# Introduction to Control Theory

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**Course grade breakdowns**

Labs - 40%

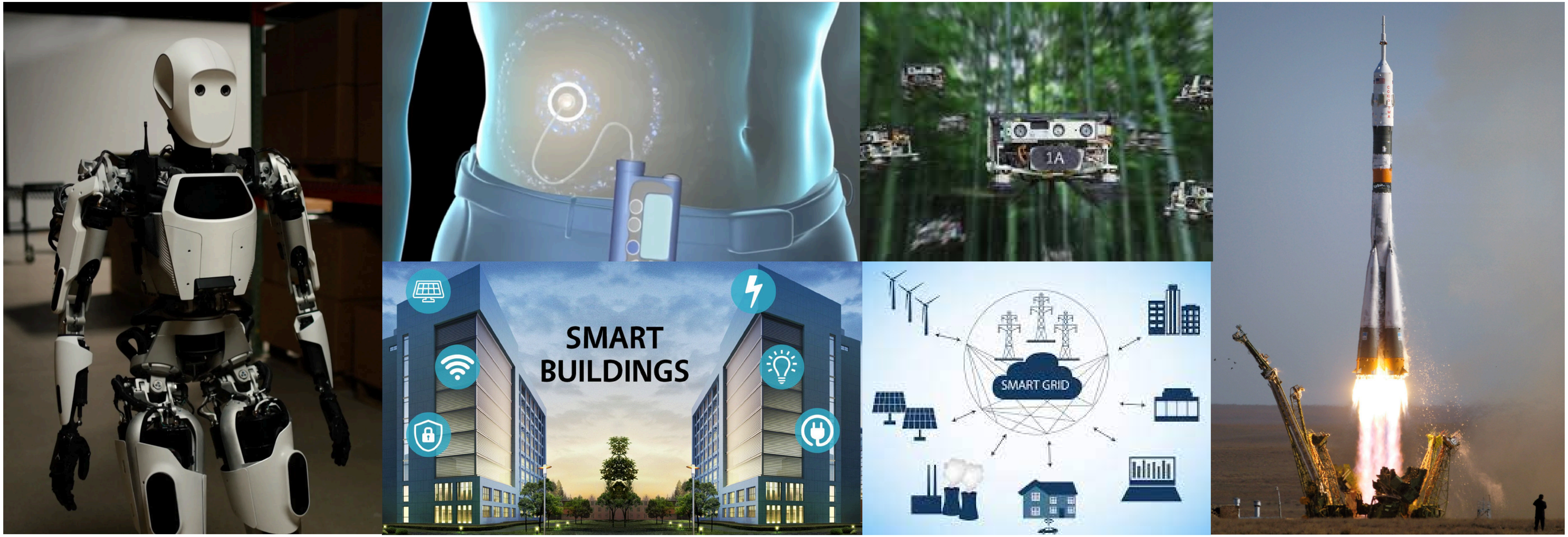
Final test - 30%

Final project - 30 %

# What is a control system?


Link to the video: **[Prof. Jeff Hoffman, MIT Open Learning](#)**

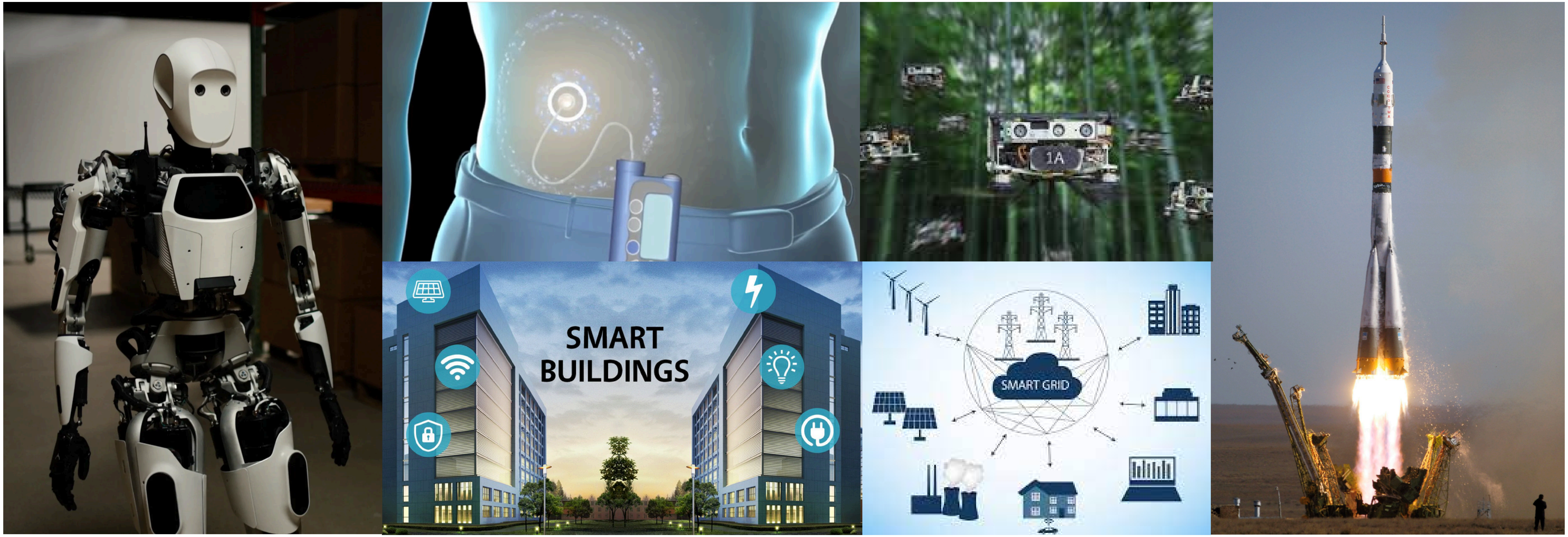
# Why automatic control ?




