Controllability and Observability

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Course grade breakdowns

Labs - 40%

Final test - 30%

Final project - 30 %

PID controller

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \dot{e}(t)$$

where e(t) = r(t) - y(t)

Proportional (P) Control:
Effect: Faster response but steady-state error remains.

Integral (I) Control:
Effect: Improves accuracy but may cause overshoot

Derivative (D) Control:
Effect: Reduces overshoot and improves stability.

PID: Pros

Stability

PID controllers are capable of providing stable and accurate control over systems, ensuring that they reach and maintain the desired setpoint efficiently.

Tuning Flexibility

PID controllers offer flexibility in tuning parameters (Proportional, Integral, and Derivative gains) to achieve optimal performance for different systems and operating conditions.

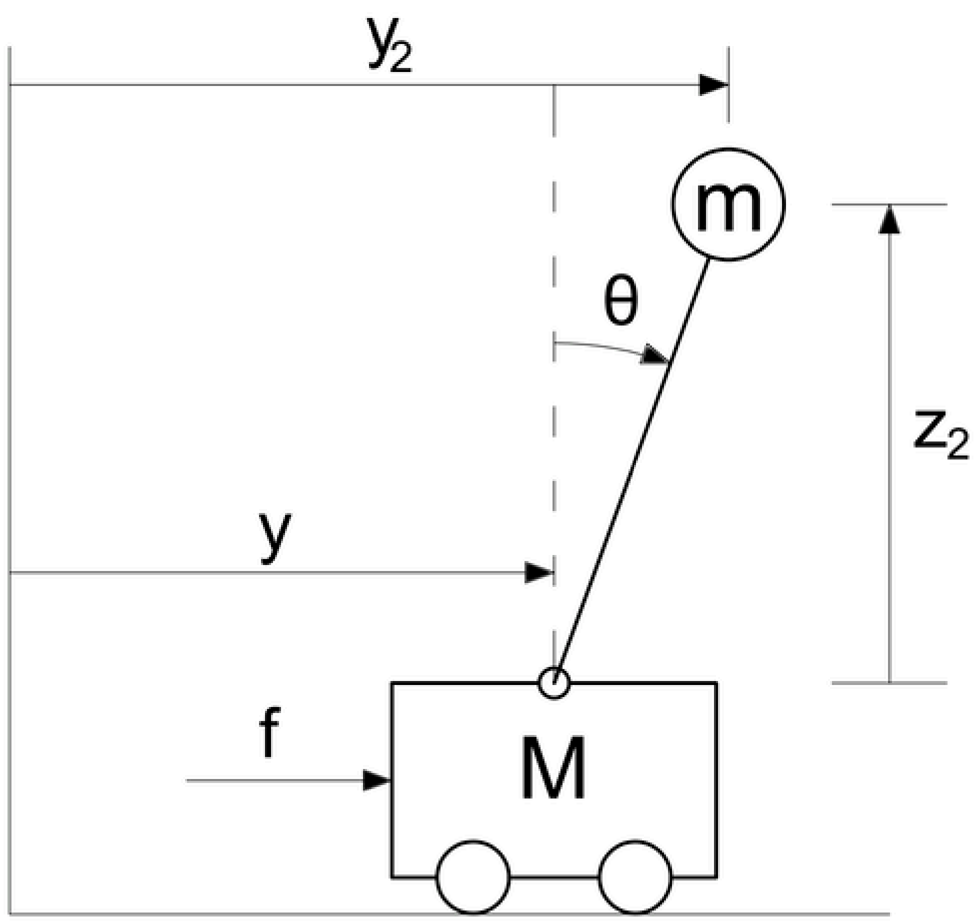
Simple Implementation

Compared to more complex control algorithms, PID controllers are relatively simple to implement, making them suitable for a wide range of applications and accessible to engineers and technicians with basic control theory knowledge.

Real-Time Control

PID controllers are well-suited for real-time control applications due to their simplicity and efficiency, making them suitable for controlling systems with fast response

Cart-pole control

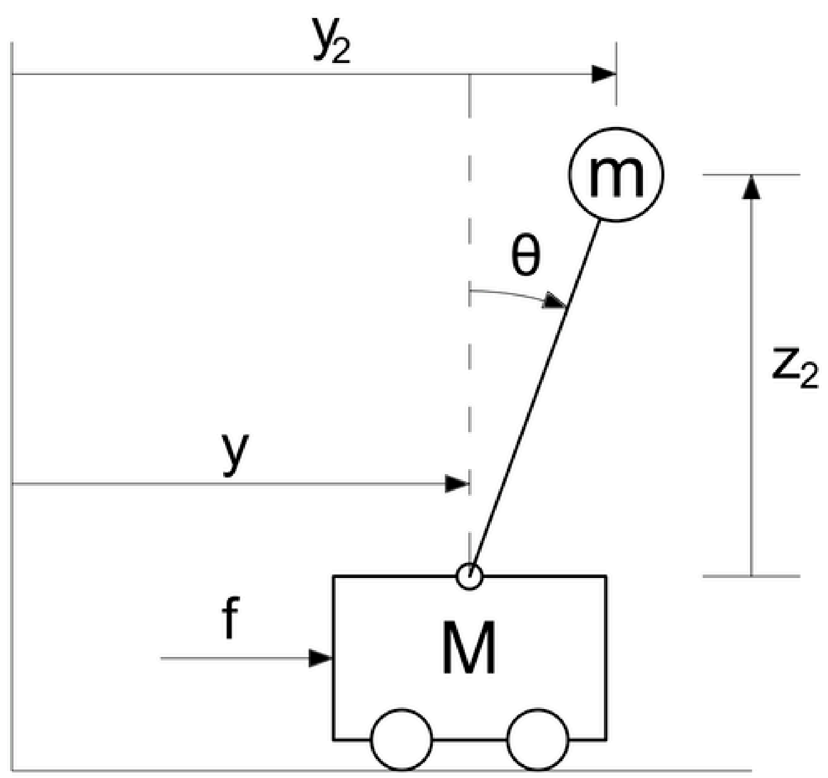


Inverted pendulum on the cart can be modeled as follows

$$(M+m)\ddot{y} + b\dot{y} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin(\theta) = F$$

$$ml\cos(\theta)\ddot{y} + (I+ml^2)\ddot{\theta} - mgl\sin\theta = 0$$

Cart-pole control



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where $F = u + w$, i.e. control + disturbance

Or in canonical state space ODE form

$$\begin{cases} \dot{y} = y_1 \\ \dot{y_1} = \dfrac{-m^2l^2g\cos\theta\sin\theta + (I+ml^2)(ml\theta_1^2\sin\theta + F - by_1)}{(I+ml^2)(M+m) - m^2l^2\cos^2\theta} \\ \dot{\theta} = \theta_1 \\ \dot{\theta_1} = \dfrac{(M+m)mgl\sin\theta + by_1ml\cos\theta - m^2l^2\theta_1^2\cos\theta\sin\theta - mlF\cos\theta}{(M+m)(I+ml^2) - m^2l^2\cos^2\theta} \end{cases}$$

Cart-pole control

Lineralized model

$$\underbrace{\begin{bmatrix} \dot{y} \\ \dot{y}1 \\ \dot{\theta} \\ \dot{\theta}_1 \end{bmatrix}}_{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{-gm^2 l^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} y \\ y_1 \\ \theta \\ \theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{-ml}{I(M+m)+Mml^2} \end{bmatrix} (u + w)$$

Design a PID controller such that

$$\theta(t) \to 0,$$

i.e. $C = [0\ 0\ 1\ 0]$, $y = Cx$, $x = \begin{bmatrix} y \\ y_1 \\ \theta \\ \theta_1 \end{bmatrix}$

Cart-pole control

$$\dot{x} = Ax + Bu + Dw$$
$$y = Cx$$
$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \dot{e}(t)$$

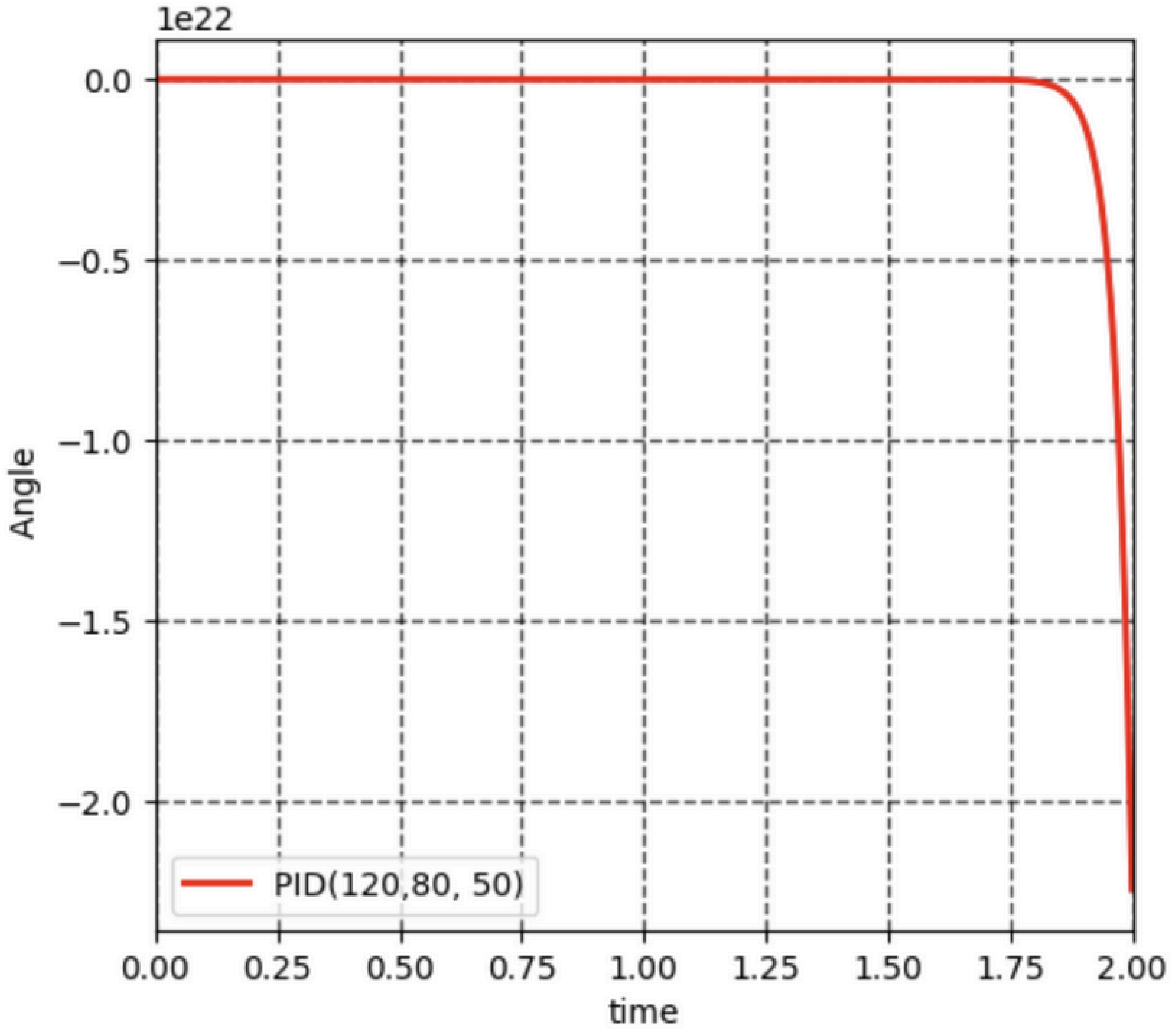
where e(t) = r(t) - y(t)

Let us tune PID for

$x_0 = (0, 0, 0, 0)'$

$W = 1.0$

Cart-pole control



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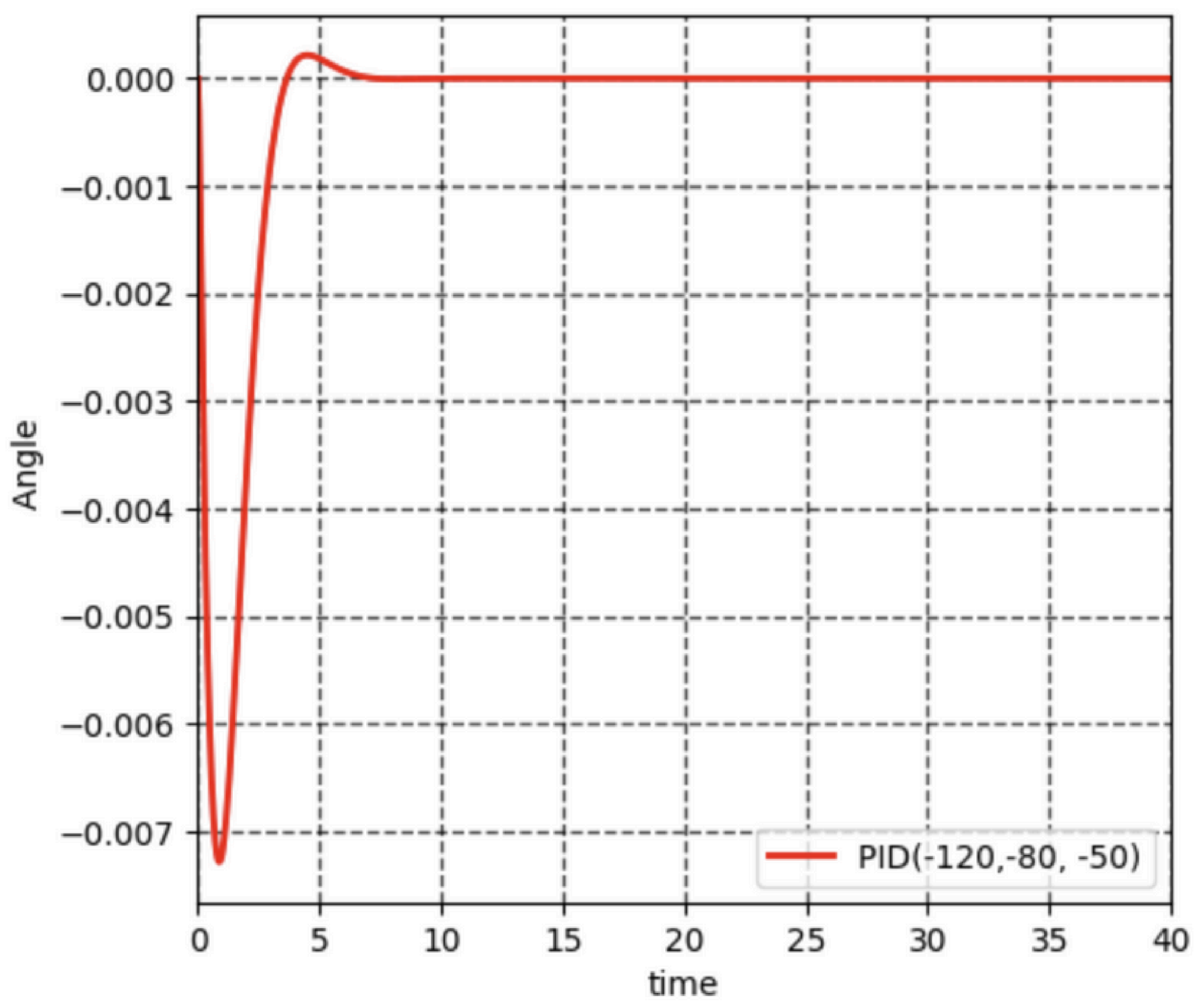


