# State observers.
# Dynamic output stabilisation.
# Discrete time control systems.


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**Course grade breakdowns**

**Labs - 40%**

**Final test - 30%**

**Final project - 30 %**

# State feedback design

Linear state space control theory involves modifying the behaviour of an $m$-input, $p$-output, $n$-state system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \qquad \text{(OL)}$$

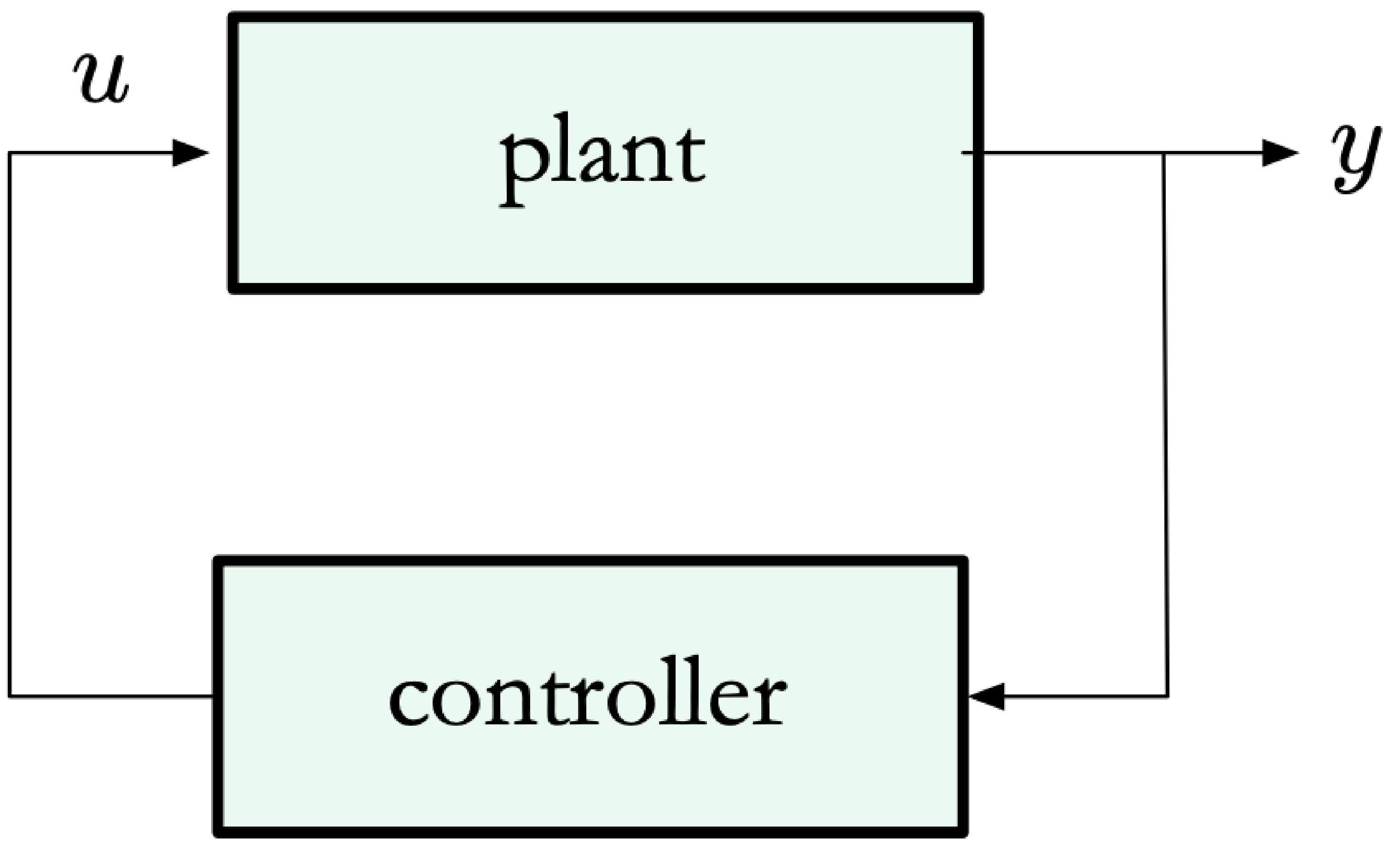
which we call **the plant**, or **open loop state equation**, by application of a control law of the form

$$\mathbf{u}(t) = \mathbf{N}\mathbf{r}(t) - \mathbf{K}\mathbf{x}(t), \qquad \text{(U)}$$

in which $\mathbf{r}(t)$ is the new (reference) input signal. The matrix $\mathbf{K}$ is the state feedback gain and $\mathbf{N}$ the feedforward gain.

# Is Full State Feedback Always Available?

In a typical system, measurements are provided by sensors:




Full state feedback $u = -Kx$ is *not implementable*!!

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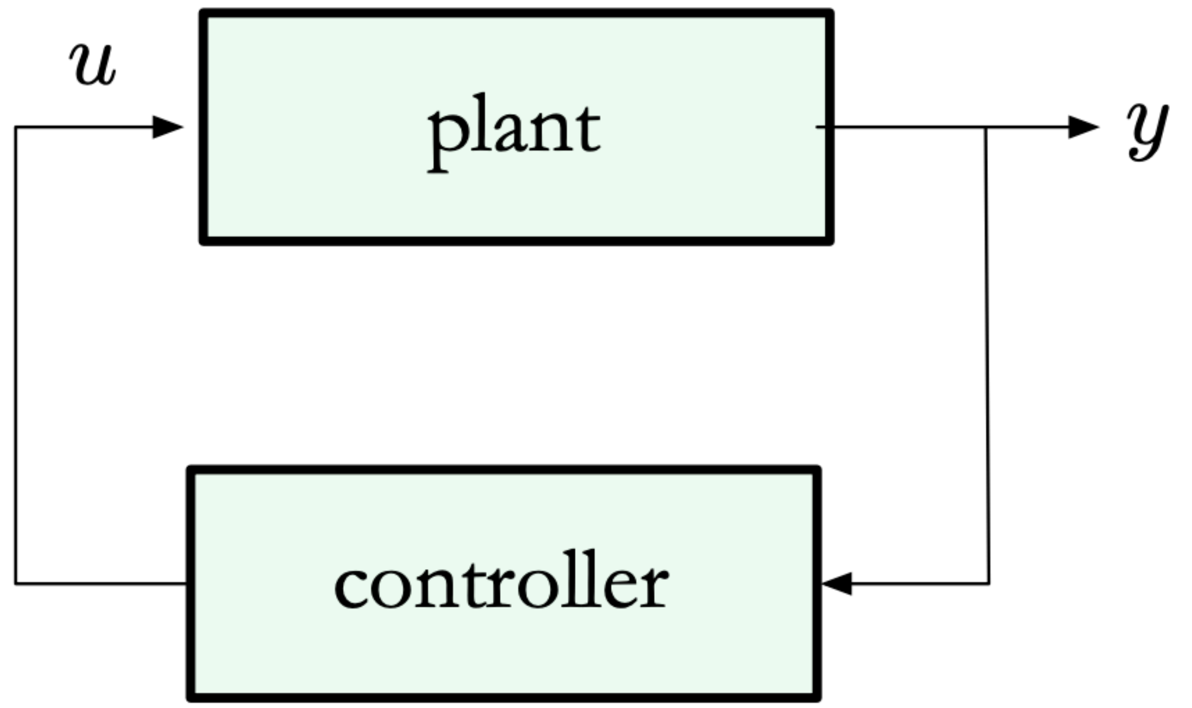


Full state feedback $u = -Kx$ is *not implementable*!!