**A brief history of control theory...**

https://www.youtube.com/watch?v=FD6Fz9cYy5I

# “smart” means “automatically controlled”…

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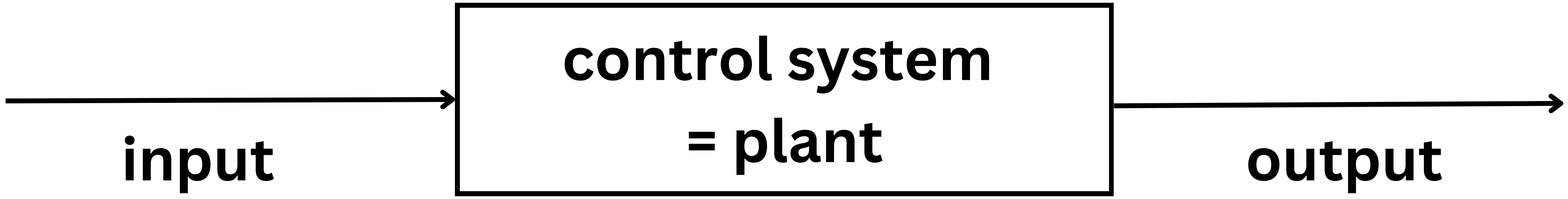
# What is a control system?


**Control system =**

**mechanism that alters the future state of the system**

# What is a control theory?




how do I change this ?

to get what I want?

Control system =
mechanism that alters the future state of the system

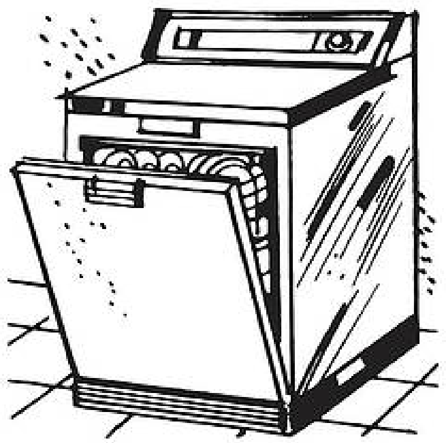
Control theory =
a strategy to select appropriate input

# Open-loop vs Closed-loop

Open-loop control systems are typically reserved for simple processes that have well-defined input to output behaviors.

washing program

time

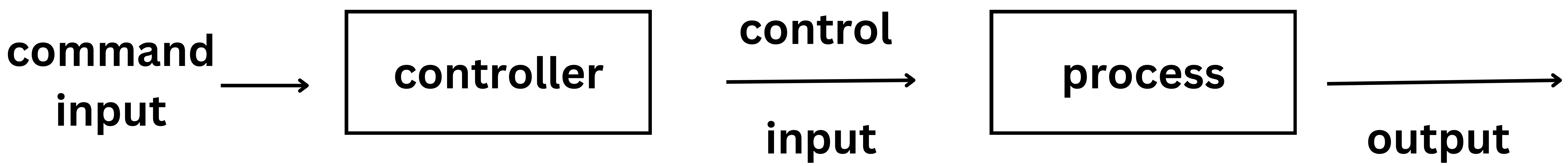


clean dishes

Once the user sets the wash timer the dishwasher will run for that set time, regardless of whether the dishes are actually clean or not when it finishes running.

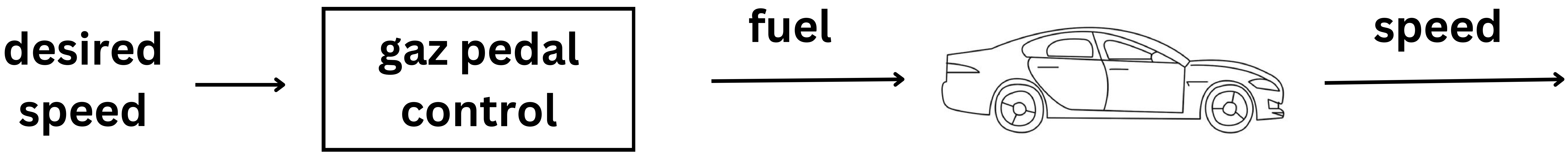
# Open-loop vs Closed-loop

**Open-loop control systems are typically reserved for simple processes that have well-defined input to output behaviors.**



# Open-loop vs Closed-loop

For any arbitrary process, though, an open-loop control system is typically not sufficient.