Hm

$K_p = -120$

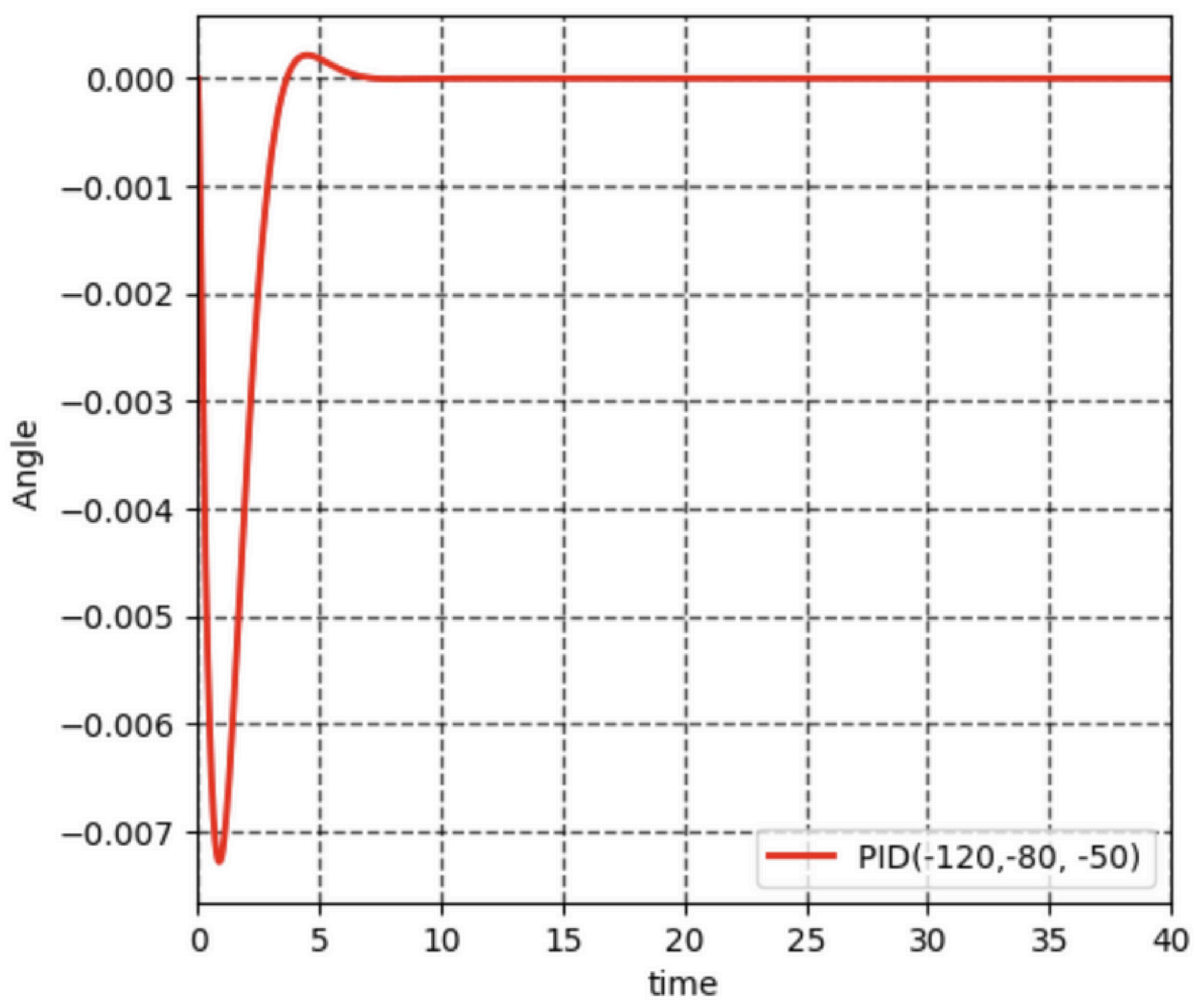
$K_i = -80$

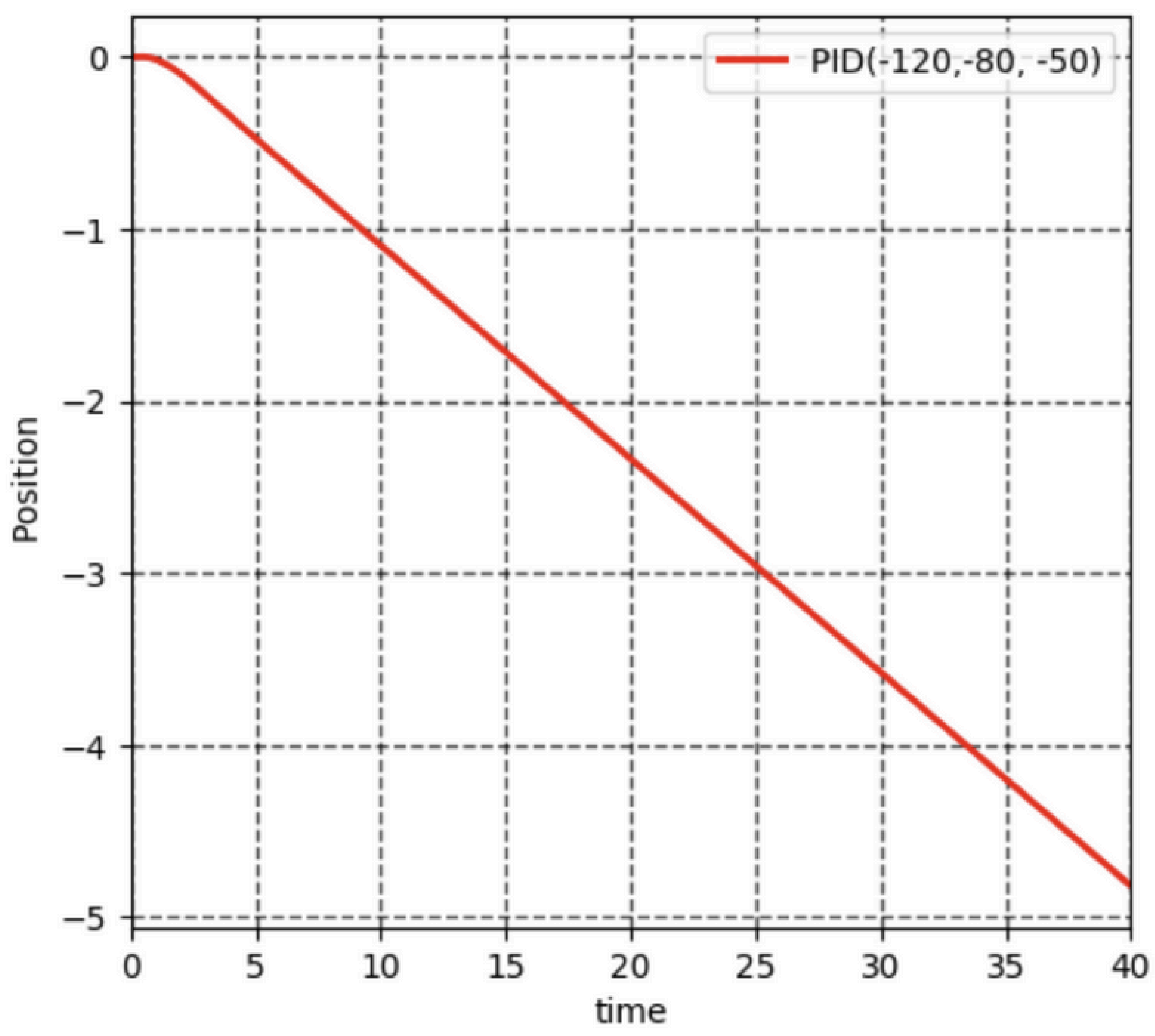
$K_d = -50$

seems to be an option...

Cart-pole control

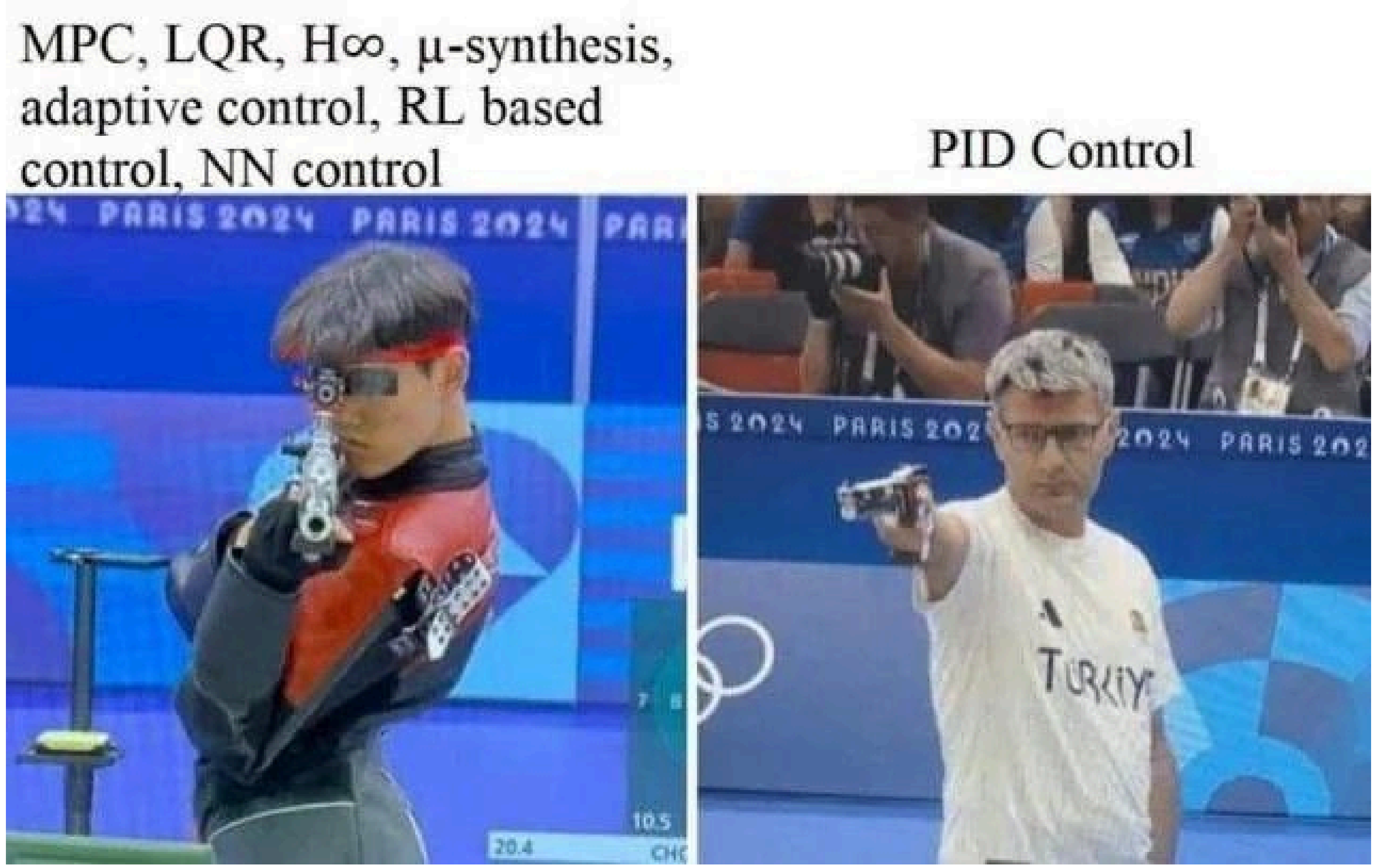
$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \dot{e}(t)$$





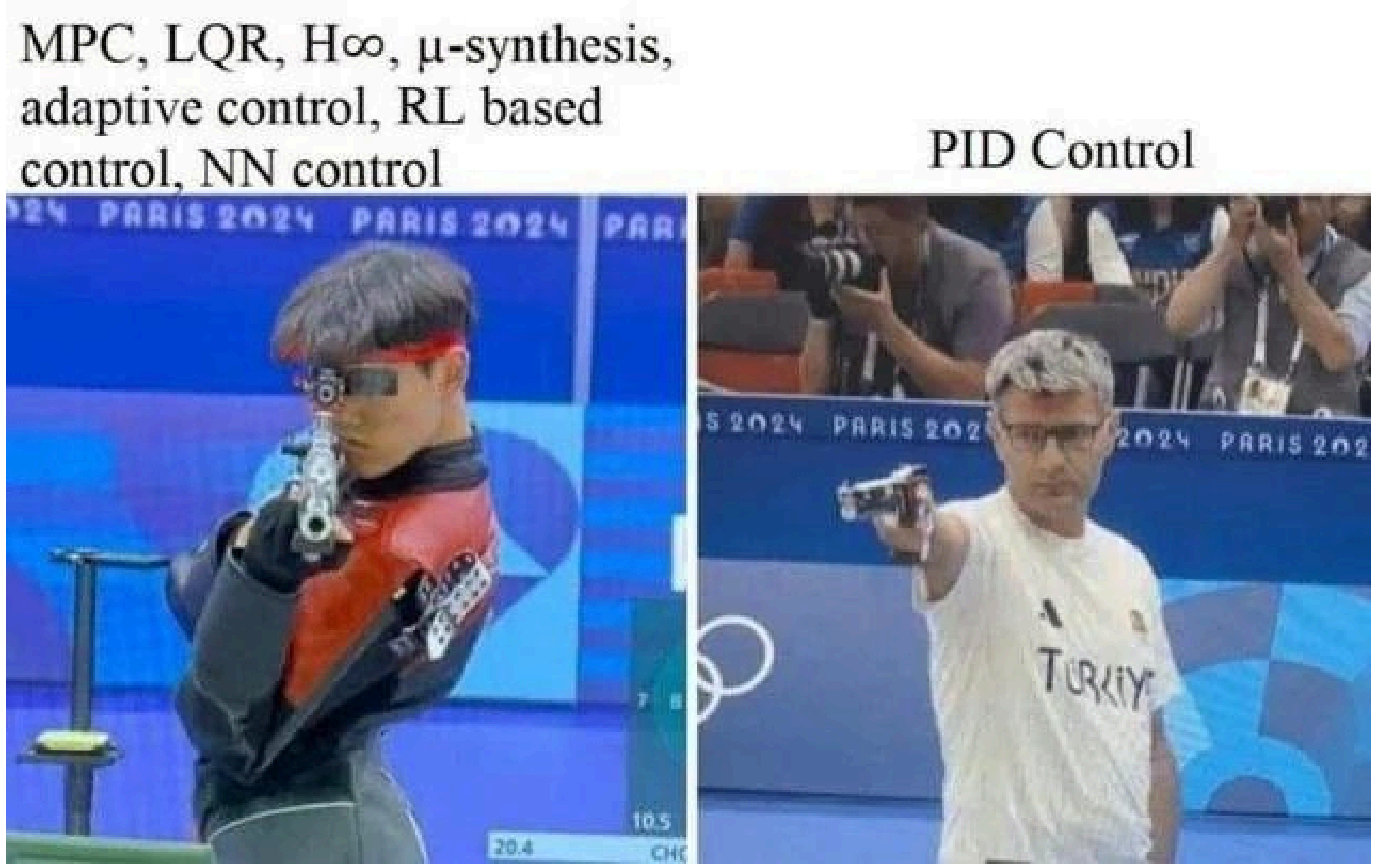
The controller keeps pendulum in up right position, but position of the cart goes to infinity....

PID: Pros



PID is easy to implement, real-time controller which works for many industrial challendges

PID: Cons

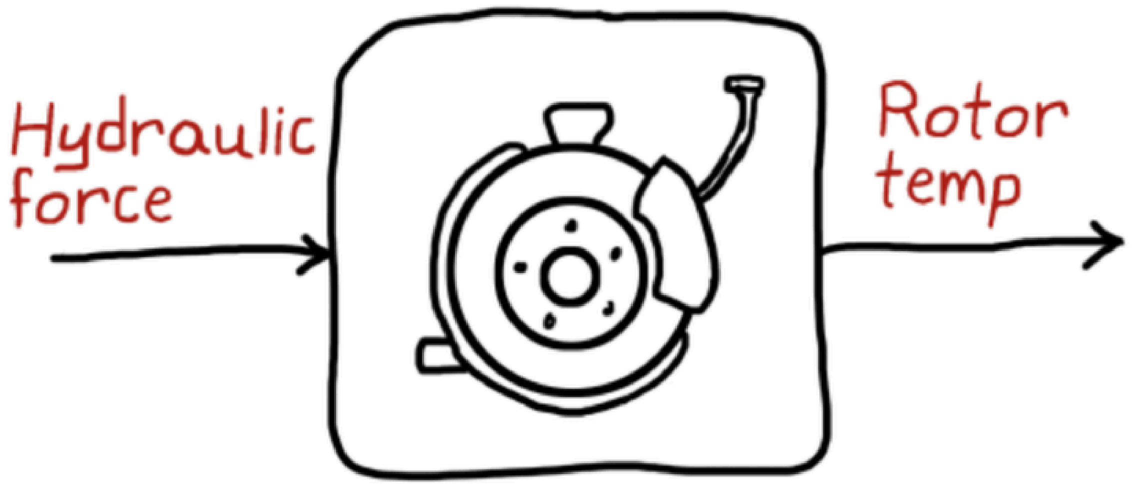


PID controllers do not work well when system is unstable or non-linear

PID controllers were designed for single input single output system, while many real-world examples are multi inputs multi outputs systems

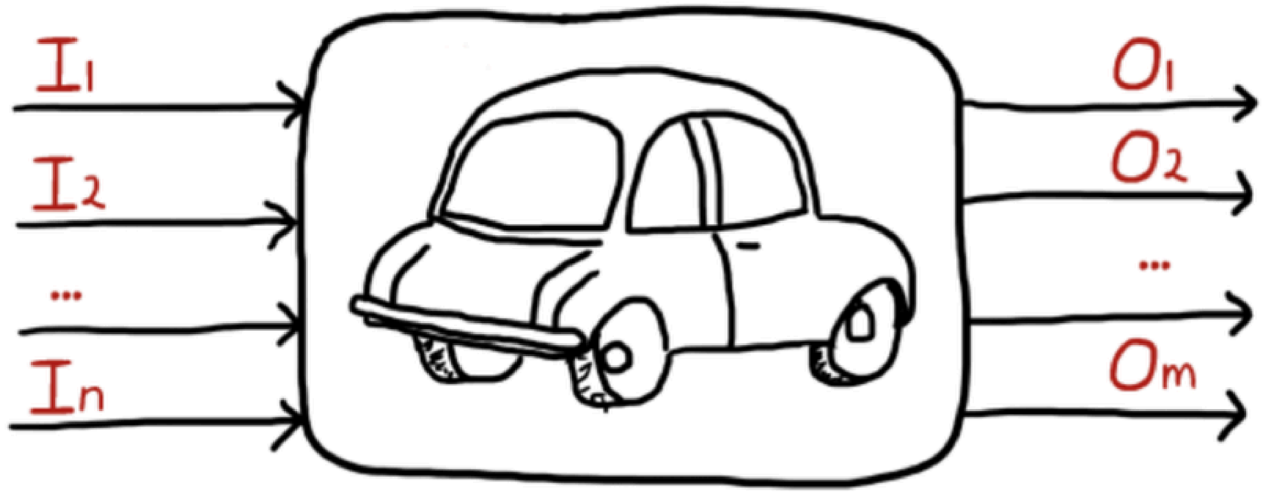
SISO system VS MIMO system

SISO



Single Input Single Output

MIMO



Multiple Inputs Multiple Outputs

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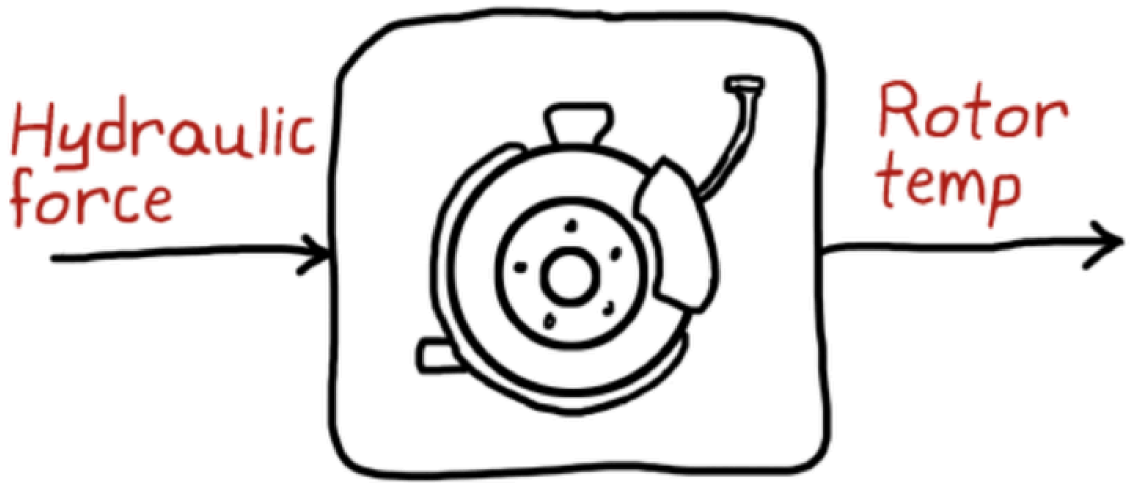