# When Full State Feedback Is Unavailable ...

... we need an Observer!!

# Is Full State Feedback Always Available?

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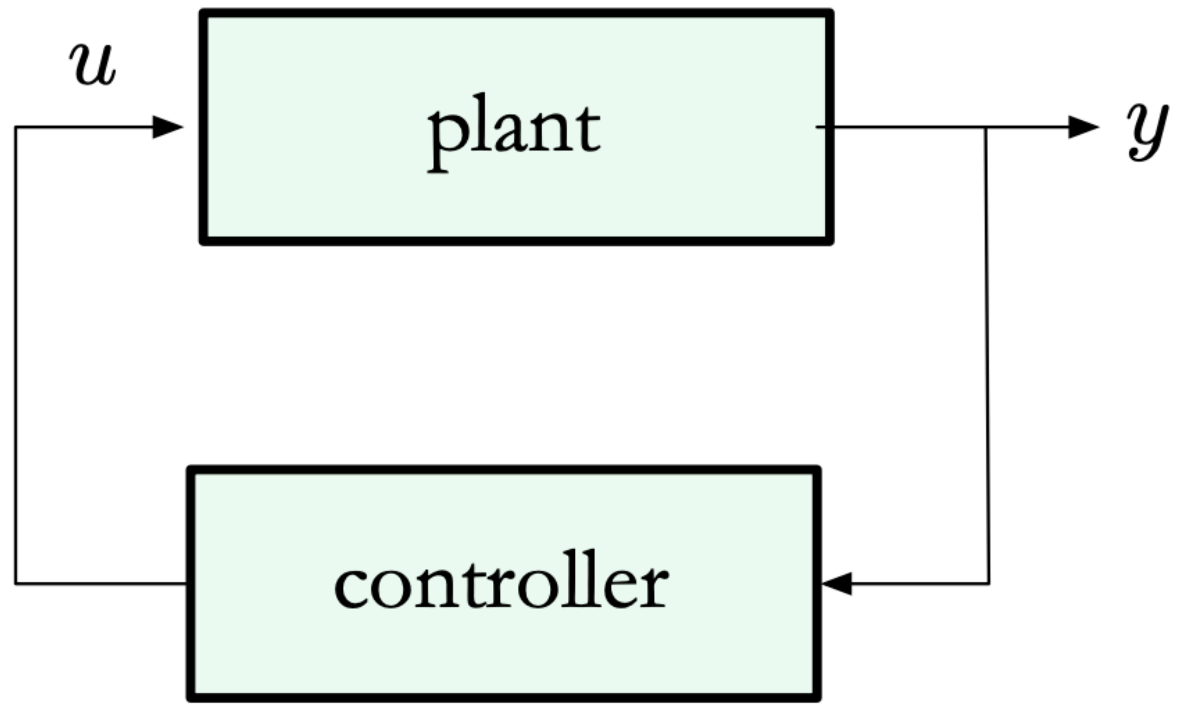
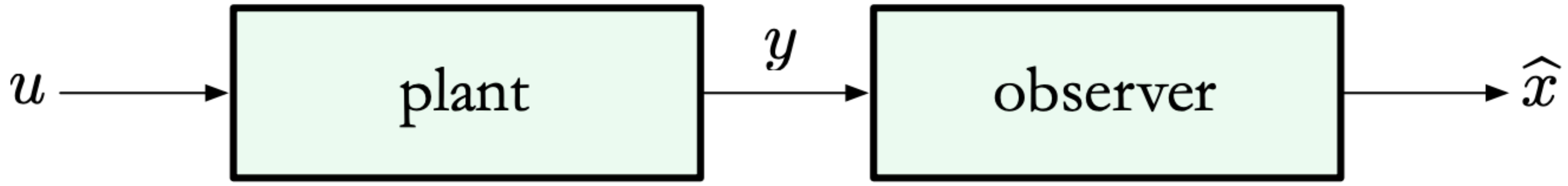



Full state feedback $u = -Kx$ is *not implementable*!!

In that case, an observer is used to estimate the state $x$:

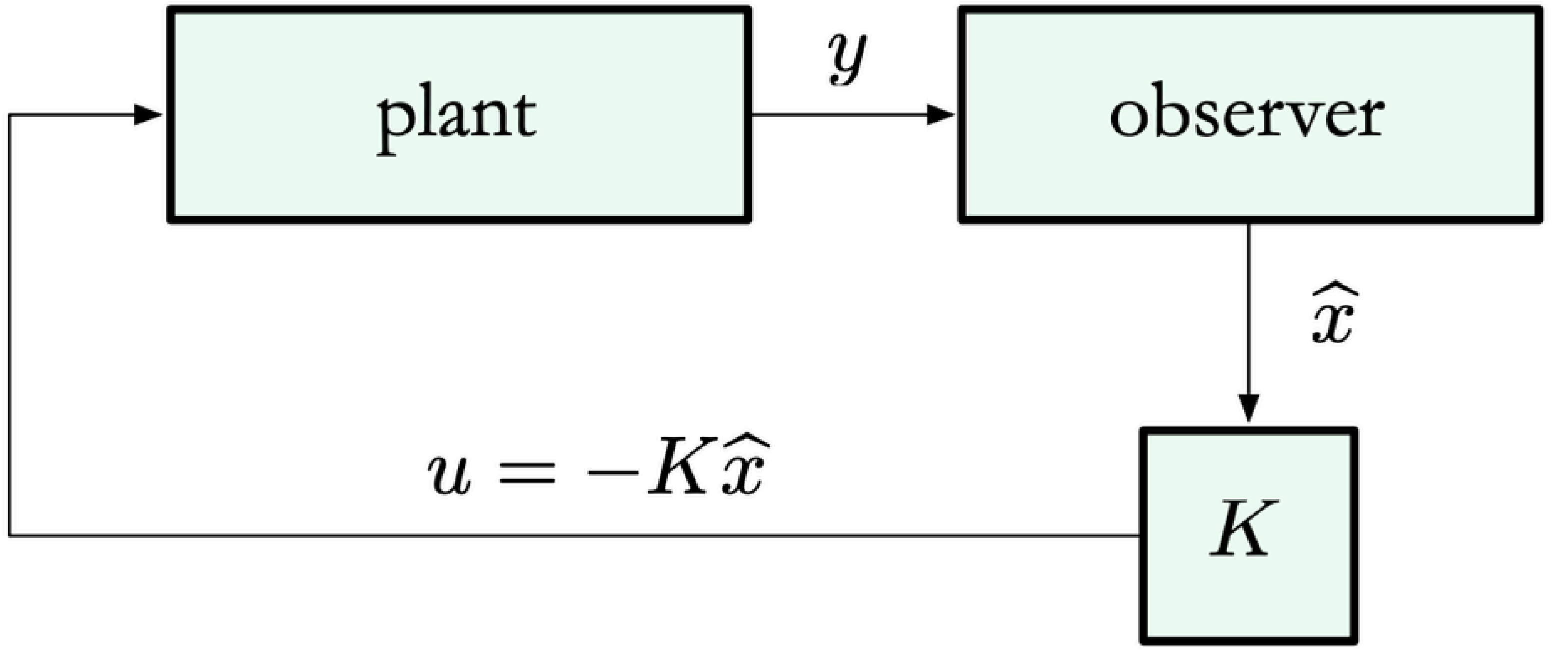


# State Estimation Using an Observer

If the system is observable, the state estimate $\widehat{x}$ is *asymptotically accurate*:

$$\|\widehat{x}(t) - x(t)\| = \sqrt{\sum_{i=1}^{n} (\widehat{x}_i(t) - x_i(t))^2} \xrightarrow{t\to\infty} 0$$

If we are successful, then we can try estimated state feedback:



# The Luenberger Observer

$$\text{System:} \quad \dot{x} = Ax$$
$$y = Cx$$
$$\text{Observer:} \quad \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly.$$

What happens to state estimation error $e = x - \widehat{x}$ as $t \to \infty$?

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\widehat{x}} \\ &= Ax - [(A - LC)\widehat{x} + LCx] \\ &= (A - LC)x - (A - LC)\widehat{x} \\ &= (A - LC)e \end{aligned}$$

Does $e(t)$ converge to zero in some sense?

# The Luenberger Observer

$$\begin{aligned} \text{System:} \quad & \dot{x} = Ax \\ & y = Cx \\ \text{Observer:} \quad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly \\ \text{Error:} \quad & \dot{e} = (A - LC)e \end{aligned}$$

Recall our assumption that $A - LC$ is Hurwitz (all eigenvalues are in LHP). This implies that

$$\|x(t) - \widehat{x}(t)\|^2 = \|e(t)\|^2 = \sum_{i=1}^{n} |e_i(t)|^2 \xrightarrow{t \to \infty} 0$$

at an exponential rate, determined by the eigenvalues of $A - LC$.

For fast convergence, want eigenvalues of $A - LC$ far into LHP!!

# Observability and Estimation Error

Fact: If the system

$$\dot{x} = Ax, \qquad y = Cx$$

is observable, then we can arbitrarily assign eigenvalues of $A - LC$ by a suitable choice of the output injection matrix $L$.

This is similar to the fact that controllability implies arbitrary closed-loop pole placement by state feedback.

In fact, these two facts are closely related because CCF is dual to OCF.

# Controllability–Observability Duality

Claim: The system

$$\dot{x} = Ax, \qquad y = Cx$$

is observable if and only if the system

$$\dot{x} = A^T x + C^T u$$

is controllable.

Proof: $\mathcal{C}(A^T, C^T) = [C^T \,|\, A^T C^T \,|\, \ldots \,|\, (A^T)^{n-1} C^T]$

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T = [\mathcal{O}(A, C)]^T$$

Thus, $\mathcal{O}(A, C)$ is nonsingular if and only if $\mathcal{C}(A^T, C^T)$ is.

# Observer Pole Placement, O/C Duality Version

Given an observable pair $(A, C)$:

1. For $F = A^T$, $G = C^T$, consider the system $\dot{x} = Fx + Gu$ (this system is controllable).
2. Use our earlier procedure to find $K$, such that

$$F - GK = A^T - C^T K$$

   has desired eigenvalues.
3. Then

$$\mathrm{eig}(A^T - C^T K) = \mathrm{eig}(A^T - C^T K)^T = \mathrm{eig}(A - K^T C),$$

   so $L = K^T$ is the desired output injection matrix.

Final answer: use the observer

$$\begin{aligned} \dot{\widehat{x}} &= (A - LC)\widehat{x} + Ly \\ &= (A - K^T C)\widehat{x} + K^T y. \end{aligned}$$

## Recall: infinite-horizon Linear Quadratic Regulator (LQR)

**Problem formulation:** optimal control for integral-quadratic cost

$$\underset{u(t)}{\text{minimize}}\ J(u(t)) = \int_0^\infty x(t)^\top Q\, x(t) + u(t)^\top R\, u(t)\, dt$$

$$\text{subject to}\ \ \dot{x}(t) = Ax(t) + Bu(t)\,,\ \ x(0) = x_0$$

**Feasible if** $Q \succeq 0$, $R \succ 0$, $(A, B)$ stabilizable, & $(A, Q^{1/2})$ detectable.

**Solution:** independent of the initial condition $x_0$, the linear state feedback

$$u^\star(t) = -\,K^\star x(t) = -\underbrace{R^{-1}B^\top P}_{=K^\star} x$$

where $P \succ 0$ solves the algebraic matrix Riccati equation

$$A^\top P + PA + Q = PBR^{-1}B^\top P$$

**Equivalent problem formulation**:

In hindsight, LQR can be interpreted as *optimal pole placement* for

$$\dot{x}(t) = (A - BK^\star)\, x(t)$$

trading off minimal state deviation and minimal control energy:

$$K^\star = \mathrm{argmin}_K \int_0^\infty \underbrace{x(t)^\top Q\, x(t)}_{\text{state deviation}} + \underbrace{x(t)^\top K^\top R K\, x(t)}_{\text{control energy}}\, dt$$

# Recall: dual notions of controllability/observability

The following statements are equivalent for **controller design**:

1. the system $(A, B)$ is controllable
2. the controllability matrix

$$W_c = [B\ AB\ \dots\ A^{n-1}B]$$

   has full rank $n$

3. the eigenvalues of $A - BK$ can be assigned via the matrix $K$
4. $\dots$

The following statements are equivalent for **observer design**:

1. the system $(A, C)$ is observable
2. the observability matrix

$$W_o = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}$$

   has full rank $n$

3. the eigenvalues of $A - LC$ can be assigned via the matrix $L$
4. $\dots$

**idea:** use duality $(A, B, K) \leftrightarrow (A^\top, C^\top, L^\top)$ to design *optimal observers*

# Optimal design by duality $(A, B, K) \leftrightarrow (A^\top, C^\top, L^\top)$

**LQ-optimal control** for closed-loop dynamics: $\dot{x} = (A - BK)\,x$

$$\text{minimize}_K \int_0^\infty \underbrace{x(t)^\top Q\, x(t)}_{\text{state deviation}} + \underbrace{x(t)^\top K^\top R K\, x(t)}_{\text{control energy}}\, dt$$

$\Rightarrow K^\star = R^{-1} B^\top P$ where $P \succ 0$ solves $A^\top P + PA + Q = PBR^{-1}B^\top P$

**LQ-optimal estimation** for estimation error dynamics: $\dot{\epsilon} = (A - LC)\,\epsilon$

$$\text{minimize}_L \int_0^\infty \underbrace{\epsilon(t)^\top Q\, \epsilon(t)}_{\text{estimation error}} + \underbrace{\epsilon(t)^\top L R L^\top \epsilon(t)}_{\text{output correction}}\, dt$$

$\Rightarrow L^\star = PC^\top R^{-1}$ where $P \succ 0$ solves $AP + PA^\top + Q = PC^\top R^{-1} C P$

# Role of $Q$ and $R$ in LQ observer design

**Optimal observer** for integral-quadratic cost

$$\text{minimize}_L \int_0^\infty \underbrace{\epsilon(t)^\top Q\,\epsilon(t)}_{\text{estimation error}} + \underbrace{\epsilon(t)^\top LRL^\top \epsilon(t)}_{\text{output correction}}\,dt$$

trades off **prediction and correction** $\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{prediction}} + \underbrace{L(y - C\hat{x})}_{\text{correction}}$

$\Rightarrow$ $R \succ 0$ quantifies correction through measurement:

$R$ "large" $\implies$ $L$ "small" $\implies$ trust prediction

$R$ "small" $\implies$ $L$ "large" $\implies$ trust measurement

$\Rightarrow$ $Q \succeq 0$ quantifies prediction error: $Q$ "large" $\implies$ "smaller" error $\epsilon$

# Summary: LQ optimal estimation (LQE)

**Problem formulation:** optimal observer for integral-quadratic cost

$$\text{minimize}_L \int_0^\infty \underbrace{\epsilon(t)^\top Q\,\epsilon(t)}_{\text{estimation error}} + \underbrace{\epsilon(t)^\top LRL^\top \epsilon(t)}_{\text{output correction}} \, dt$$

$$\text{subject to} \;\; \dot{\epsilon}(t) = (A - LC)\,\epsilon(t)$$

**Feasible if** $Q \succeq 0$, $R \succ 0$, $(A, C)$ detectable, & $(A, Q^{1/2})$ stabilizable.

**Solution:** independent of the initial condition $\epsilon_0$, the output feedback

$$L^\star = PC^\top R^{-1}$$

where $P \succ 0$ solves the algebraic matrix Riccati equation

$$AP + PA^\top + Q = PC^\top R^{-1} CP$$

# Combining Full-State Feedback with an Observer

- So far, we have focused on autonomous systems ($u = 0$).
- What about nonzero inputs?

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

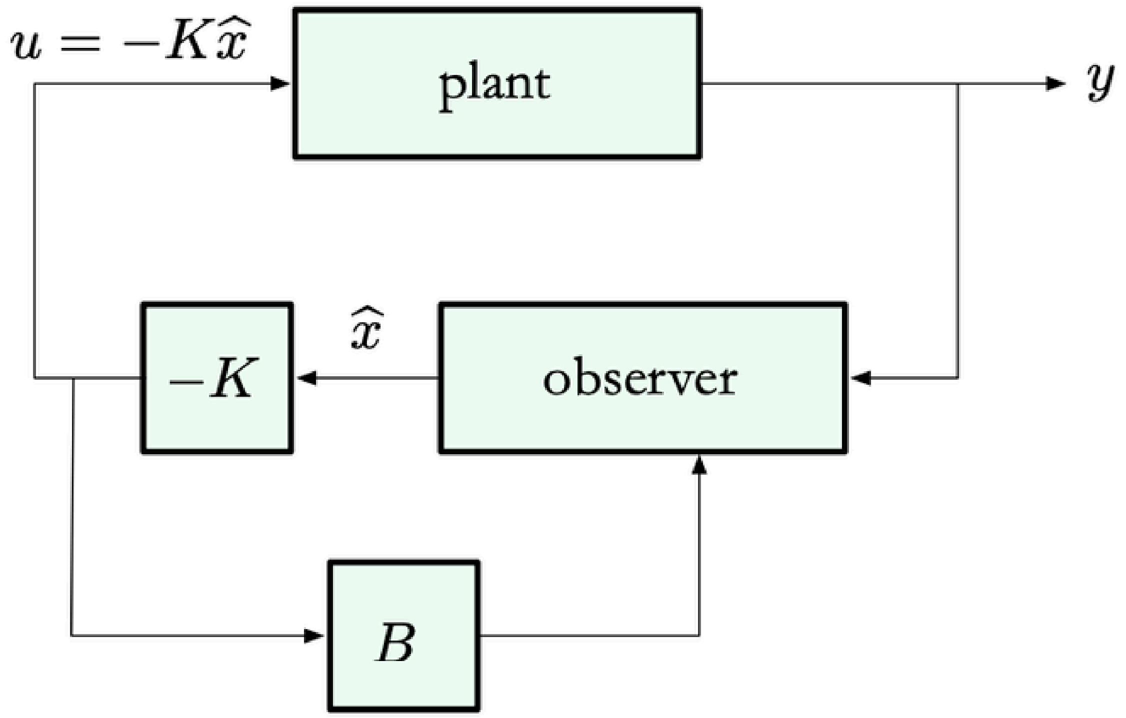
# Combining Full-State Feedback with an Observer

- So far, we have focused on autonomous systems ($u = 0$).
- What about nonzero inputs?

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

— assume $(A, B)$ is controllable and $(A, C)$ is observable.

- Today, we will learn how to use an observer together with estimated state feedback to (approximately) place closed-loop poles.