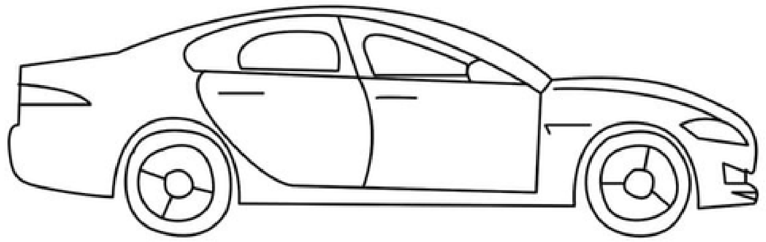



Imagine you are trying to move your car with a constant speed

# Open-loop vs Closed-loop

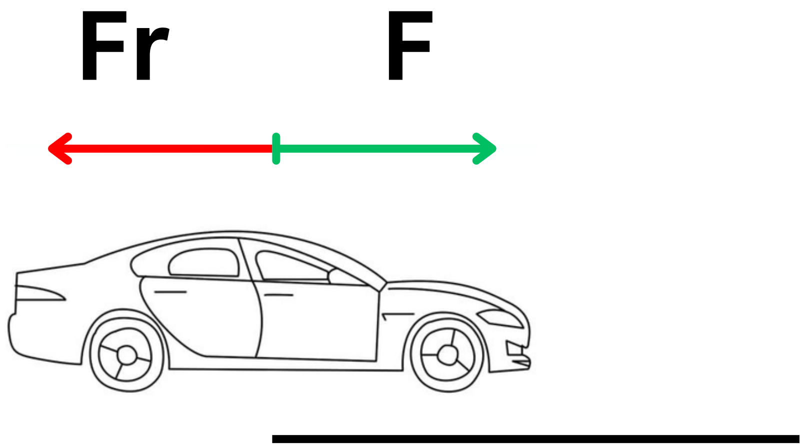
For any arbitrary process, though, an open-loop control system is typically not sufficient.

desired speed → gaz pedal control → fuel →



→ speed

Moving flat road you can apply the force F which is balanced by the force of friction Fr at this point



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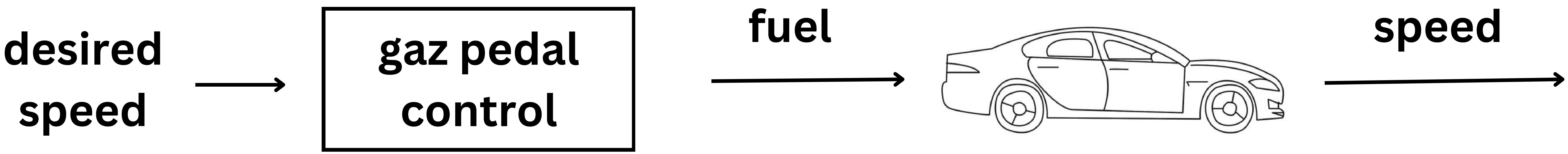


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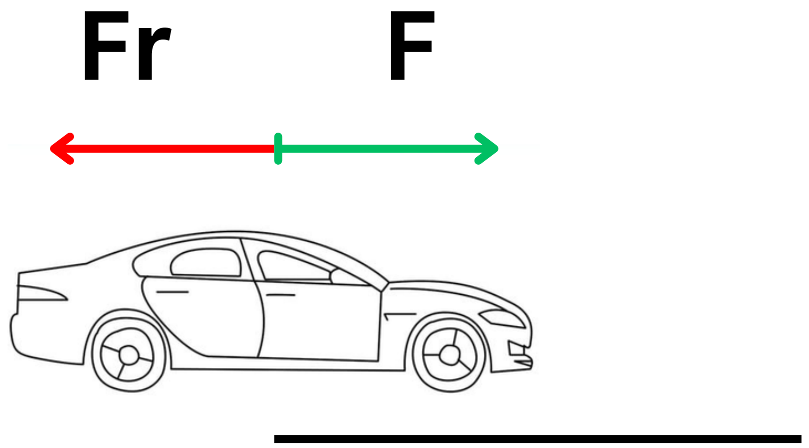

# Open-loop vs Closed-loop

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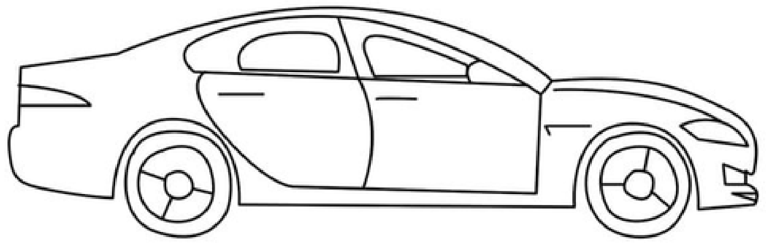
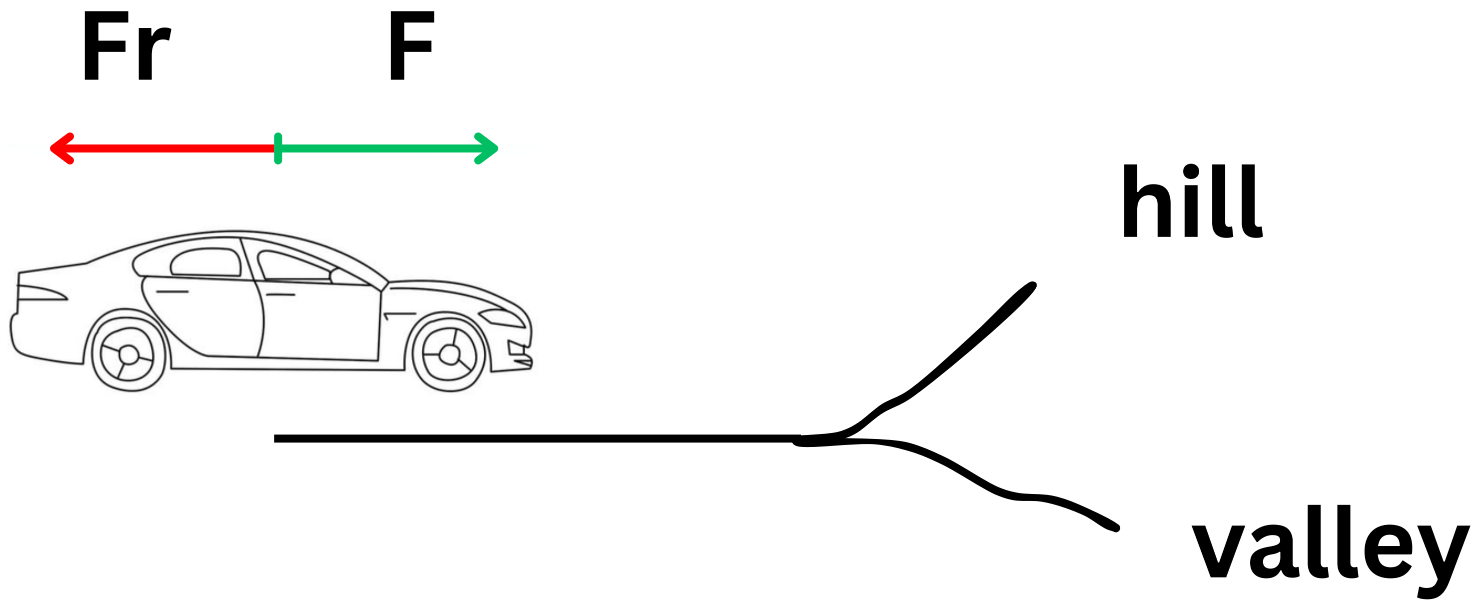
desired speed → gaz pedal control → fuel →  → speed

