the system is controllable.

Controllability & Observability of LTI system

State equation

$$\dot{x} = Ax + Bu$$

Output equation

$$y = Cx + Du$$

Dimensions

n states

p controls m outputs

Solution of a state equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

matrix exponential

Input - output relation

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Observability of LTI system

- Observability: Can we reconstruct $x(0)$ by knowing $y(\tau)$ and $u(\tau)$ over some finite time interval $[0, t]$? (By knowing the initial condition, we can reconstruct the entire state $x(t)$)
- Let us introduce notation

$$\tilde{y}(t) = y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t)$$

then

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \Leftrightarrow \tilde{y}(t) = Ce^{At}x(0)$$

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observability matrix

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\tilde{y}(t) &= Ce^{At}x(0) \\
\tilde{y}(t)^{(1)} &= CAe^{At}x(0) \\
&\cdots \\
\tilde{y}(t)^{(n-1)} &= CA^{n-1}e^{At}x(0)
\end{aligned}
\quad \Leftrightarrow \quad
\begin{bmatrix} \tilde{y}(t) \\ \tilde{y}(t)^{(1)} \\ \cdots \\ \tilde{y}(t)^{(n-1)} \end{bmatrix}
=
\underbrace{\begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}(A, C)}
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Kalman's Observability Rank Condition

The LTI system $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^{n\times 1}$ with measurements $y = Cx + Du$ is observable if and only if the observability matrix $\mathcal{O}(A, C)$ has full rank, i.e. $rank(\mathcal{O}(A, C)) = n$.

Let me summarize

State equation

$$\dot{x} = Ax + Bu$$

Output equation

$$y = Cx + Du$$

Dimensions

n states

p controls **m outputs**

- The LTI system is controllable if and only if $rank(\mathcal{C}(A, B)) = n$.
- The LTI system is observable if and only if $rank(\mathcal{O}(A, C)) = n$.

Let me summarize

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Duality of controllability & observability

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Duality of Controllability and observability

The pair of matrices (A, B) is controllable if and only if the pair of matrices (A^T, B^T) is observable.

Invariance Under Change of Coordinates

- Consider $\dot{x} = Ax + Bu, y = Cx + Du$ and similarity transformation $\tilde{x} = Tx$, where T is invertible.
- The system $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, y = \tilde{C}\tilde{x} + Du$ with matrices

$$\tilde{A} = TAT^{-1}, \; \tilde{B} = TB, \tilde{C} = CT^{-1}$$

is then called an equivalent system.

Invariance Under Nonsingular Transformations

The LTI system is controllable if and only if the equivalent system is controllable.
The LTI system is observable if and only if the equivalent system is observable.

Kalman Decomposition

The Kalman decomposition is defined as the realization of this system obtained by transforming the original matrices as follows:

$$\hat{A} = TAT^{-1},$$
$$\hat{B} = TB,$$
$$\hat{C} = CT^{-1},$$
$$\hat{D} = D,$$

where T^{-1} is the coordinate transformation matrix defined as

$$T^{-1} = \begin{bmatrix} T_{r\bar{o}} & T_{ro} & T_{\overline{ro}} & T_{\bar{r}o} \end{bmatrix},$$

and whose submatrices are

- $T_{r\bar{o}}$: a matrix whose columns span the subspace of states which are both reachable and unobservable.
- T_{ro} : chosen so that the columns of $\begin{bmatrix} T_{r\bar{o}} & T_{ro} \end{bmatrix}$ are a basis for the reachable subspace.
- $T_{\overline{ro}}$: chosen so that the columns of $\begin{bmatrix} T_{r\bar{o}} & T_{\overline{ro}} \end{bmatrix}$ are a basis for the unobservable subspace.
- $T_{\bar{r}o}$: chosen so that $\begin{bmatrix} T_{r\bar{o}} & T_{ro} & T_{\overline{ro}} & T_{\bar{r}o} \end{bmatrix}$ is invertible.

It can be observed that some of these matrices may have dimension zero. For example, if the system is both observable and controllable, then $T^{-1} = T_{ro}$, making the other matrices zero dimension.

Kalman Decomposition

By using results from controllability and observability, it can be shown that the transformed system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ has matrices in the following form:

$$\hat{A} = \begin{bmatrix} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{ro} & 0 & A_{24} \\ 0 & 0 & A_{\overline{ro}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{r}o} \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix}$$

$$\hat{D} = D$$

This leads to the conclusion that

- The subsystem $(A_{ro}, B_{ro}, C_{ro}, D)$ is both reachable and observable.
- The subsystem $\left(\begin{bmatrix} A_{r\bar{o}} & A_{12} \\ 0 & A_{ro} \end{bmatrix}, \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \end{bmatrix}, \begin{bmatrix} 0 & C_{ro} \end{bmatrix}, D\right)$ is reachable.
- The subsystem $\left(\begin{bmatrix} A_{ro} & A_{24} \\ 0 & A_{\bar{r}o} \end{bmatrix}, \begin{bmatrix} B_{ro} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{ro} & C_{\bar{r}o} \end{bmatrix}, D\right)$ is observable.

Please check the following resource for the proof