# Combining Full-State Feedback with an Observer

- Consider

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

  where $(A, B)$ is controllable and $(A, C)$ is observable.
- We know how to find $K$, such that $A - BK$ has desired eigenvalues (controller poles).
- Since we do not have access to $x$, we must design an observer. But this time, we need a slight modification because of the $Bu$ term.

# Observer in the Presence of Control Input

- Let's see what goes wrong when we use the old approach:

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly$$

- For the estimation error $e = x - \hat{x}$, we have

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu - [(A - LC)\hat{x} + LCx] \\ &= (A - LC)e + Bu && \text{– not good} \end{aligned}$$

- Idea: since $u$ is a signal we can access, let's use it as an input to the observer to cancel the $Bu$ term from $\dot{x}$.
- Modified observer:

$$\begin{aligned} \dot{\hat{x}} &= (A - LC)\hat{x} + Ly + Bu \\ \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu - [(A - LC)\hat{x} + LCx + Bu] \\ &= (A - LC)e && \text{regardless of } u \end{aligned}$$

# Observer and Controller

$$\begin{aligned} \text{System:} \quad & \dot{x} = Ax + Bu \\ & y = Cx \\ \text{Observer:} \quad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly + Bu \\ \text{Error:} \quad & \dot{e} = (A - LC)e \end{aligned}$$

- By observability, we can arbitrarily assign $\text{eig}(A - LC)$; these should be farther into LHP than desired controller poles.

$$\text{Controller:} \quad u = -K\widehat{x} \quad (\text{estimated state feedback})$$

- By controllability, we can arbitrarily assign $\text{eig}(A - BK)$.

# Observer and Controller

$$\begin{aligned} \text{System:} \quad & \dot{x} = Ax + Bu \\ & y = Cx \\ \text{Observer:} \quad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly + Bu \\ \text{Controller:} \quad & u = -K\widehat{x} \end{aligned}$$

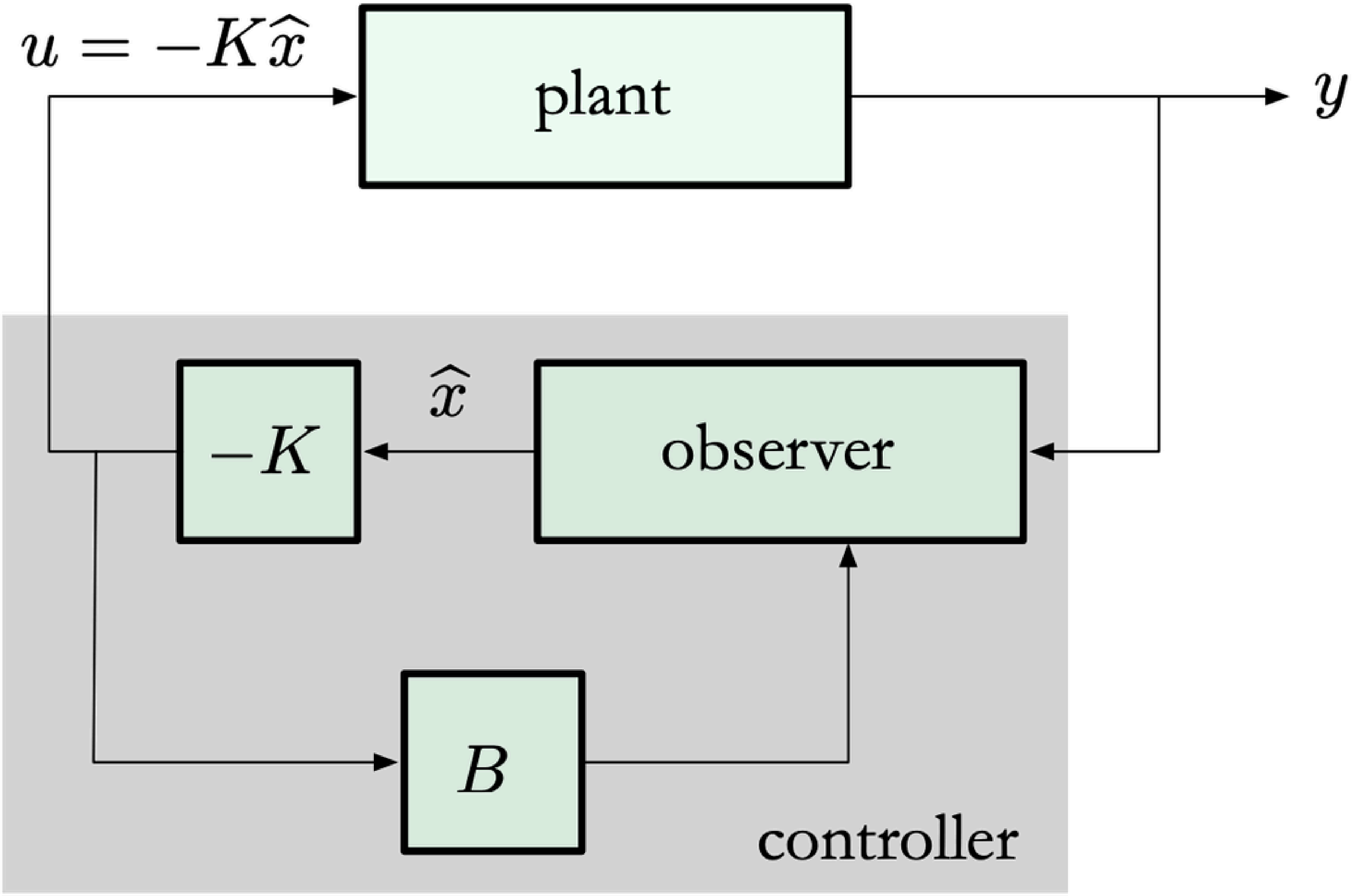
The overall observer-controller system is:

$$\begin{aligned} \dot{\widehat{x}} &= (A - LC)\widehat{x} + Ly + B\underbrace{(-K\widehat{x})}_{=u} \\ &= (A - LC - BK)\widehat{x} + Ly \\ u &= -K\widehat{x} \qquad \qquad \text{(dynamic output feedback)} \end{aligned}$$

— this is a dynamical system with input $y$ and output $u$

# Dynamic Output Feedback

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$\dot{\widehat{x}} = (A - LC - BK)\widehat{x} + Ly$$
$$u = -K\widehat{x}$$



# Dynamic Output Feedback: Does It Work?

Summarizing:

- When $y = x$, full state feedback $u = -Kx$ achieves desired pole placement.
- How do we know that $u = -K\widehat{x}$ achieves similar objectives?

Here is our overall closed-loop system:

$$\dot{x} = Ax - BK\widehat{x}$$
$$\dot{\widehat{x}} = (A - LC - BK)\widehat{x} + LCx$$

We can write it in block matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{\widehat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - LC - BK \end{pmatrix} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$$

How do we relate this to "nominal" behavior, $A - BK$?