but what happens when
the car encounters a hill or a valley?



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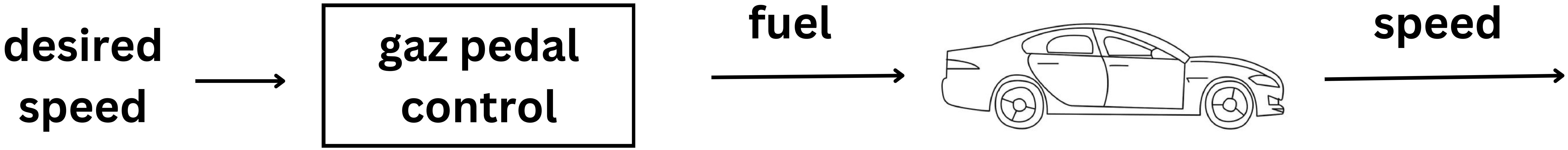


**to account for road gradient changes you must vary the input to your system with respect to the output**

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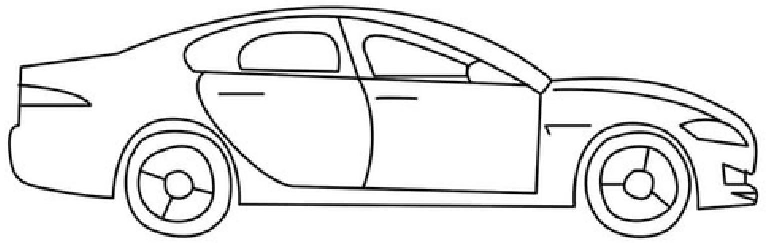
desired speed → gaz pedal control → fuel →



→ speed

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# Open-loop vs Closed-loop

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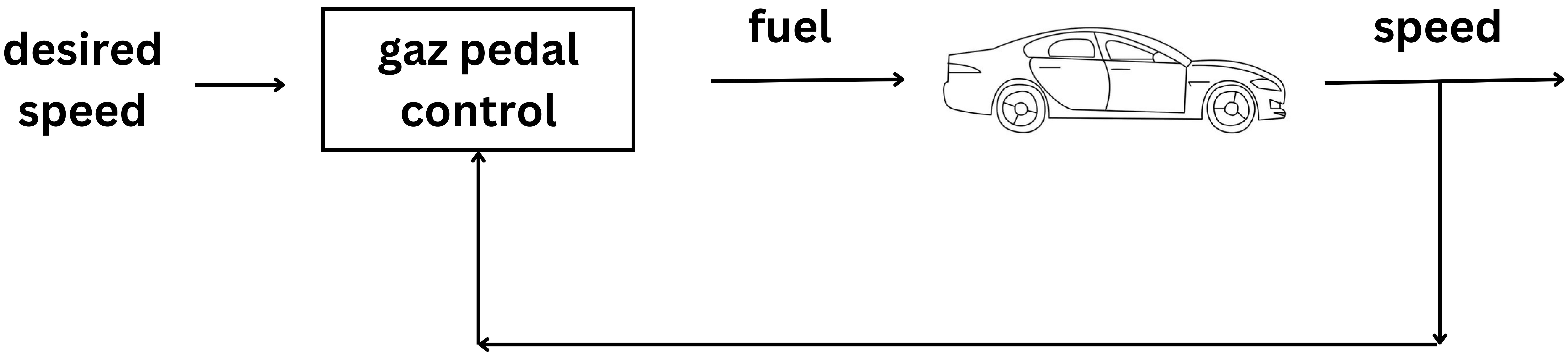