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Multiple Inputs Multiple Outputs

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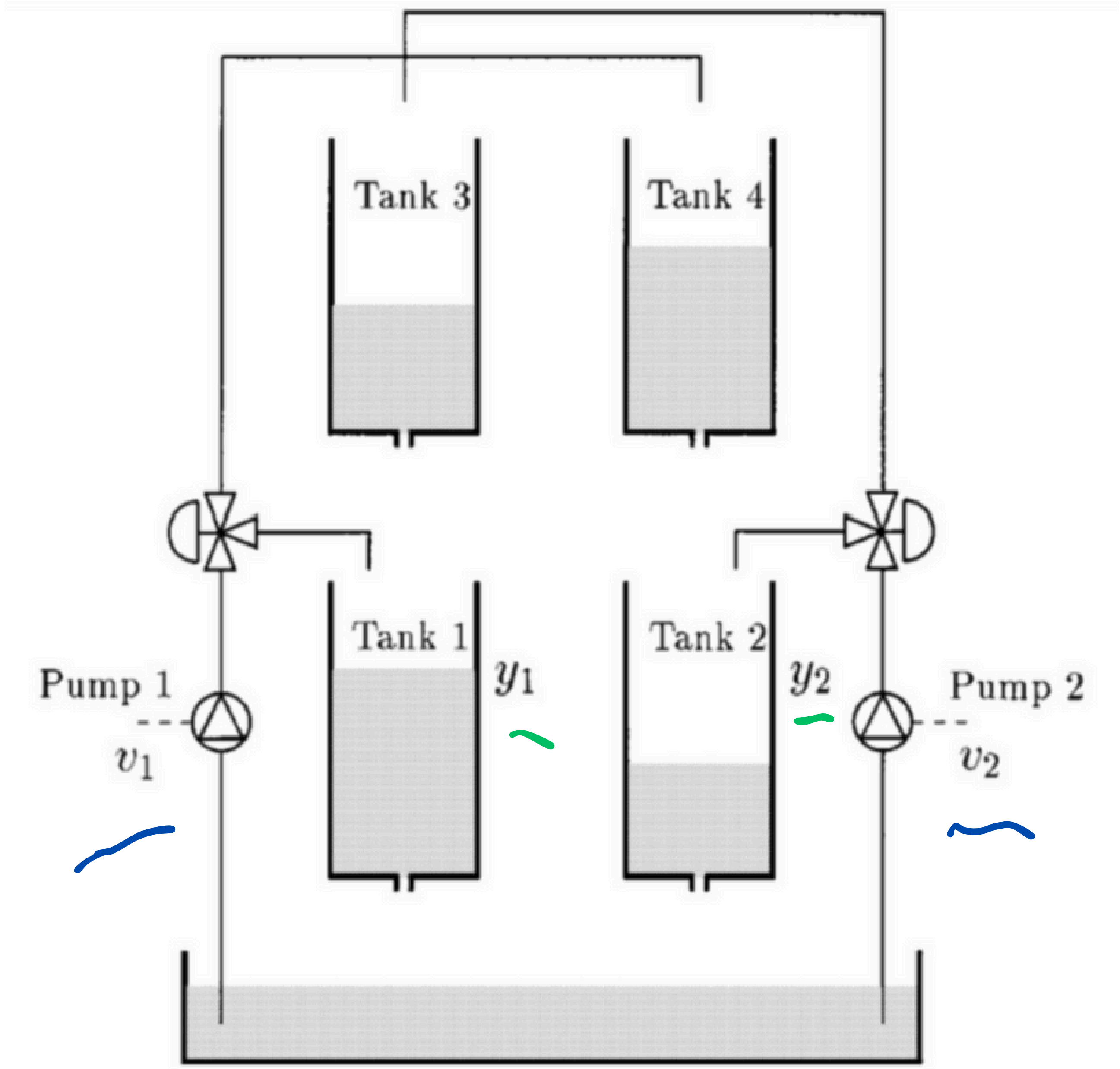
$$

Design a PID controller such that

$\theta(t) \to 0$, and $y(t) \to 0$

i.e. $C = [0\ 0\ 1\ 0]$, $[1\ 0\ 0\ 0]$, $y = Cx$, $x = \begin{bmatrix} y \\ y_1 \\ \theta \\ \theta_1 \end{bmatrix}$

Quadruple-Tank Process



The process inputs are

v_1, v_2

(input voltages to the pumps)

the outputs

y_1, y_2

(water levels level measurement devices).

The target is to control the level in the lower two tanks with two pumps.

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \rightarrow \begin{bmatrix} y_1^{ref}(t) \\ y_2^{ref}(t) \end{bmatrix}$$

Paris unveils massive underground water storage basin to clean up Seine River ahead of Olympics

Paris 2024: Why the Seine's high flow rate threatens the Games' opening ceremony

Heavy spring and summer rainfall has swelled the river and its tributaries, as well as the four artificial reservoirs responsible for regulating them. The ceremony scheduled for July 26 could be adapted.

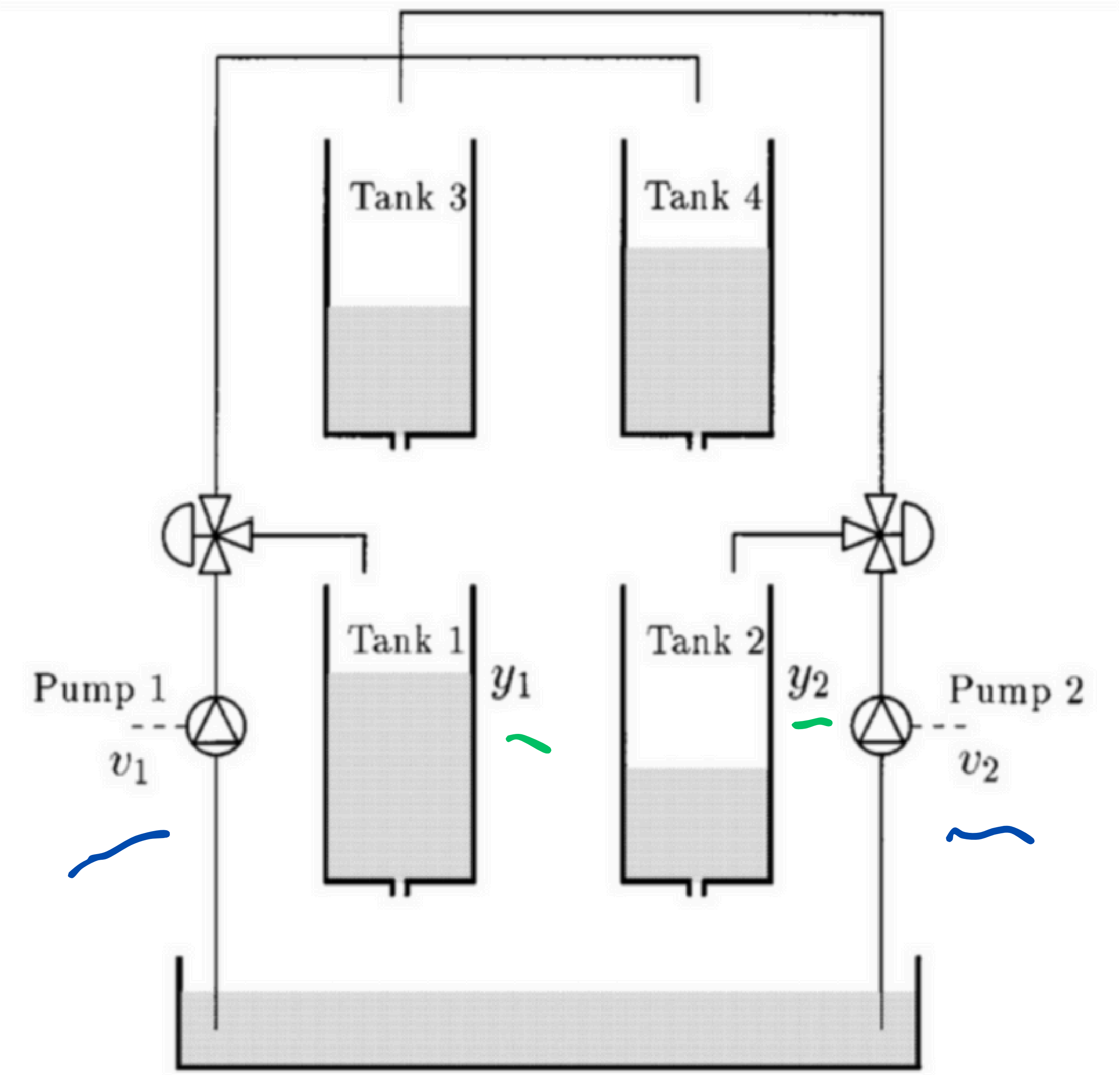
By Nicolas Lepeltier
Published on July 12, 2024, at 5:30 am (Paris) • 4 min read • Lire en français

What did Paris do to clean up?

To prepare for the Paris Games, the city built a giant basin to capture excess rainwater and keep untreated waste from flowing into the river, renovated the sewage system and upgraded water treatment plants.

Heavy rain may still swamp the system.

Quadruple-Tank Process



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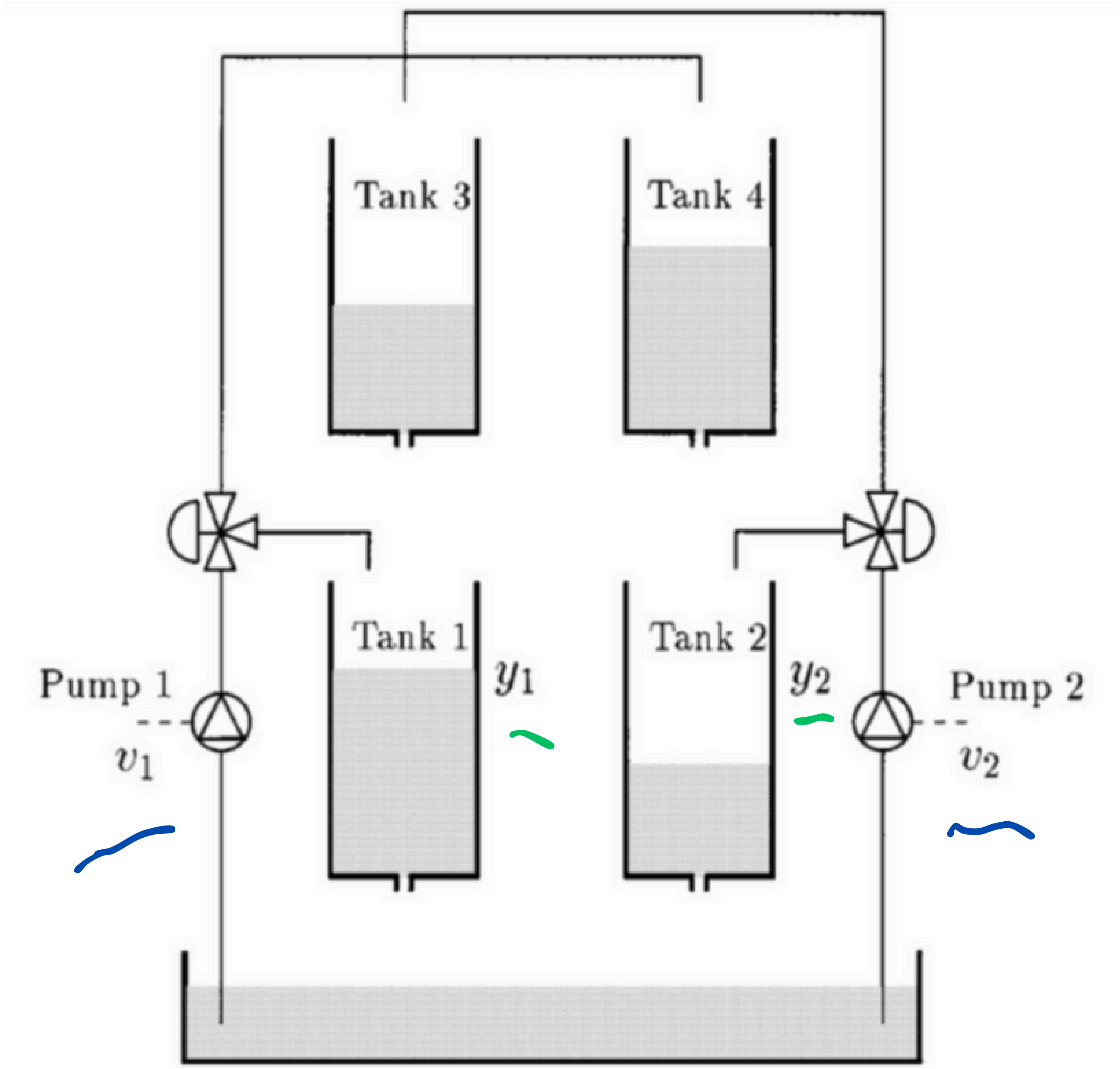

Could we design such a controller?
And what if one of the pumps are broken?

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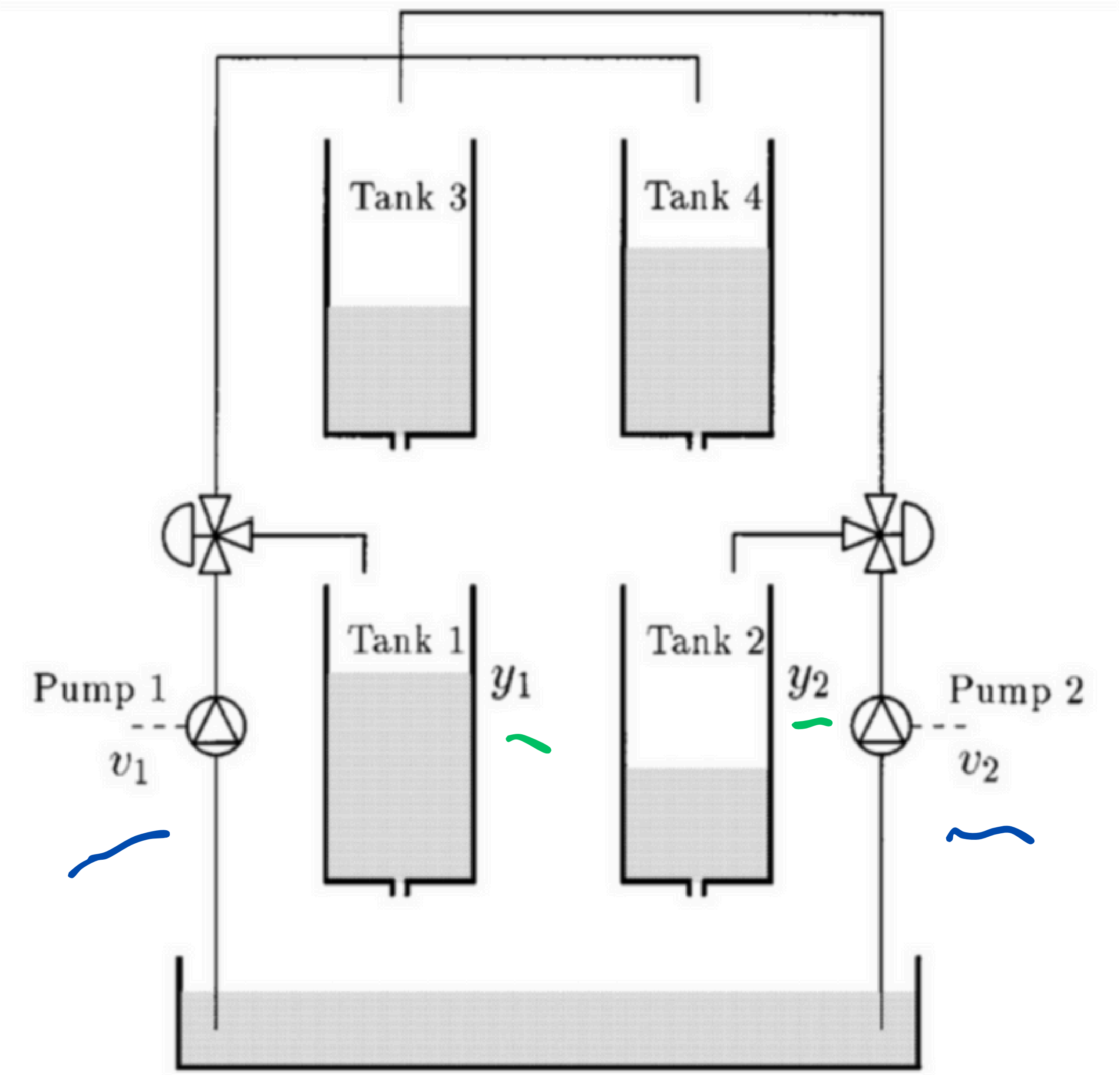

Could we design such a controller?
And what if one of the pumps are broken?