# Dynamic Output Feedback

$$\begin{pmatrix} \dot{x} \\ \dot{\widehat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - LC - BK \end{pmatrix} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$$

Let us transform to new coordinates:

$$\begin{pmatrix} x \\ \widehat{x} \end{pmatrix} \longmapsto \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} x \\ x - \widehat{x} \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ I & -I \end{pmatrix}}_{T} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$$

Two key observations:

- $T$ is invertible, so the new representation is equivalent to the old one
- in the new coordinates, we have

$$\begin{aligned} \dot{x} &= Ax - BK\widehat{x} \\ &= (A - BK)x + BK(x - \widehat{x}) \\ &= (A - BK)x + BKe \\ \dot{e} &= (A - LC)e \end{aligned}$$

# The Main Result: Separation Principle

So now we can write

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \underbrace{\begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}}_{\text{upper triangular matrix}} \begin{pmatrix} x \\ e \end{pmatrix}$$

The closed-loop characteristic polynomial is

$$\det \begin{pmatrix} Is - A + BK & -BK \\ 0 & Is - A + LC \end{pmatrix}$$
$$= \det(Is - A + BK) \cdot \det(Is - A + LC)$$

**Separation principle.** The closed-loop eigenvalues are:

$$\{\text{controller poles (roots of } \det(Is - A + BK))\}$$
$$\cup \{\text{observer poles (roots of } \det(Is - A + LC))\}$$

— this holds only for linear systems!!

# Separation Principle

Separation principle. The closed-loop eigenvalues are:

$$\begin{aligned}&\{\text{controller poles (roots of } \det(Is - A + BK))\}\\ &\quad \cup \{\text{observer poles (roots of } \det(Is - A + LC))\}\end{aligned}$$

— this holds only for linear systems!!

Moral of the story:

- If we choose observer poles to be several times faster than the controller poles (e.g., 2–5 times), then the controller poles will be dominant.
- Dynamic output feedback gives essentially the same performance as (nonimplementable) full-state feedback — provided observer poles are far enough into LHP.
- Remember: the system must be controllable and observable!!

# Control of
# Discrete-time Systems

# Space model of discrete-time system

## Continuous-time systems

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

# Space model of discrete-time system

**Continuous-time systems**

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

**Dicrite-time systems**

$$x_k = Ax_{k-1} + Bu_{k-1}$$

$$y_k = Cx_k + Du_k$$

# Space model of discrete-time system

**Continuous-time systems**

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

**Dicrite-time systems**

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

**Discrete-time systems are either inherently discrete (e.g. models of bank accounts, national economy growth models, population growth models, digital words)**

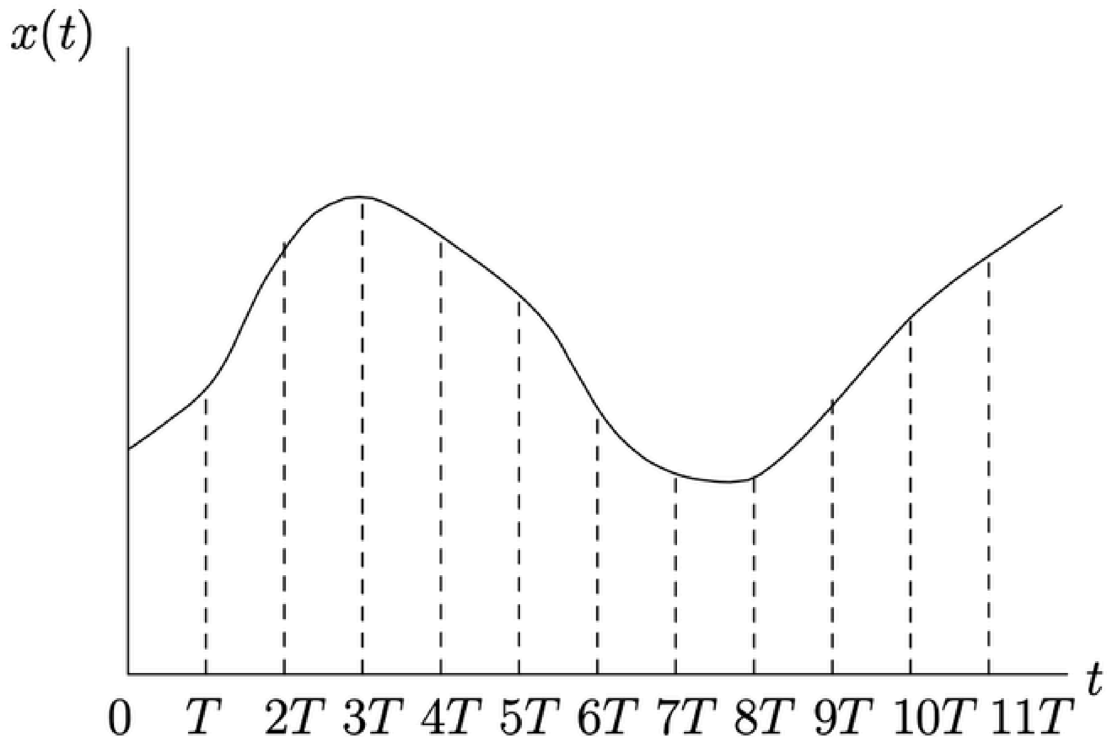
# Disretization of continuous-time system