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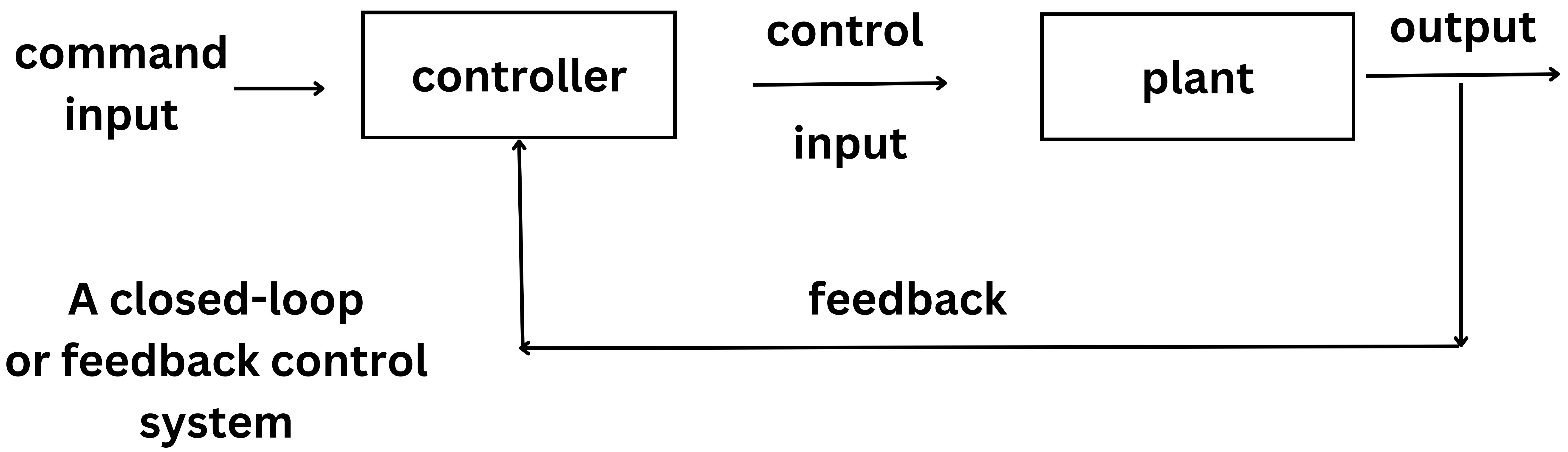

# Modeling

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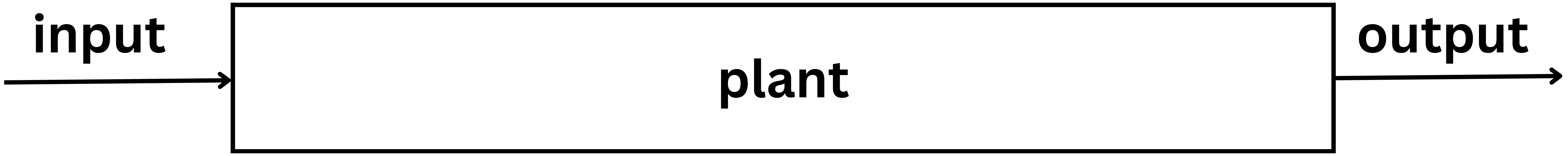
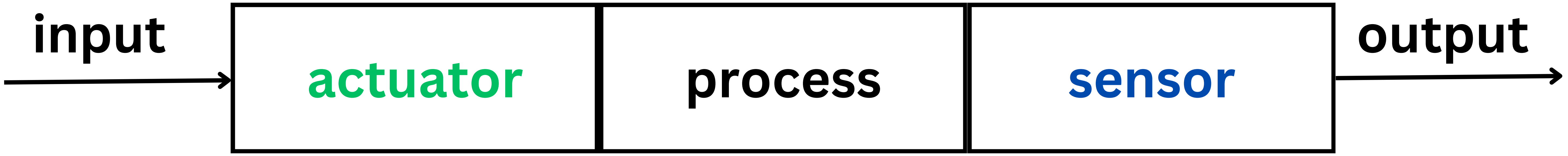


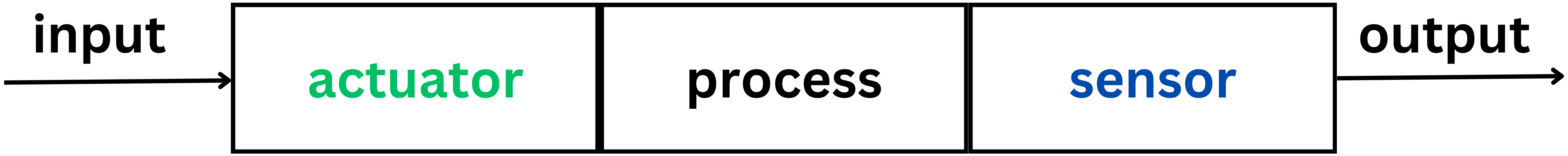
# Modeling



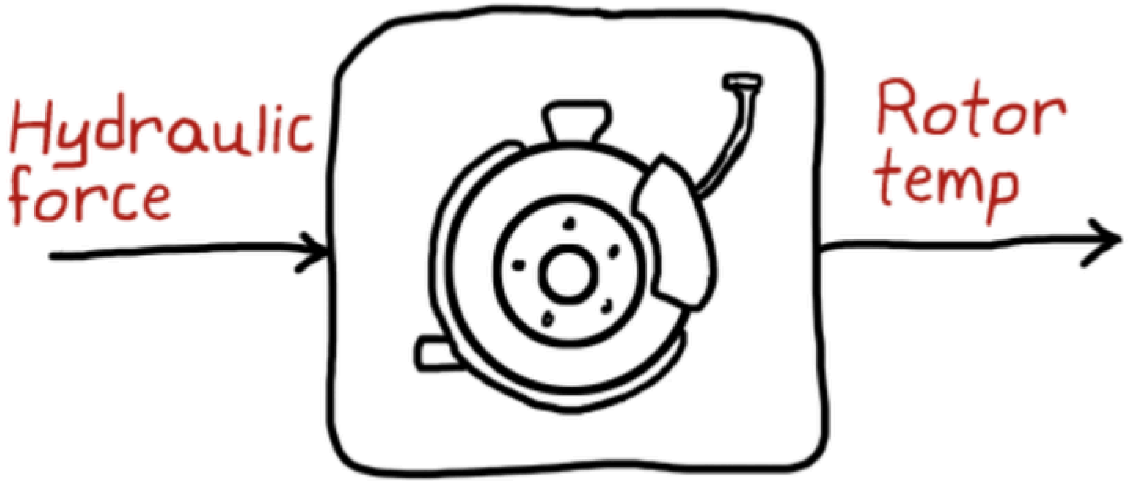

An actuator is a part of a device or machine that convert energy, often electrical, air, or hydraulic, into mechanical force. It is the component in any machine that enables movement.