Should we measure the level of the water in all for tanks? Or it is enough to measure the water level only in two lover tanks?

The target is to control the level in the lower two tanks with two pumps.

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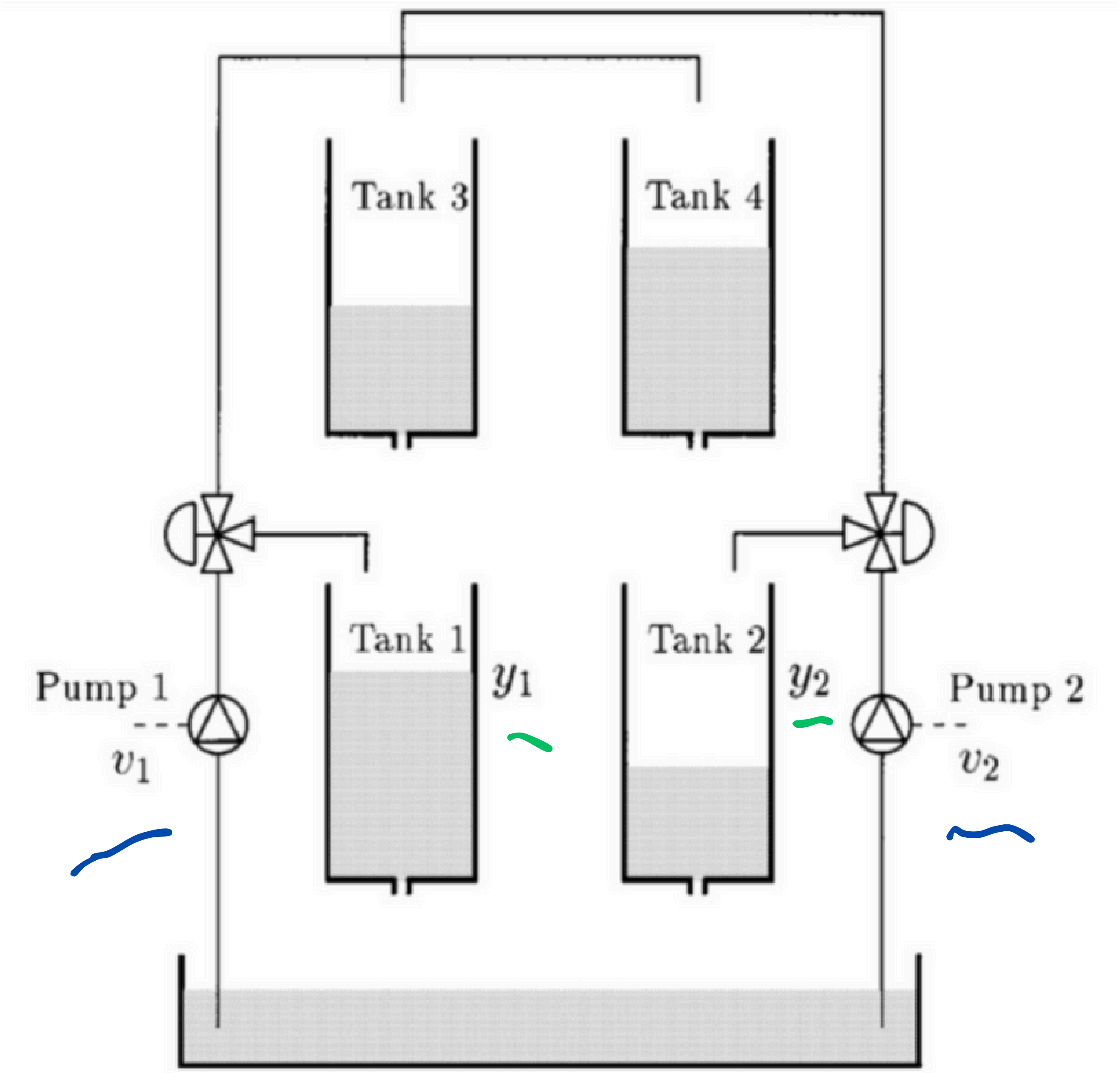

Could we design such a controller?
And what i[illegible]re broken?

Controllability

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Observability

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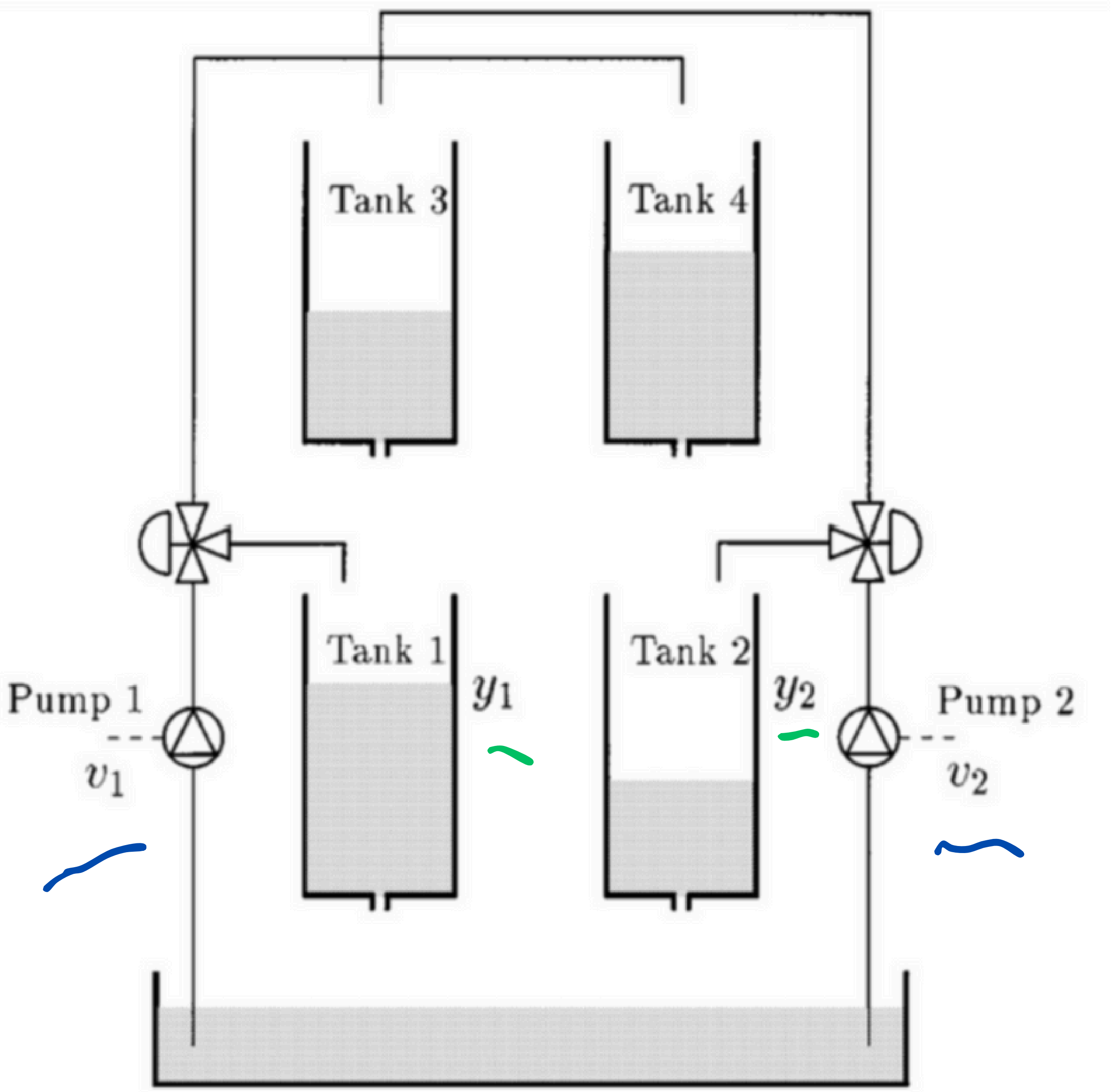

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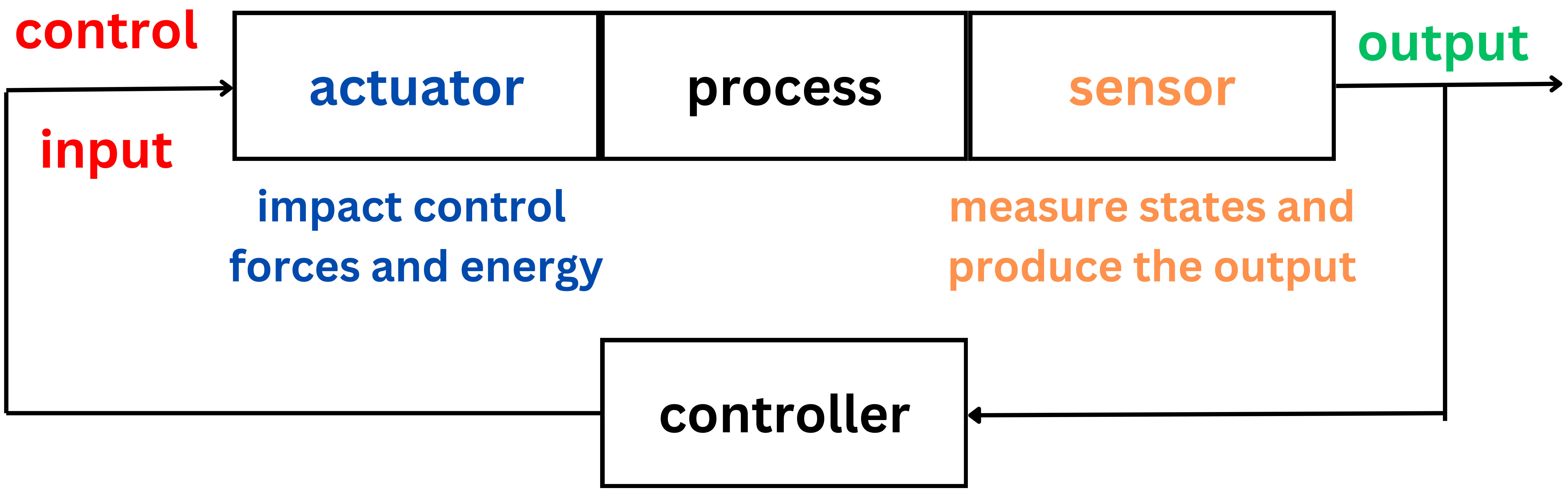
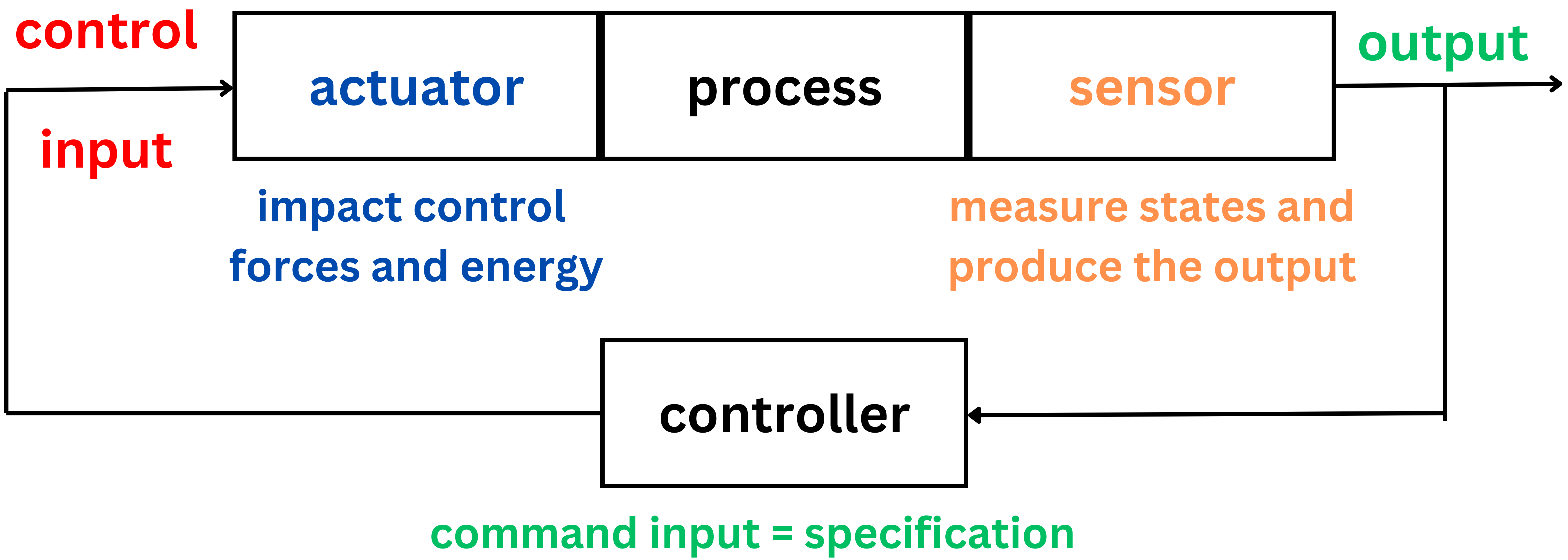
Controllability & Observability



command input = specification

To design controller

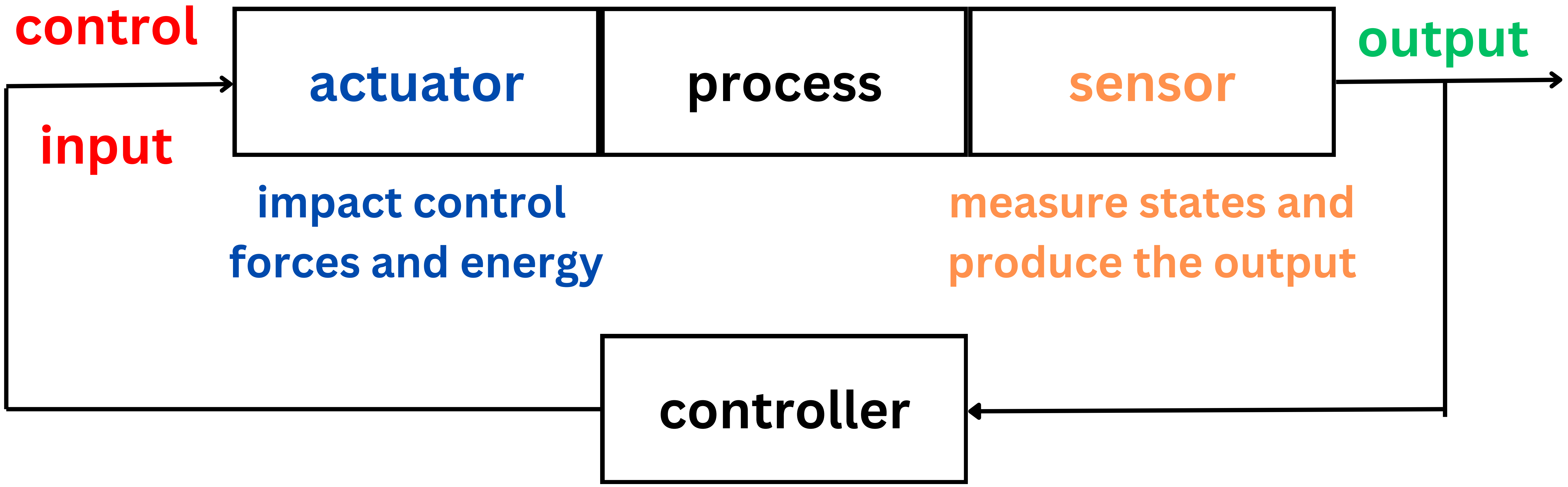
Controllability & Observability



To design controller

you need to be able to
influence the system

Controllability & Observability



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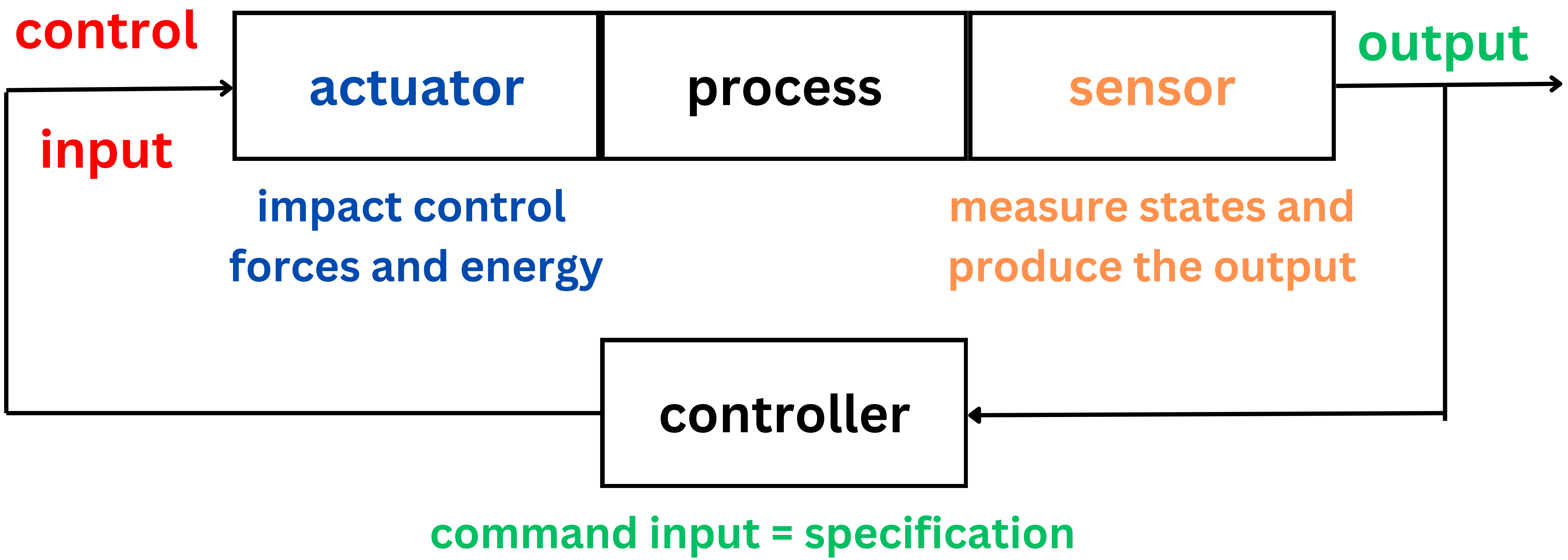