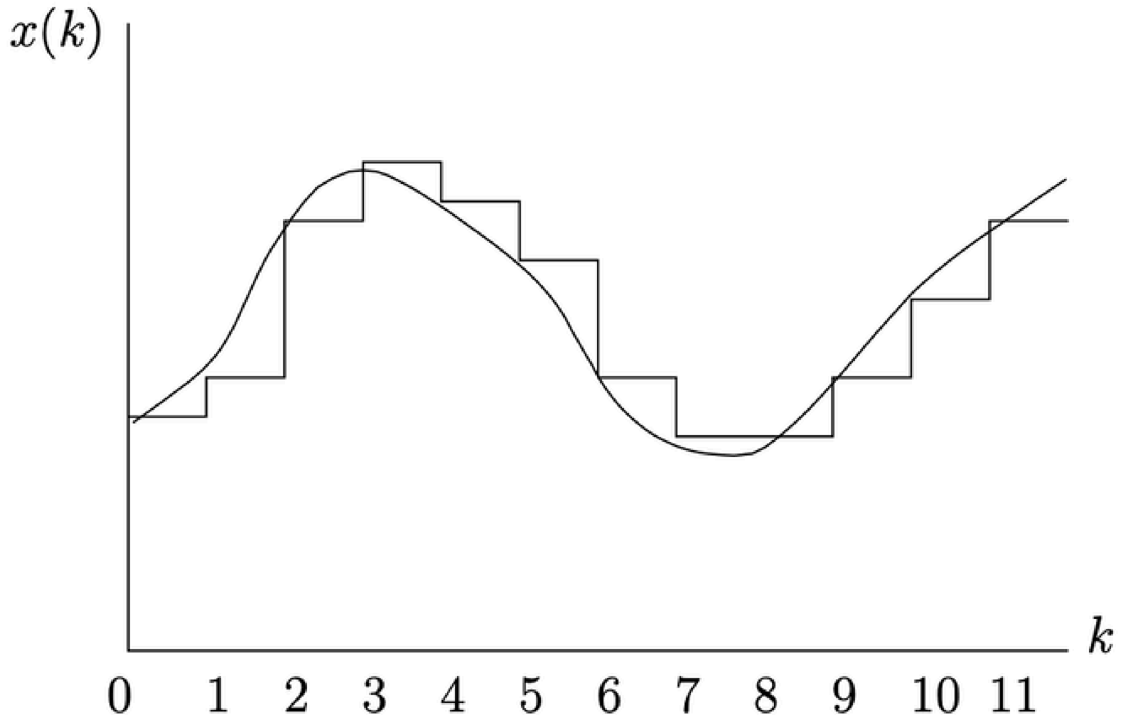
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

with sampling-time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$




**or they are obtained as a result of sampling (discretization) of continuous-time systems.**

# Controllability of discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

## Definition of Controllability

A discrete-time linear system $x_{k+1} = Ax_k + Bu_k$ is called controllable at $k = 0$ if there exists a finite time $k_N$ such that for any initial state $x_0$ and target state $x_t$, there exists a control sequence $\{u_k; k = 0, 1, , k_N\}$ that will transfer the system from $x_0$ at $k = 0$ to $x_t$ at $k = k_N$

# Observability of discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

## Definition of Observability

A discrete-time linear system is called observable at $k = 0$ if there exists a finite time $k_N$ such that for any initial state $x_0$, the knowledge of input $\{u_k; k = 0, 1, \ldots, k_N\}$ and $\{y_k; k = 0, 1, \ldots, k_N\}$ suffice to determine the state $x_0$.

# Internal stability of discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

## Definition of internal stability

A discrete-time system is stable if and only if when the input $u_k = 0$ for all $k \geq 0$, the state $x_k$ is bounded for all $k \geq 0$ for any initial state $x_0 \in \mathbb{R}^n$

A discrete-time system is asymptotically stable if and only if it is stable and $\lim_{k\to+\infty} \|x_k\| = 0$ for any initial state $x_0 \in R^n$ n.

# Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

→

with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$

**Continuous-time system**

**It's sampled version**

# Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

$\longrightarrow$

with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$

**Attention!**

**Continuous-time system**

Controllable

$\not\Longrightarrow$

**It's sampled version**

Controllable

# Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

$\longrightarrow$

with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
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## Attention!

| Continuous-time system | | It's sampled version |
|---|---|---|
| Controllable | $\not\Longrightarrow$ | Controllable |
| Observable | $\not\Longrightarrow$ | Observable |

# Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method → with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$

## Attention!

| Continuous-time system | | It's sampled version |
|---|---|---|
| Controllable | $\not\Longrightarrow$ | Controllable |
| Observable | $\not\Longrightarrow$ | Observable |
| Stable | $\not\Longrightarrow$ | Stable |

# Disretization of continuous-time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

euler method

with sampling time T

$$x_{k+1} = (I + AT)x_k + BTu_k$$
$$y_k = Cx_k + Du_k$$

## Attention!

**Continuous-time system**

Controllable

Observable

Stable

**It's sampled version**

Controllable ?

Observable ?

Stable ?

# Criterion of controllability for discrete-time system

$$x_{k+1} = Ax_k + Bu_k \quad \textbf{(1)}$$

$$y_k = Cx_k + Du_k$$

## Controllability matrix

$$[B, AB, \ldots, A^{n-1}B]$$

## Kalman's Criterion

**The linear discrete-time system (1) is controllable if and only if the controllability matrix has rank equal to n, where n is a number of state variables.**

# Criterion of observability for discrete-time system

$$x_{k+1} = Ax_k + Bu_k \quad \textbf{(1)}$$

$$y_k = Cx_k + Du_k \quad \textbf{(2)}$$

**Observability matrix**

$$\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}$$

## Kalman’s Criterion

**The linear discrete-time system (1) with measurements (2) is observable if and only if the observability matrix has rank equal to n, where n is a number of state variables.**

# Criterion of Stability for discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

## Criterion of stability

A discrete-time LTI system is asymptotically (internally) stable if and only if $|\lambda_j| < 1$ for all $j \in 1, \ldots, s$ where $\lambda_1, \ldots, \lambda_s$ is the set of distinct eigenvalues of A.

# PID controller

**Control system**

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

# PID controller

**SISO Control system**

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$u \in \mathbb{R}, \; y \in \mathbb{R}$

# PID controller

## SISO Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$u \in \mathbb{R}, \ y \in \mathbb{R}$

## Specification

**output of closed-loop system should track the given reference trajectory:**

$$\lim_{t \to +\infty} (y_{ref}(t) - y(t)) = 0$$

# PID controller

## SISO Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

## Specification

output of closed-loop system should track the given reference trajectory:

$$\lim_{t \to +\infty} (y_{ref}(t) - y(t)) = 0$$

## PID controller

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) \mathrm{d}\tau + K_d \frac{\mathrm{d}}{\mathrm{d}t} e(t).$$

$y_{ref}(t) - y(t)$

# PID controller

## SISO Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

## Specification

output of closed-loop system should track the given reference trajectory:

$$\lim_{t \to +\infty} (y_{ref}(t) - y(t)) = 0$$

## PID controller

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) \mathrm{d}\tau + K_d \frac{\mathrm{d}}{\mathrm{d}t} e(t).$$

$y_{ref}(t) - y(t)$

# Digital PID controller

## SISO Control system

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k$$

## Specification

**output of closed-loop system should track the given reference trajectory:**

$$\lim_{k \to +\infty} \underbrace{(y_{ref,k} - y_k)}_{e_k} = 0$$

## PID controller

$$u_k = K_\mathrm{p} e_k + K_\mathrm{i} \sum_{n=1}^{k} e_n + K_\mathrm{d} \left[ e_k - e_{k-1} \right]$$