A sensor is a device that produces an output signal for the purpose of sensing a physical phenomenon.

# Modeling

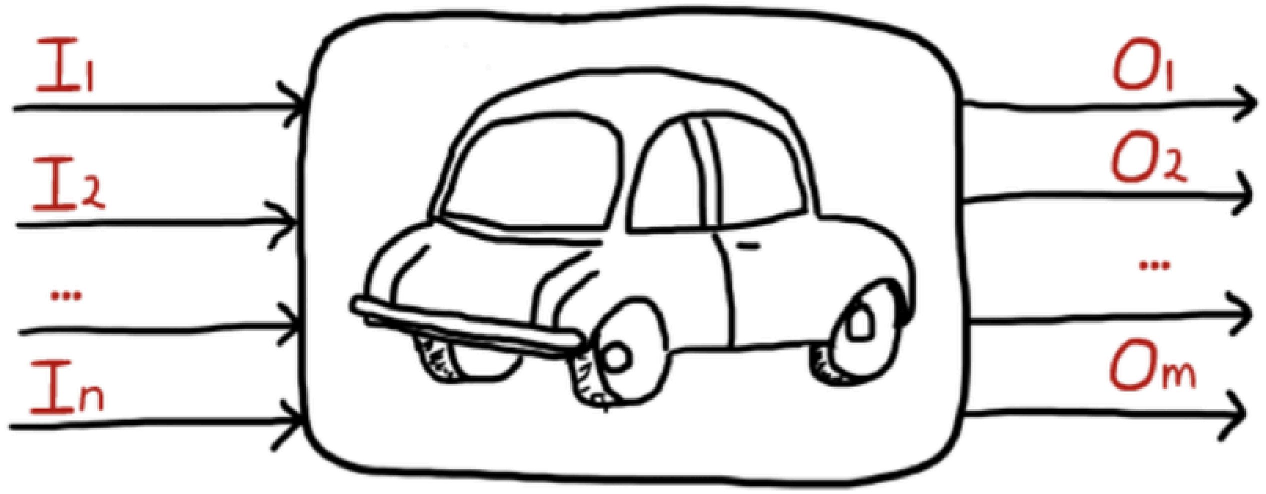



**SISO**



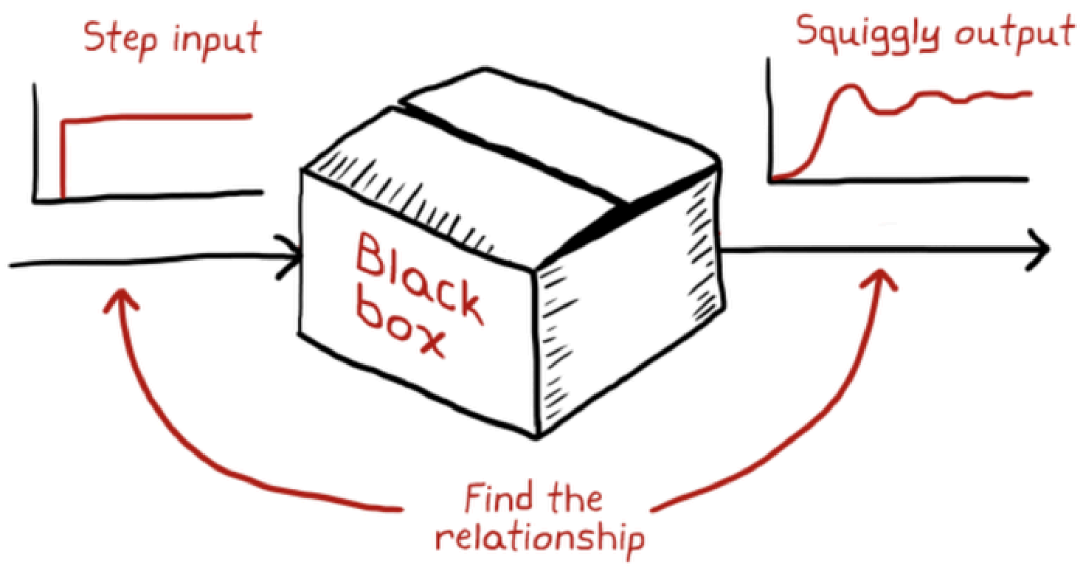

Single Input Single Output