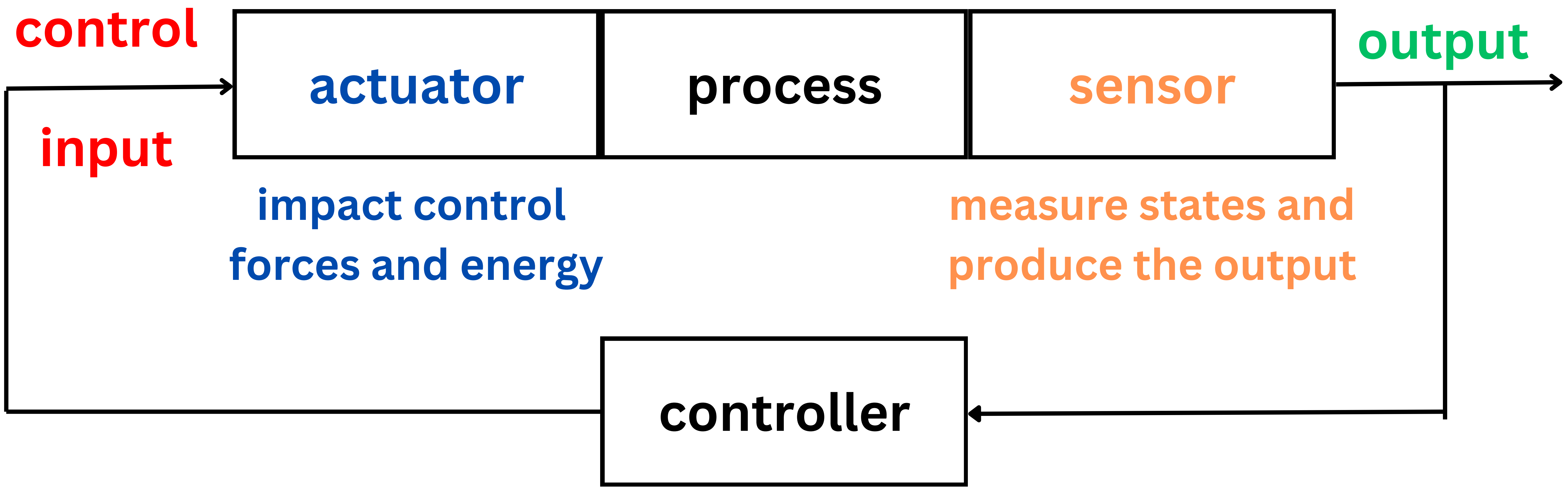
Controllable

To design controller

you need to be able to influence the system

and know it's changing

Controllability & Observability



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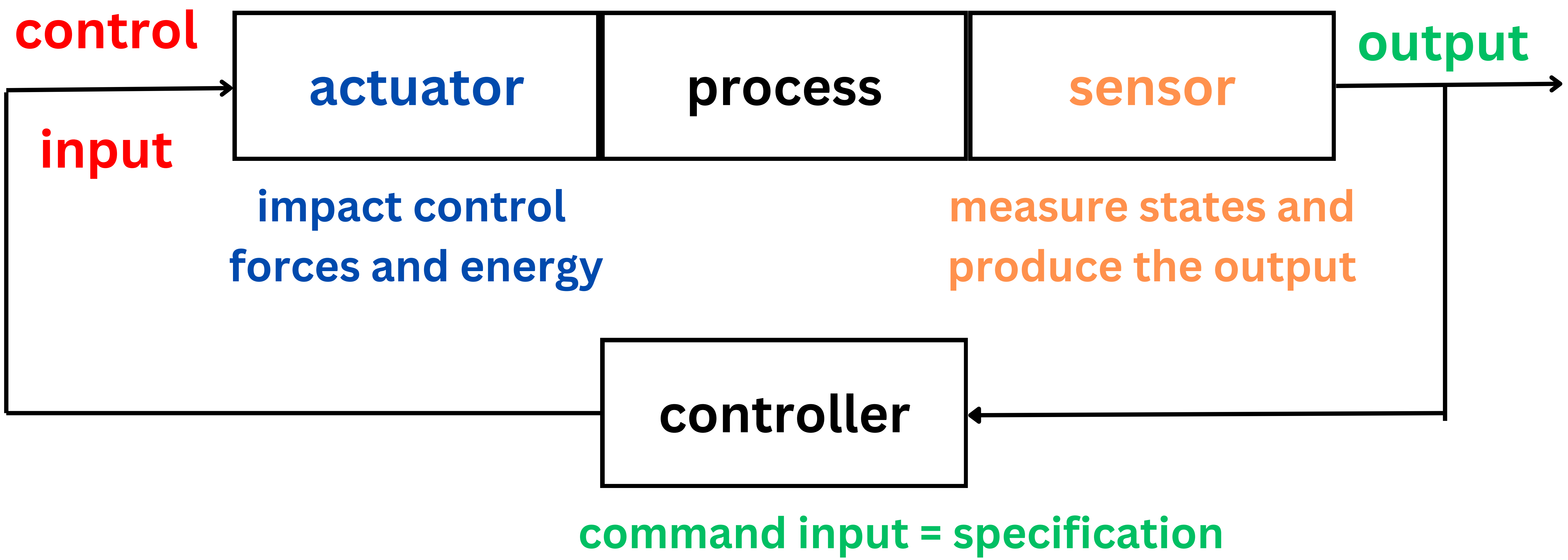
Controllability & Observability



Controllability and observability are conditions of how the system works with the actuators and sensors, and it's not tied to a specific control technique

Controllability

Controllability (null reachability) **means that there exists control signal which allows the system to move from any any initial state to any final state in a finite time interval**

Controllability

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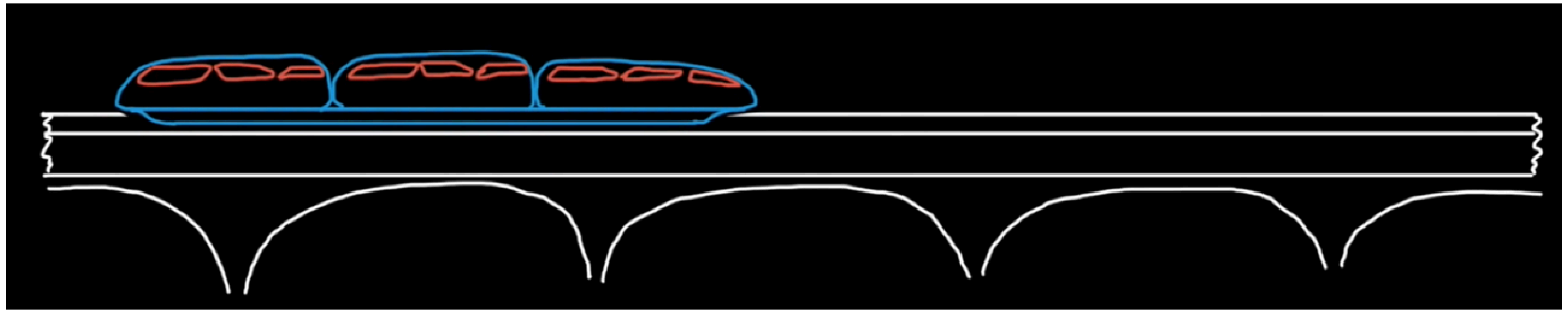
Monorail

$$\dot{v} = u$$

$$\dot{p} = v$$



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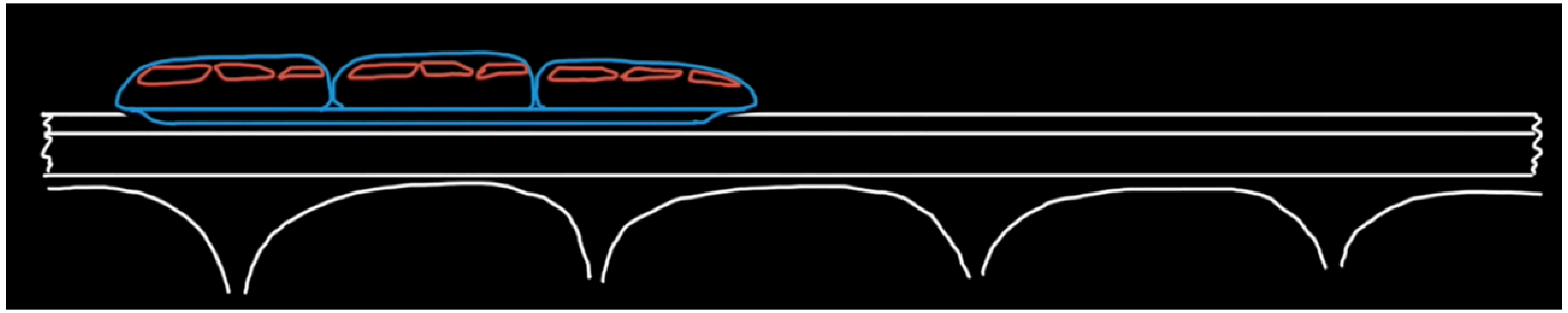

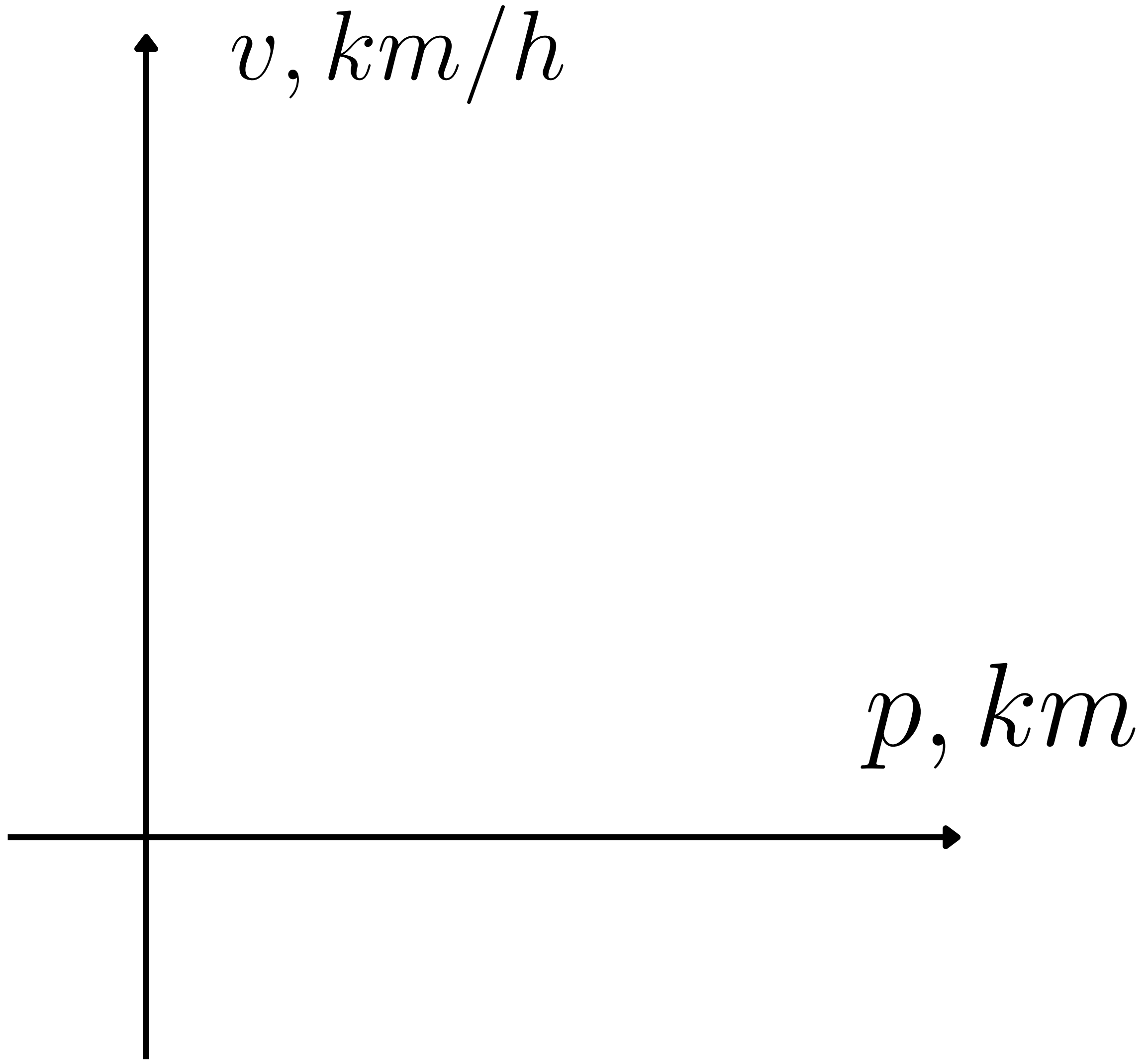

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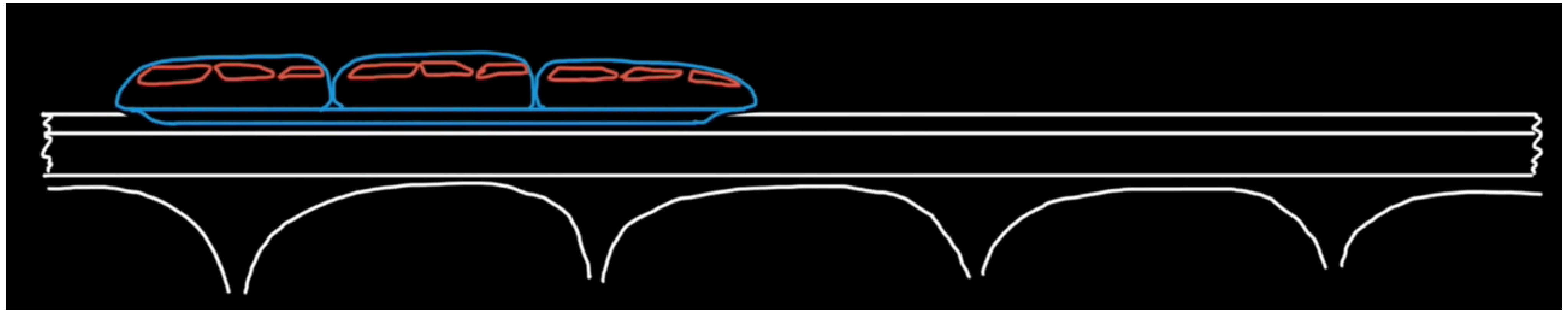