# Digital PID controller

## SISO Control system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k$$

$y_{ref,k}$

## Specification

**output of closed-loop system should track the given reference trajectory:**

$$\lim_{k \to +\infty} (y_{ref,k} - y_k) = 0$$

$e_k$

## PID controller

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## PID controller

The digital PID-controller is usually implemented using the so-called velocity form

$$u_k = u_{k-1} + K_p\left[e_k - e_{k-1}\right] + K_i e_k + K_d\left[e_k - 2e_{k-1} + e_{k-2}\right]$$

to avoid keep track of the sum

# PID: Summary

$$u_k = u_{k-1} + K_p \left[e_k - e_{k-1}\right] + K_i e_k + K_d \left[e_k - 2e_{k-1} + e_{k-2}\right]$$

## PID: Pros

- Real-Time Control
- Simple Implementation
- Tuning flexibilty

## PID: Cons

- Requires Tuning
- Wrongly tuned might be unstable
- Not Ideal for Complex Processes
- Don’t take into account state and input constraints

# Stabilisation by full feedback

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**MIMO** **Control system**

$$\dot{x} = Ax + Bu$$

$$y = x$$

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**The closed-loop system should be asymptotically stable**

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## Closed-loop system

$$\dot{x} = (A - BK)x$$

**Theorem (Eigenvalue assignment — MIMO).** All eigenvalues of $(\mathbf{A} - \mathbf{BK})$ can be assigned arbitrarily (provided complex eigenvalues are assigned in conjugated pairs) by selecting a real constant $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

**To make closed-loop system stable assign eigenvalues with negative real part**

# Digital full feedback regulator

## MIMO Control system

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k$$

## Specification

**The closed-loop system should be asymptotically stable**

$$\lim_{k->+\infty} \|x_k\| = 0$$

**Linear Full-State Feedback Controller:** $u_k = -Kx_k$

**Closed-loop system**

$$x_{k+1} = (A - BK)x_k$$

**Theorem (Eigenvalue assignment — MIMO).** All eigenvalues of $(\mathbf{A} - \mathbf{BK})$ can be assigned arbitrarily (provided complex eigenvalues are assigned in conjugated pairs) by selecting a real constant $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

**To make closed-loop system stable assign eigenvalues, s.t.** $|\lambda_i| \leq 1, i = 1, \ldots n$

# Stabilisation by dynamic feedback

**MIMO Control system**

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

**The closed-loop system should be asymptotically stable**

$$\lim_{t->+\infty} \|x(t)\| = 0$$

# Stabilisation by dynamic feedback

**MIMO Control system**

$$\dot{x} = Ax + Bu$$
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**The closed-loop system should be asymptotically stable**

$$\lim_{t -> +\infty} \|x(t)\| = 0$$

**Luenberger Observer:** $\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu$

**Feedback controller:** $u = K\hat{x}$

# Stabilisation by dynamic feedback

**MIMO Control system**

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**Luenberger Observer:** $\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu$

**Feedback controller:** $u = K\hat{x}$

If pair (A, B) is controllable we can choose $K$, such that for all $\lambda_i \in eig(A - BK)$ we have $Re(\lambda_i) < 0$.

If pair (A,C) is observable we can choose $L$, such that for all $\lambda_i \in eig(A - LC)$ we have $Re(\lambda_i) < 0$.

# Stabilisation by dynamic feedback

**MIMO Control system**

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k$$

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If pair (A,C) is observable we can choose $L$, such that for all $\lambda_i \in eig(A - LC)$ we have $|\lambda_i| < 1$.

# LQR: continuous system

For a continuous-time linear system described by:

$$\dot{x} = Ax + Bu$$

with a cost function defined as:

$$J = \int_0^\infty \left(x^T Q x + u^T R u + 2x^T N u\right) dt$$

the feedback control law that minimizes the value of the cost is:

$$u = -Kx$$

where $K$ is given by:

$$K = R^{-1}(B^T P + N^T)$$

and $P$ is found by solving the continuous time algebraic Riccati equation:

$$A^T P + PA - (PB + N)R^{-1}(B^T P + N^T) + Q = 0$$

# LQR: discrete system

For a discrete-time linear system described by:

$$x_{k+1} = Ax_k + Bu_k$$

with a performance index defined as:

$$J = \sum_{k=0}^{\infty} \left( x_k^T Q x_k + u_k^T R u_k + 2x_k^T N u_k \right)$$

the optimal control sequence minimizing the performance index is given by:

$$u_k = -F x_k$$

where:

$$F = (R + B^T P B)^{-1} (B^T P A + N^T)$$

and $P$ is the unique positive definite solution to the discrete time algebraic Riccati equation (DARE):

$$P = A^T P A - (A^T P B + N) \left( R + B^T P B \right)^{-1} (B^T P A + N^T) + Q.$$

# Why LQR is “better” than PID?

- It can handle multiple-input multiple-output (MIMO) systems.
- It is an optimal control,  taking into account the system dynamics and control effort. This can lead to better performance and efficiency compared to PID, which focuses on reducing error but doesn't optimize a specific criterion.
- LQR more robust than PID in uncertain environments.

# What are the limitations?

**Don't take into account state and input constraints!**

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$$\lim_{t->+\infty} \|x(t)\| = 0$$

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**but, also ensure**

$$x \in \mathcal{X}, \;\; u \in \mathcal{U}$$

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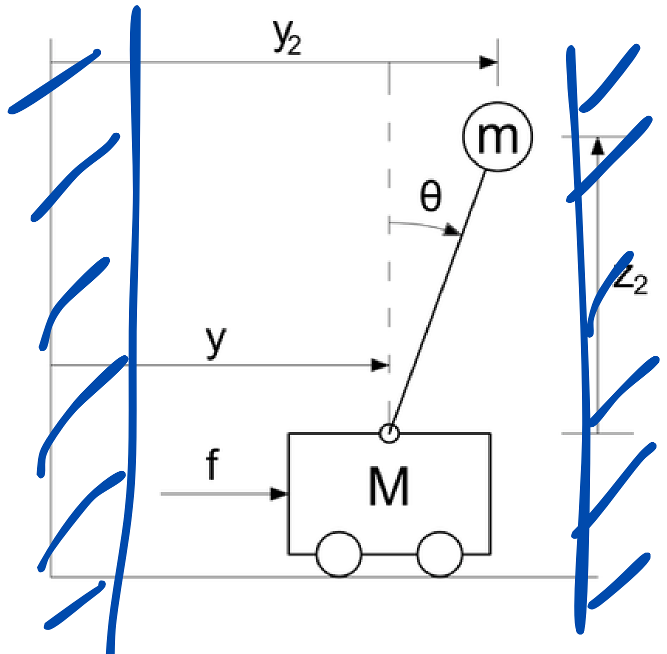


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$$x \in \mathcal{X}, \quad u \in \mathcal{U}$$