**MIMO**

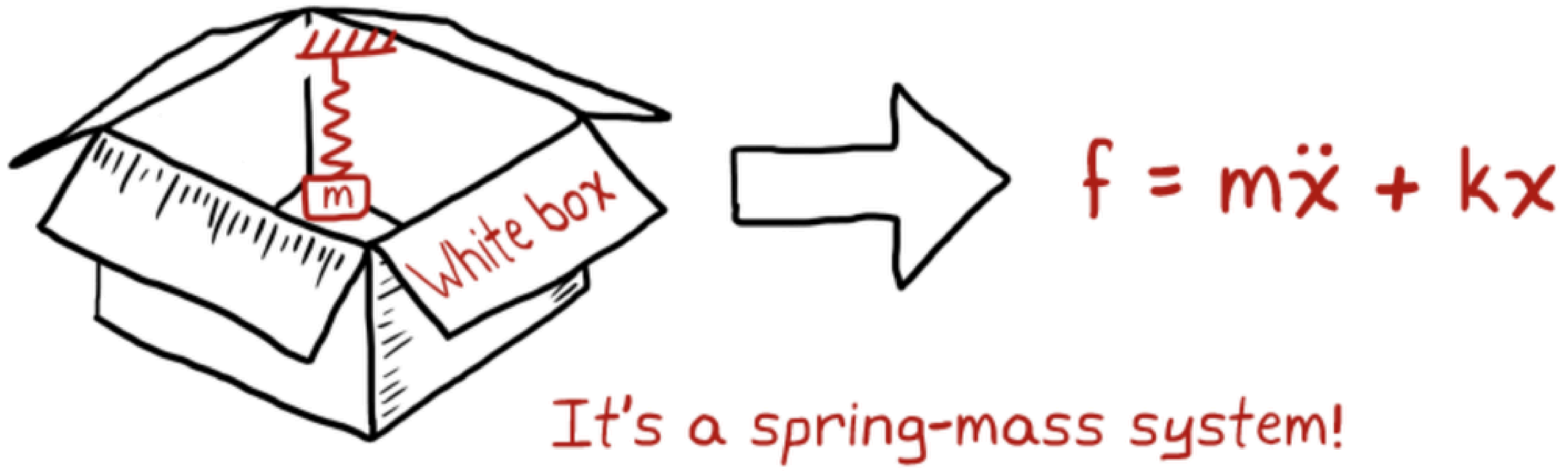



Multiple Inputs Multiple Outputs

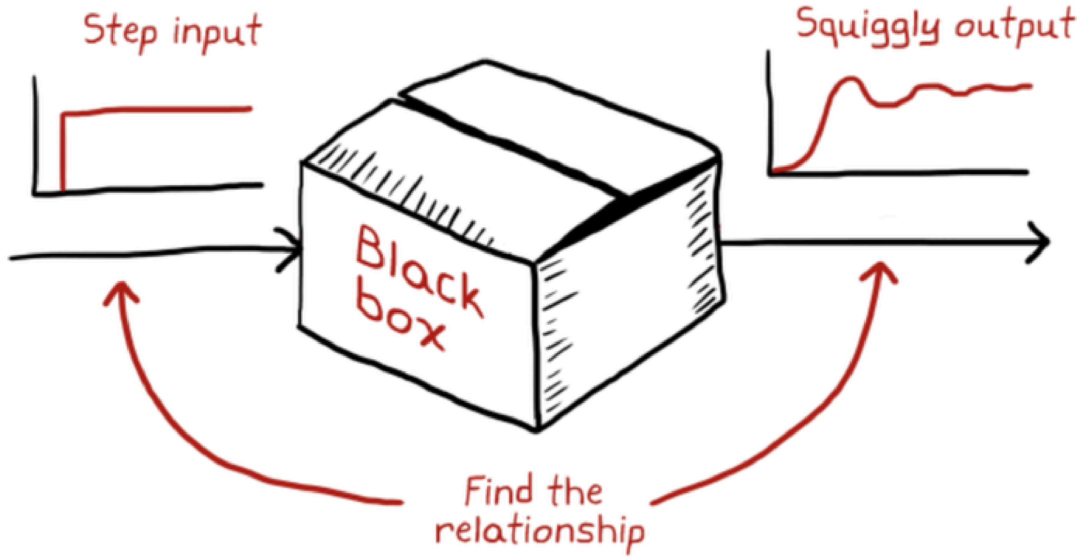
# Modeling




A **black box** model receives inputs and produces outputs but its workings are unknowable.
For example: neural networks