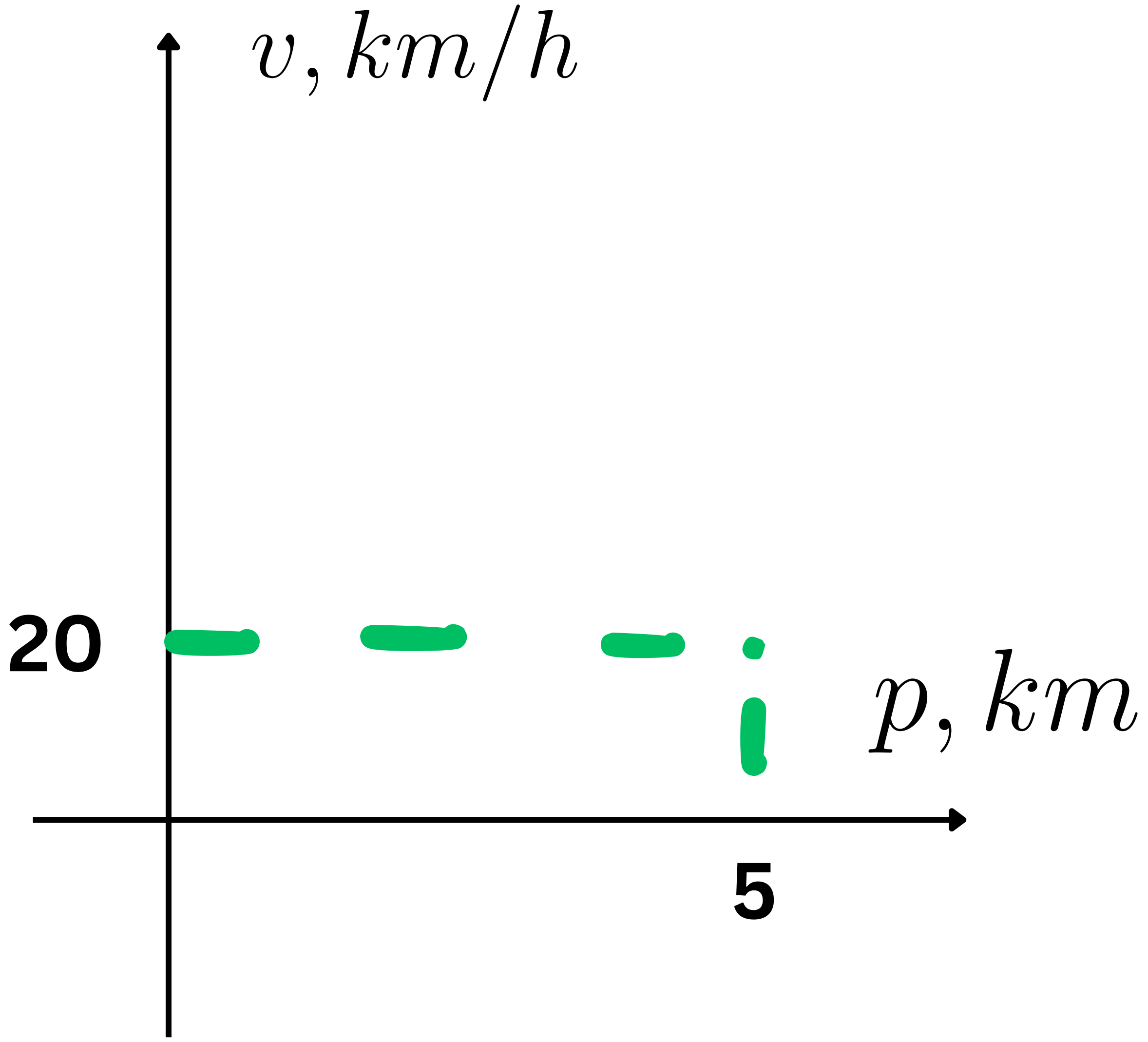

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Monorail

$$\dot{v} = u$$

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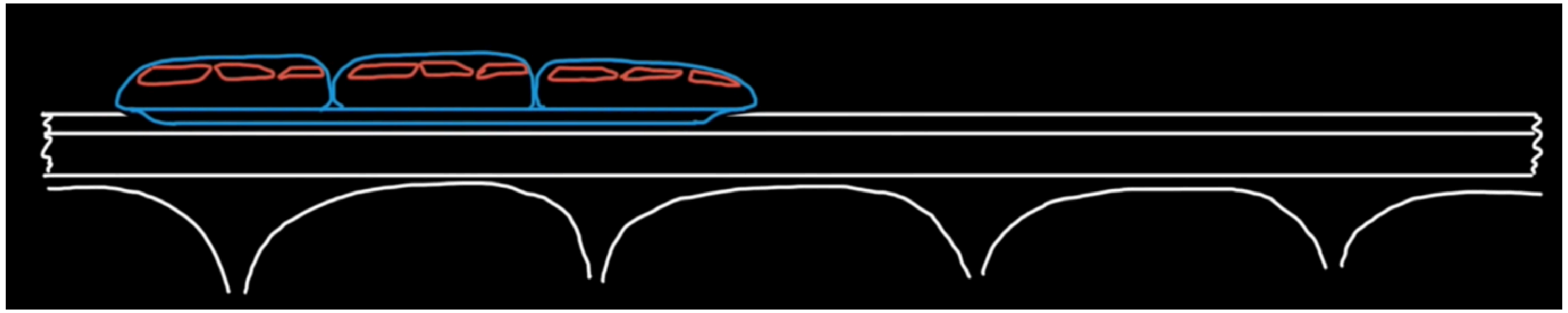
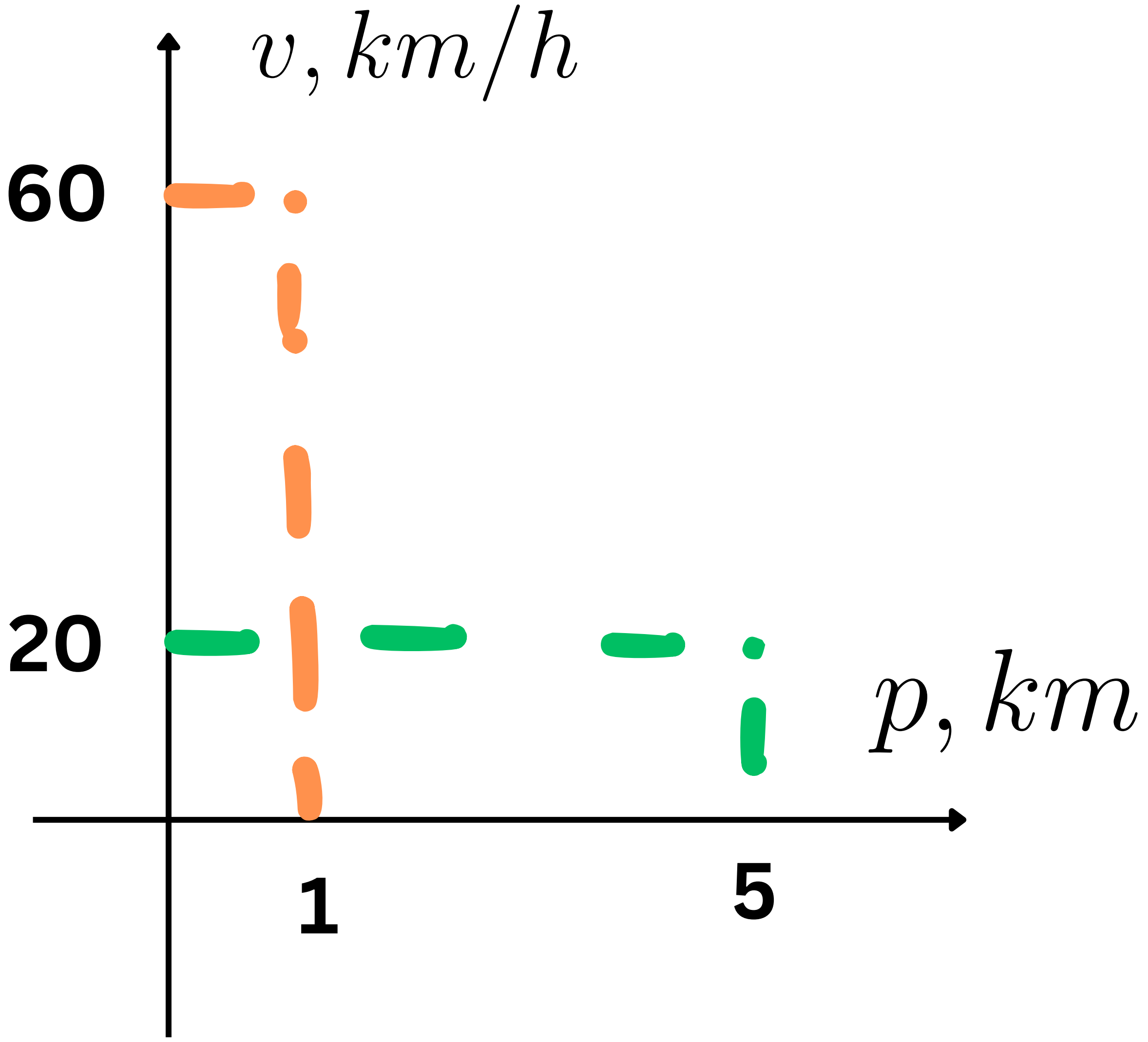
controllability does not mean that the state must be maintained, only that it can be reached...

even if infinite amount of energy is required for that....





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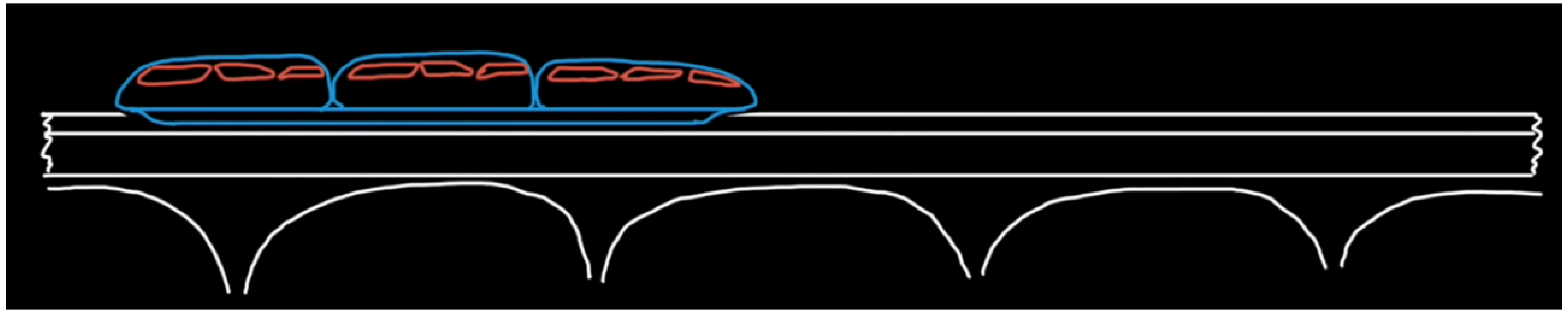

Example of uncontrollable system

imagine we lost control of gaz pedal

$$\dot{v} = 0\,u$$

$$\dot{p} = v$$

Observability

Observability means that all states can be known from the outputs of the system

Monorail

$$\dot{v} = u$$

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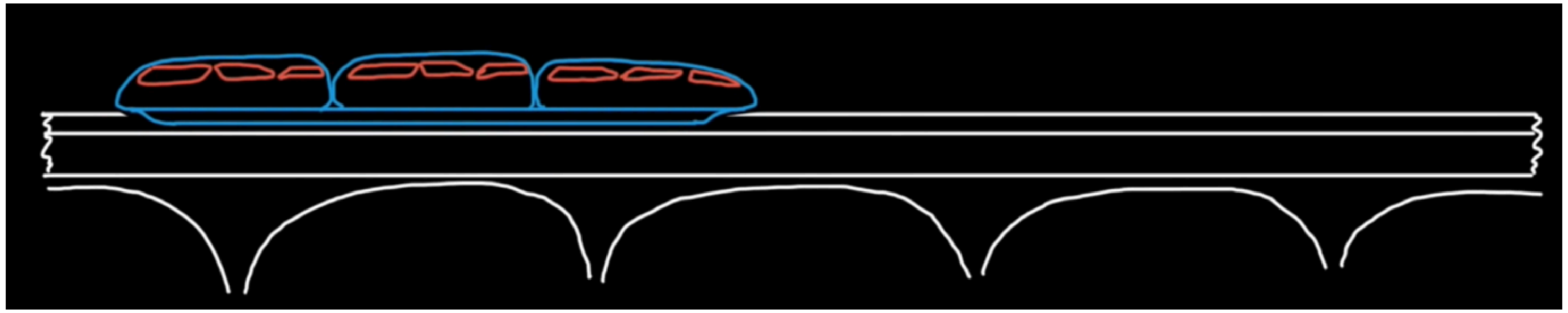

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impractical to know every state of the system

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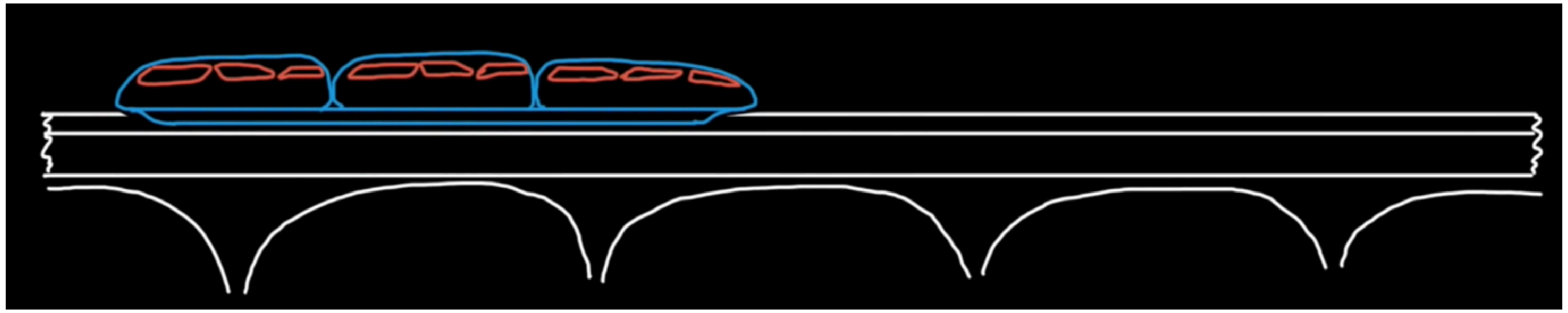

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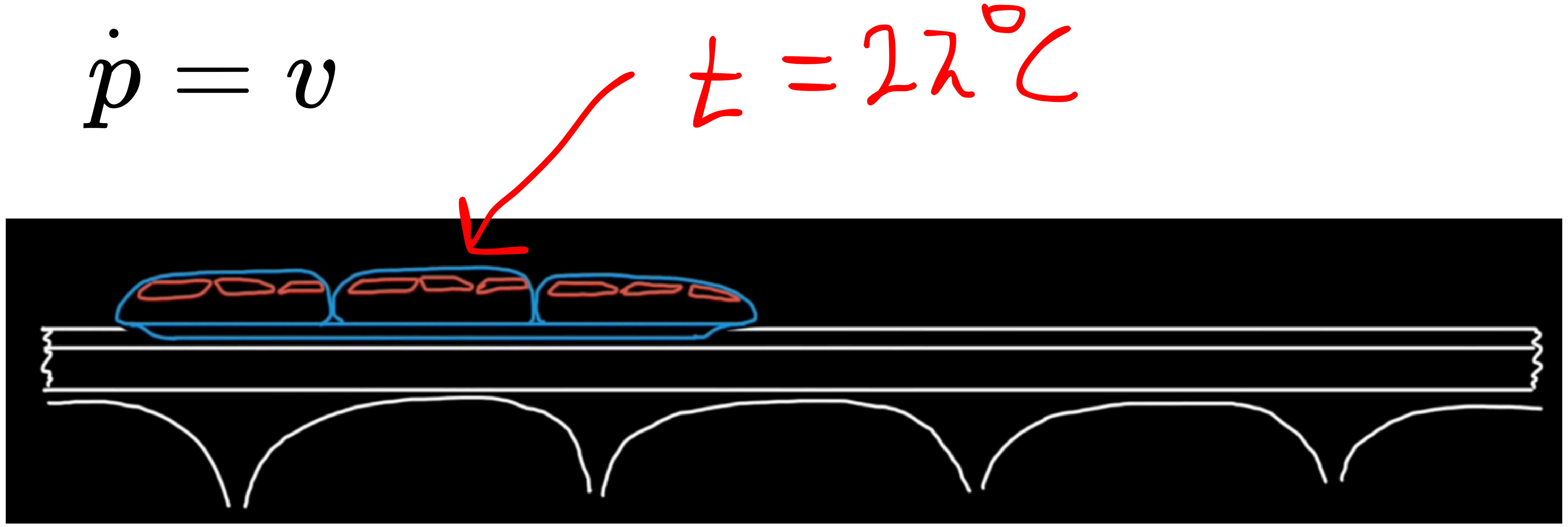


Observability

Observability means that all critical states can be known from the outputs of the system

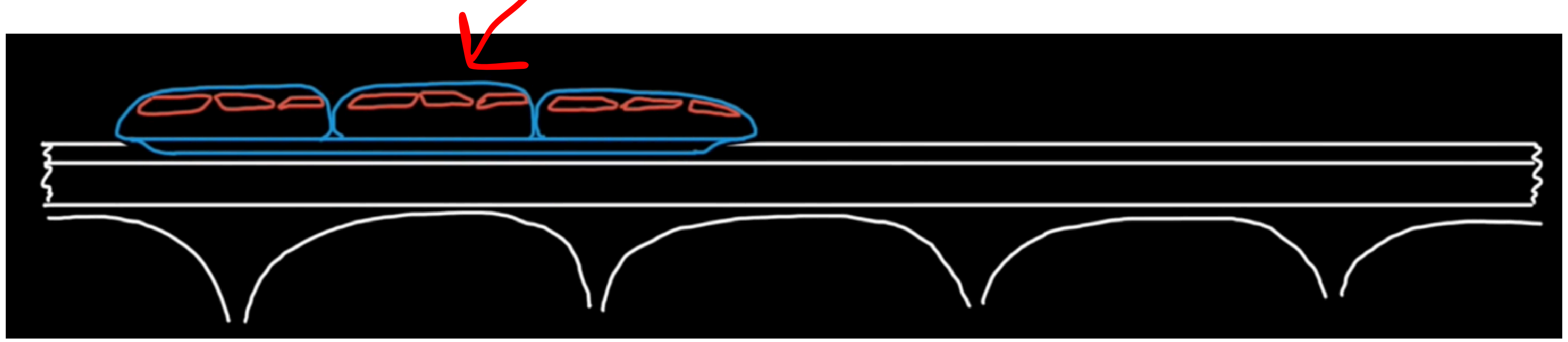
Monorail

$$\dot{v} = u$$

$$\dot{p} = v$$

most states don't impact the system in any meaningful way

$t = 22^\circ C$



Observability

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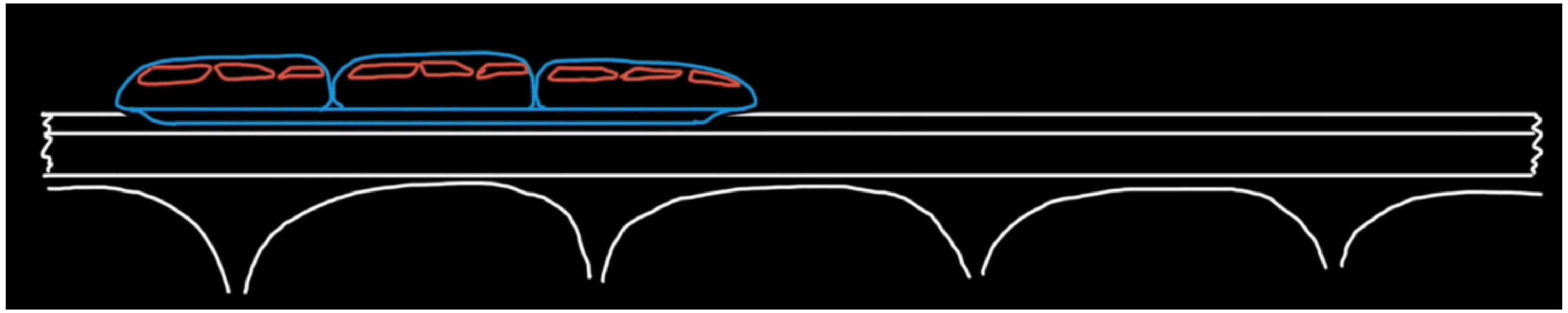
Monorail

$$\dot{v} = u$$

$$\dot{p} = v$$

and we do not consider them in the state vector of the model

$$x = (p, v, \cancel{t})$$



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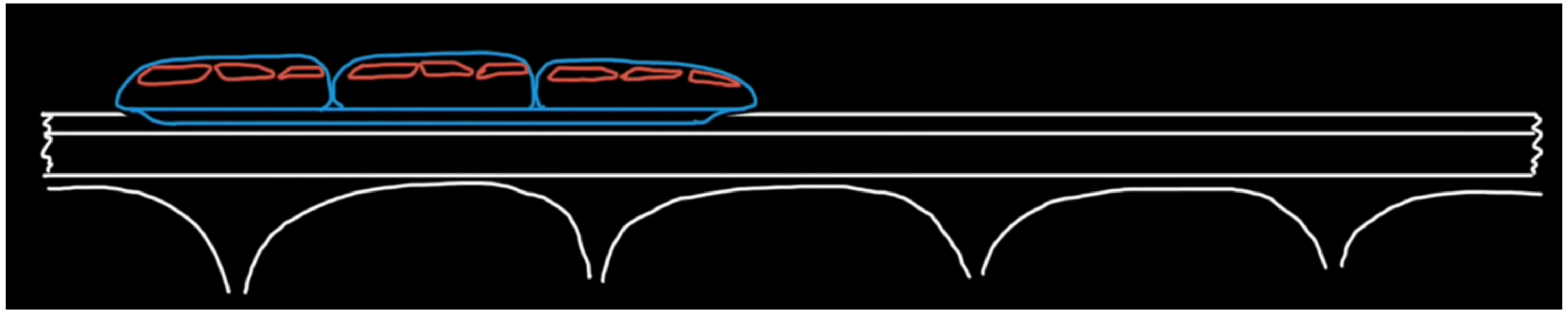


Observability

Observability means that all critical states can be known from the outputs of the system

What does it mean to observe a state?

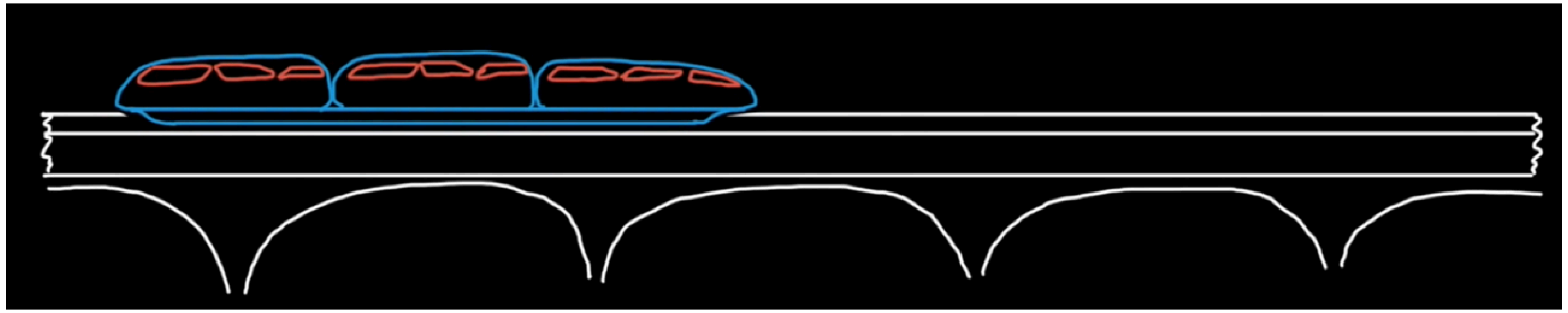
Monorail

$$\dot{v} = u$$

$$\dot{p} = v$$

we can measure both speed, and position

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} v \\ p \end{bmatrix}$$



Observability

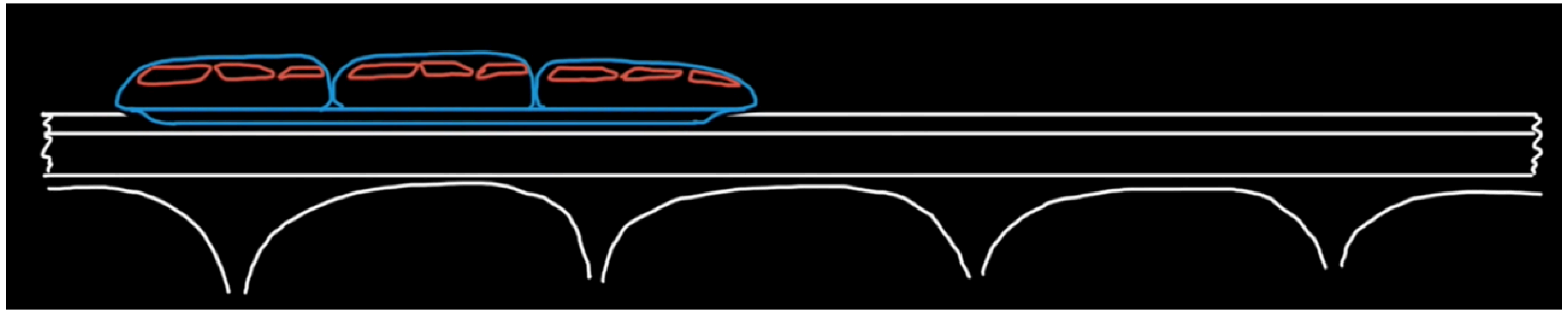
Observability means that all critical states can be known from the outputs of the system

What does it mean to observe a state?

Monorail

$$\dot{v} = u$$

$$\dot{p} = v$$



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we can estimate the whole state from available information

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}$$

$$v = \dot{p}$$

measure position | **estimate speed**

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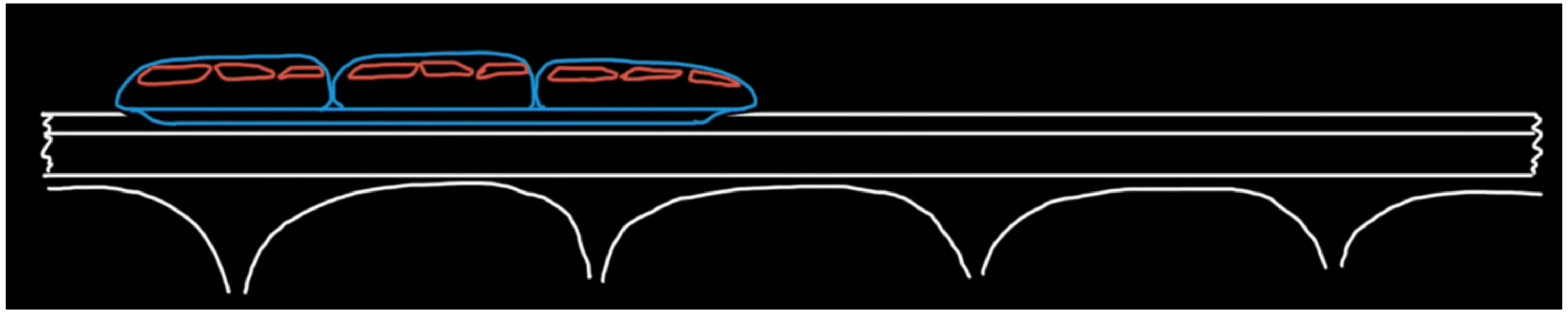

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measure position

$$v = \dot{p}$$

estimate speed

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}$$

measure speed

$$p = \int v dt + C$$

estimate position

Observability

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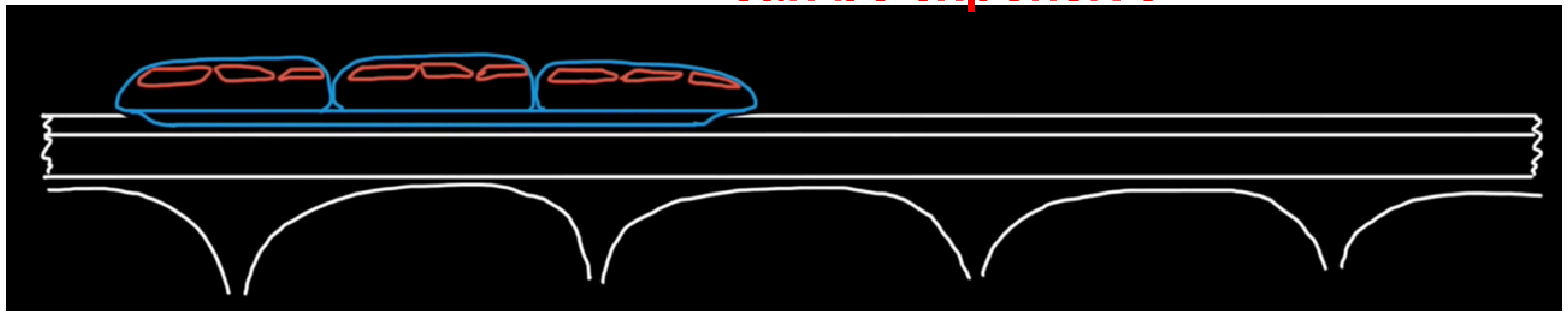
we can measure

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} v \\ p \end{bmatrix}$$

adding additional sensors can be expensive



we can estimate the whole state from available information

$$v = \dot{p}$$

estimate speed

$$p = \int v dt + C$$

estimate position

estimations are sensitive to measurement errors

Observability

Observability means that all **critical** states can be known from the outputs of the system

Monorail

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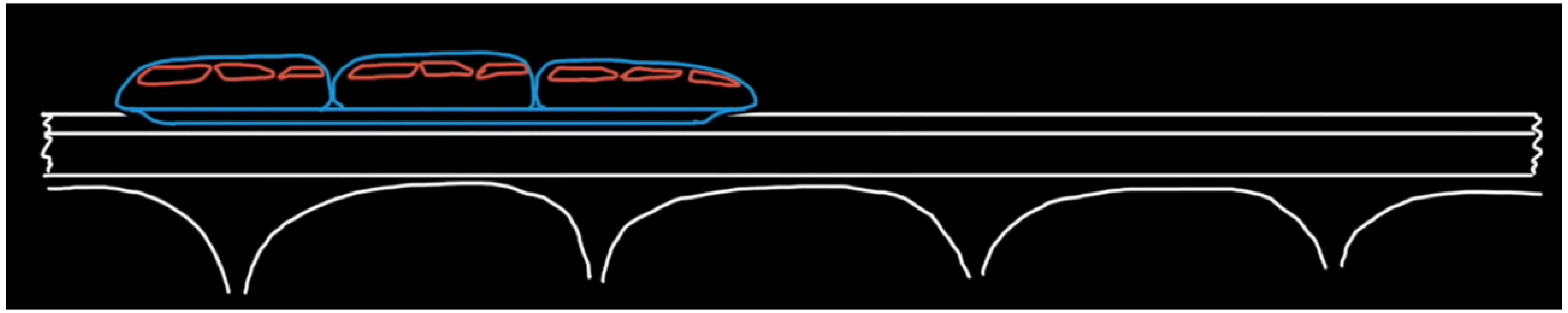

Example of unobservable system

imagine we lost all the sencors

$$y = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}$$

Controllability and observability of LTI system

Controllability & Observability of LTI system

State equation

$$\dot{x} = Ax + Bu$$

Output equation

$$y = Cx + Du$$

Dimensions

n states

p controls   m outputs

Controllability means that there exists control signal which allows the system to move from any any initial state to any final state in a finite time interval

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Controllability & Observability of LTI system

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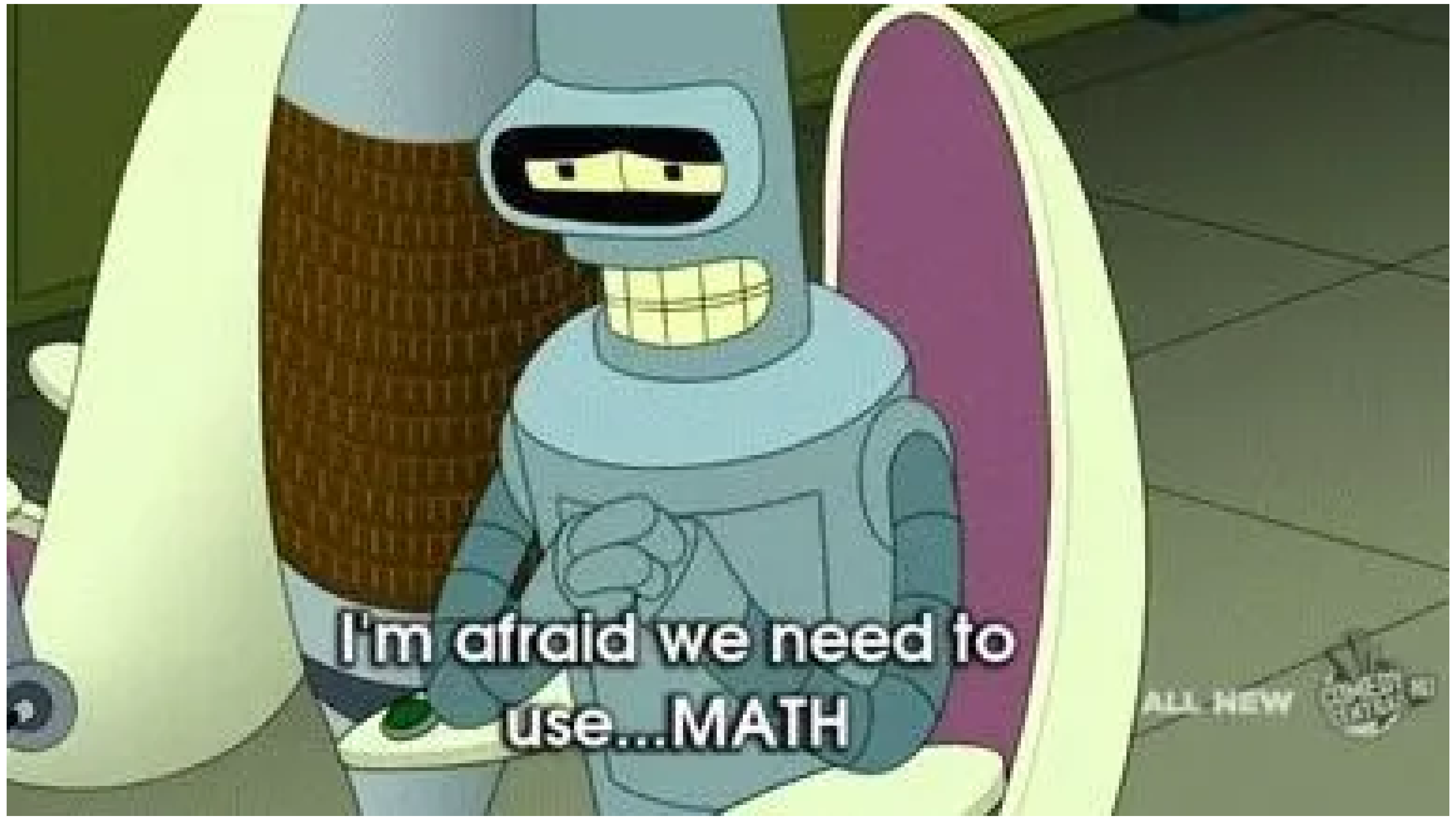

Controllability & Observability of LTI system

State equation

$$\dot{x} = Ax + Bu$$

Dimensions

n states

p controls

Solution of a state equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

<u>matrix exponential</u>

Let me remind...

- Let $A \in \mathbb{R}^{n \times n}$, the exponential of A, denoted by e^A is the $n \times n$ matrix given by the power series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

- Let $A \in \mathbb{R}^{n \times n}$ and I_n is $n \times n$ identity matrix. Then

$$p(\lambda) = det(\lambda I_n - A) = \lambda^n + \alpha_{n-1} A^{n-1} + \ldots + a_1 \lambda + \alpha_0 I_n$$

is called the characteristic polynomial of A.

Let me remind...

Theorem Caley-Hamilton

Let $A \in \mathbb{R}^{n\times n}$ then A satisfy its own characteristic polynomial equation, i.e.

$$p(A) = A^n + \alpha_{n-1}A^{n-1} + \ldots + \alpha_1 A + \alpha_0 I_n = 0.$$

- The theorem allows A^n to be expressed as a linear combination of the lower matrix powers of A

Controllability of LTI system

- The LTI system is called controllable if for any initial state x_0 and any final state x_f, there exists input signal $u(t)$ such that the system, starting from $x(0) = x_0$, reaches $x(t_f) = x_f$ in some finite time t_f.

- Starting at 0 is not a special case — if we can get to any state in finite time from the origin, then we can get from any initial condition to that state in finite time as well.

- $x(t_f) = \int_0^{t_f} e^{A(t_f-\tau)} Bu(\tau)\, d\tau$

Solution of
a state equation

Controllability of LTI system

- Change the variables $\tau_2 = \tau - t_f, d\tau = d\tau_2$ gives us a form

$$x(t_f) = \int_0^{t_f} e^{-A\tau_2} Bu(t_f + \tau_2)\, d\tau_2$$

Controllability of LTI system

- Change the variables $\tau_2 = \tau - t_f, d\tau = d\tau_2$ gives us a form

$$x(t_f) = \int_0^{t_f} e^{-A\tau_2} Bu(t_f + \tau_2)\, d\tau_2$$

- Assume the system has p inputs. From the definition of matrix exponential and Cayley-Hamilton theorem, we have

$$e^{-A\tau_2} = \sum_{i=0}^{\infty} \frac{A^i}{i!}(-\tau_2)^i = \sum_{i=0}^{n-1} A^i \alpha_i(\tau_2)$$

for some computable scalars $\alpha_i(\tau_2)$.

Controllability of LTI system

- Hence

$$x(t_f) = \int_0^{t_f} e^{-A\tau_2} Bu(t_f + \tau_2)\, d\tau_2 =$$

$$\int_0^{t_f} \left(\sum_{i=0}^{n-1} A^i \alpha_i(\tau_2) \right) Bu(t_f + \tau_2)\, d\tau_2 =$$

$$\sum_{i=0}^{n-1} (A^i B) \int_0^{t_f} \alpha_i(\tau_2) u(t_f + \tau_2)\, d\tau_2 = \sum_{i=0}^{n-1} (A^i B) \beta_i(t_f)$$

- the coefficients $\beta_i(t_f)$ depends on the input $u(\tau_2) \in \mathbb{R}^p,\ 0 < \tau_2 \leq t_f$.

Controllability of LTI system

- In matrix form, we have $x(t_f) = [B, AB, \ldots, A^{n-1}B] \begin{bmatrix} \beta_0(t_f) \\ , \ldots \\ \beta_{n-1}(t_f) \end{bmatrix}$

Controllability of LTI system

- In matrix form, we have $x(t_f) = \underbrace{[B, AB, \ldots, A^{n-1}B]}_{\mathcal{C}(A,B)} \begin{bmatrix} \beta_0(t_f) \\ , \ldots \\ \beta_{n-1}(t_f) \end{bmatrix}$

  controllability matrix

- A solution of this equation exists for any $x(t_f) \in \mathbb{R}^{n \times 1}$ if and only if

$$rank(\mathcal{C}(A, B)) = n.$$

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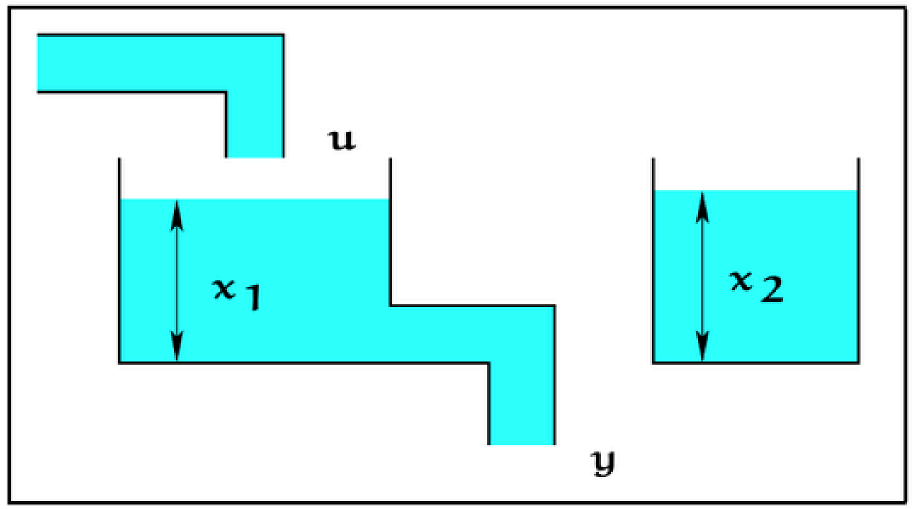
Kalman's Controllability Rank Condition

The LTI system $\dot{x} = Ax + Bu,\ x \in \mathbb{R}^{n\times 1}$ is controllable if and only if the controllability matrix $\mathcal{C}(A, B) = [B, AB, \ldots, A^{n-1}B]$ has full rank, i.e.

$$rank(\mathcal{C}(A, B)) = n.$$

Controllability Examples

Example.



In the hydraulic system on the left it is obvious that the input cannot affect the level x_2, so it is intuitively evident that the 2-tank system is not controllable.

A linearised model of this system with unitary parameters gives

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

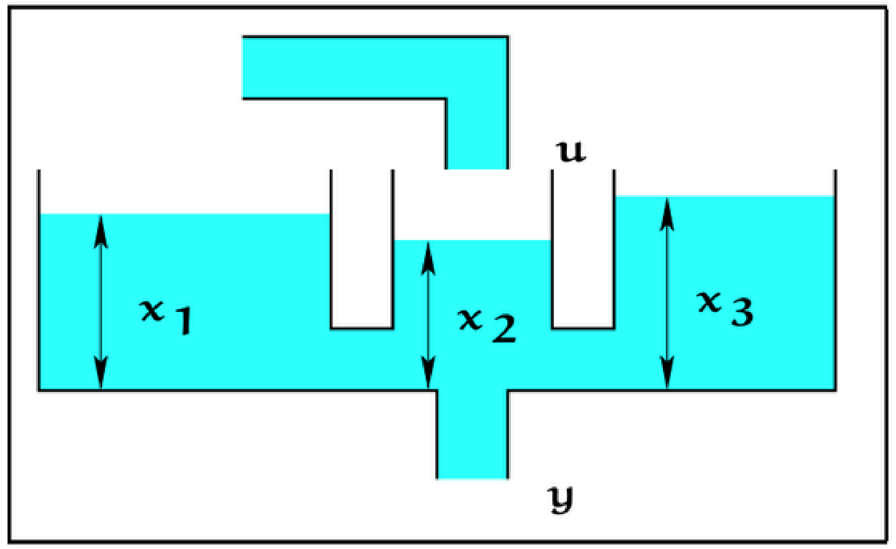
The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is not full rank, so the system is not controllable.

Controllability Examples

Example.



The controllability of the hydraulic system on the left is not so obvious, although we can see that $x_1(t)$ and $x_3(t)$ cannot be affected independently by $u(t)$.

The linearised model in this case is

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(t)$$

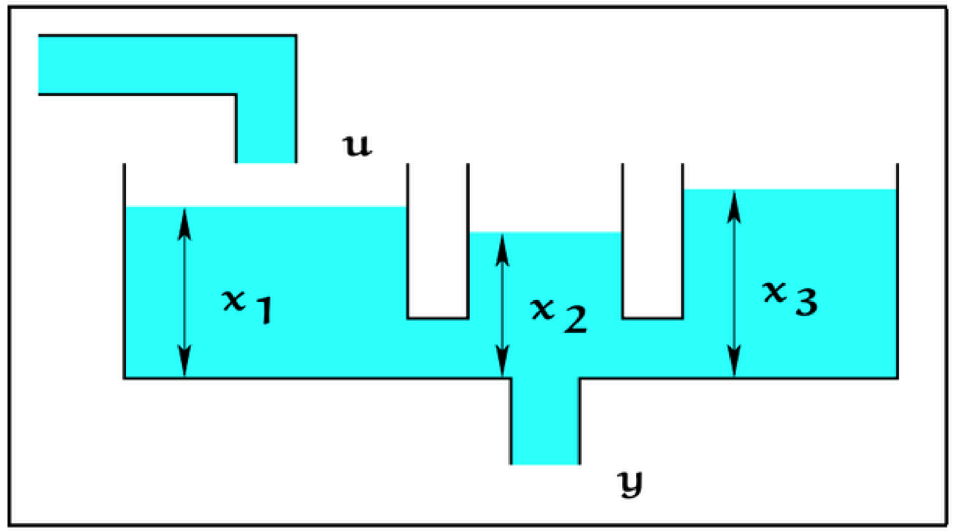
The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ 1 & -3 & 11 \\ 0 & 1 & -4 \end{bmatrix}$$

which has rank 2, showing that the system is not controllable.

Controllability Examples

Example.



Now in the previous system suppose that the input is applied in the first tank, as shown in the figure. In this case the linearised model is the same as before, except that the matrix $\mathbf{B}$ is now different

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

The controllability matrix is now

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

which has rank 3, showing that the system is controllable.

Controllability & Observability of LTI system

State equation

$$\dot{x} = Ax + Bu$$

Output equation

$$y = Cx + Du$$

Dimensions

n states

p controls m outputs

Solution of a state equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

matrix exponential

Input - output relation

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Observability of LTI system

- Observability: Can we reconstruct $x(0)$ by knowing $y(\tau)$ and $u(\tau)$ over some finite time interval $[0, t]$? (By knowing the initial condition, we can reconstruct the entire state $x(t)$)
- Let us introduce notation

$$\tilde{y}(t) = y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t)$$

then

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \Leftrightarrow \tilde{y}(t) = Ce^{At}x(0)$$

Observability of LTI system

- Since the n-dimensional vector $x(0)$ has n unknown components, we need n equations to find it.

Observability of LTI system

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- Let's differentiate $\tilde{y}(t)$ $n-1$ times:

observability matrix

$$
\begin{aligned}
\tilde{y}(t) &= Ce^{At}x(0) \\
\tilde{y}(t)^{(1)} &= CAe^{At}x(0) \\
&\dots \\
\tilde{y}(t)^{(n-1)} &= CA^{n-1}e^{At}x(0)
\end{aligned}
\Leftrightarrow
\begin{bmatrix} \tilde{y}(t) \\ \tilde{y}(t)^{(1)} \\ \dots \\ \tilde{y}(t)^{(n-1)} \end{bmatrix}
=
\underbrace{\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}(A,C)} e^{At}x(0)
$$

Observability of LTI system

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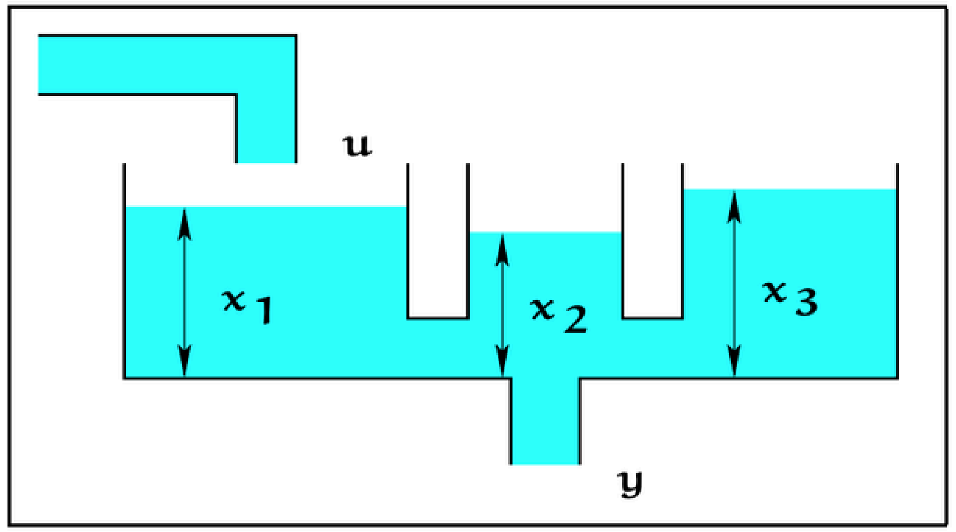
=
\underbrace{\begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}(A,C)} e^{At}x(0)
$$

Kalman's Observability Rank Condition

The LTI system $\dot{x} = Ax + Bu,\ x \in \mathbb{R}^{n\times 1}$ with measurements $y = Cx + Du$ is observable if and only if the observability matrix $\mathcal{O}(A, C)$ has full rank, i.e. $rank(\mathcal{O}(A, C)) = n$.

Observability Examples

Example.



$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

Measurements

1. $y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)$

$O(A, C) =$

```
[[ 1.  0.  0.]
 [ 0.  1.  0.]
 [ 0.  0.  1.]
 [-1.  1.  0.]
 [ 1. -3.  1.]
 [ 0.  1. -1.]
 [ 2. -4.  1.]
 [-4. 11. -4.]
 [ 1. -4.  2.]]
```

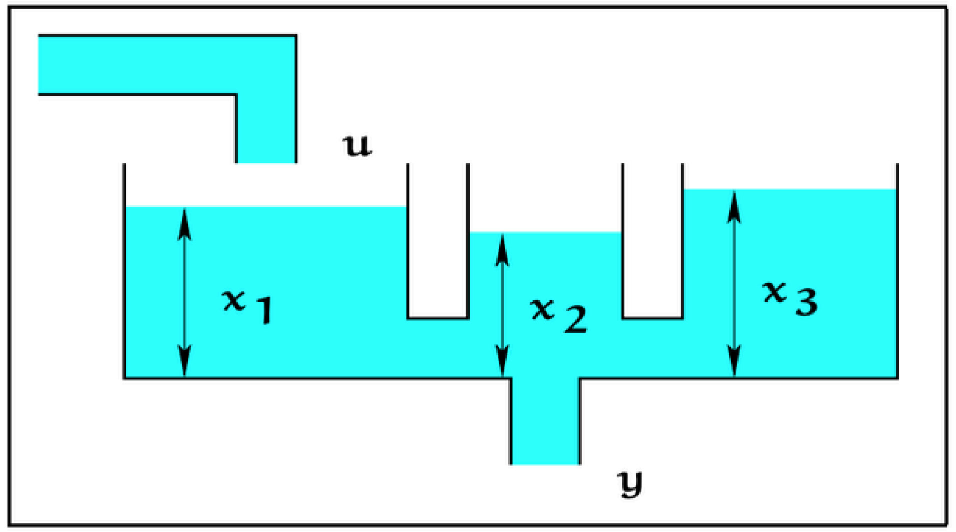
$\text{rank}(O(A, C)) = 3$

⇓

observable

Observability Examples

Example.



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```
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```
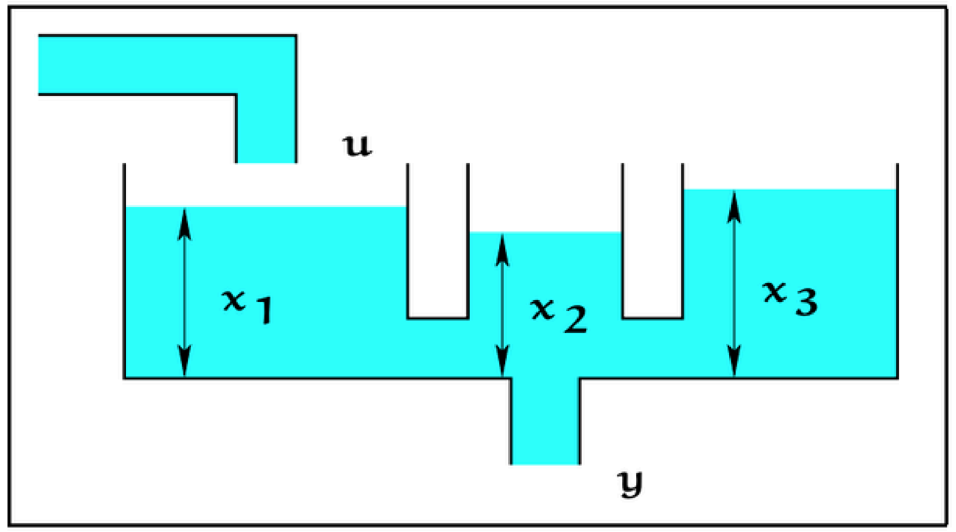

$\text{rank}(O(A,C)) = 3 \Rightarrow$ observable

i.e. 1 sensor is enough!

Observability Examples

Example.



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$O(A, C) =$

```
[[ 0.   1.   0.]
 [ 1.  -3.   1.]
 [-4.  11.  -4.]]
```

$\text{rank}(O(A, C)) = 2 \Rightarrow$ non observable.

i.e. 1 sensor is enough to estimate the state, but it shouldn't be misplaced

Let me summarize

State equation

$$\dot{x} = Ax + Bu$$

Output equation

$$y = Cx + Du$$

Dimensions

n states

p controls   m outputs

- The LTI system is controllable if and only if $rank(\mathcal{C}(A, B)) = n$.
- The LTI system is observable if and only if $rank(\mathcal{O}(A, C)) = n$.

Let me summarize

State equation

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Duality of controllability & observability

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- The pair (A, C) is observable if and only if $rank(\mathcal{O}(A, C)) = n$.

Duality of Controllability and observability

The pair of matrices (A, B) is controllable if and only if the pair of matrices (A^T, B^T) is observable.

Invariance Under Change of Coordinates

- Consider $\dot{x} = Ax + Bu, y = Cx + Du$ and similarity transformation $\tilde{x} = Tx$, where T is invertible.
- The system $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, y = \tilde{C}\tilde{x} + Du$ with matrices

$$\tilde{A} = TAT^{-1}, \ \tilde{B} = TB, \tilde{C} = CT^{-1}$$

is then called an equivalent system.

Invariance Under Nonsingular Transformations

The LTI system is controllable if and only if the equivalent system is controllable.
The LTI system is observable if and only if the equivalent system is observable.

Please complete the notebook you can find at

https://perso.ensta-paris.fr/~manzaner/Cours/AUT202/

The completed notebook should be sent to your tutor before the beginning of the next session.

Please add [APM_4AUT2_TA] to the topic of e-mail.