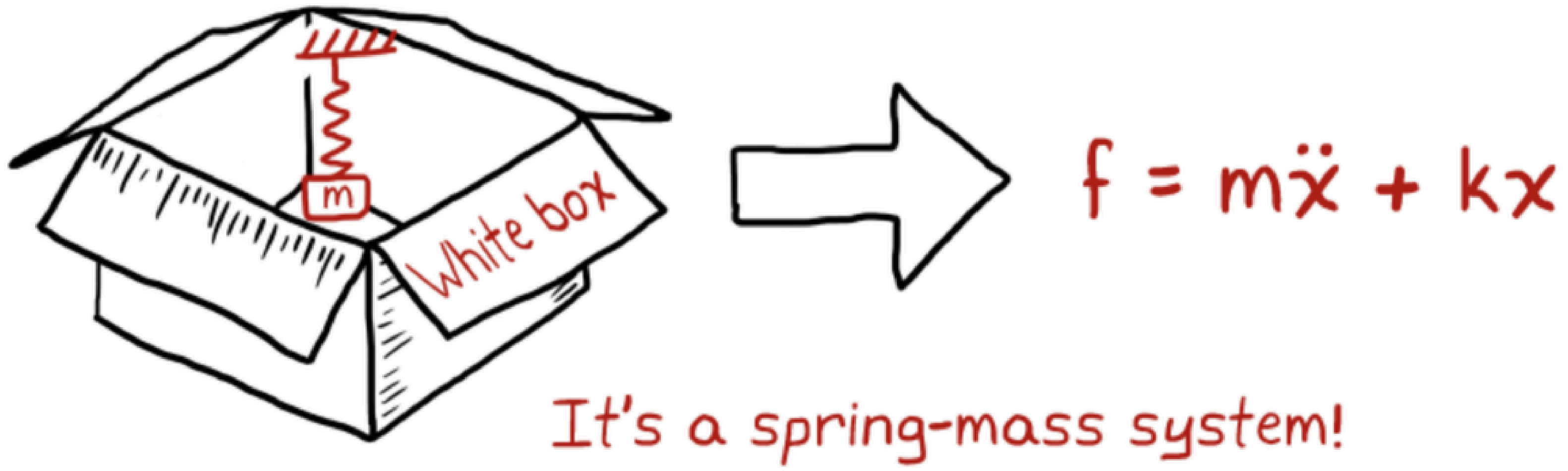


# Modeling

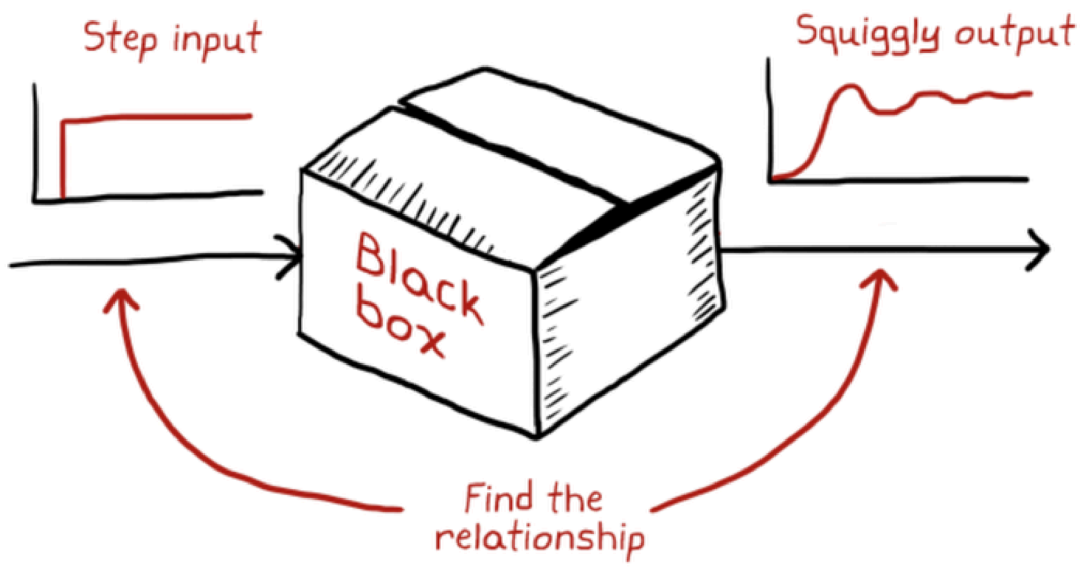



A **black box** model receives inputs and produces outputs but its workings are unknowable.
For example: neural networks

A **white box** model is a mathematical model of a physical process described by ODE or PDE



# Modeling




A **black box** model receives inputs and produces outputs but its workings are unknowable.
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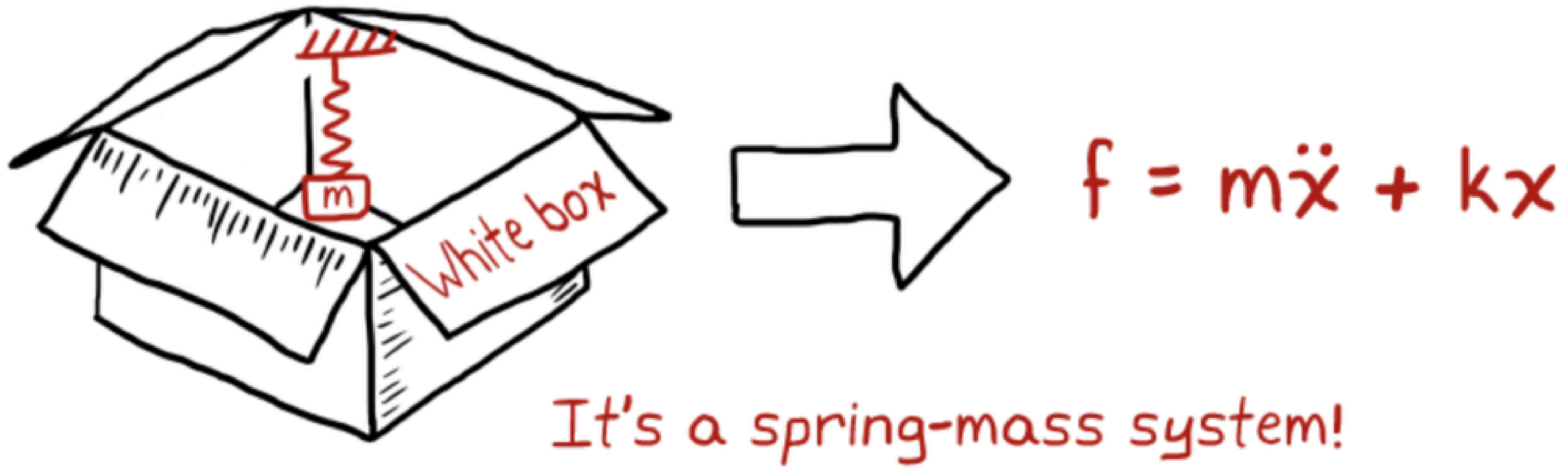
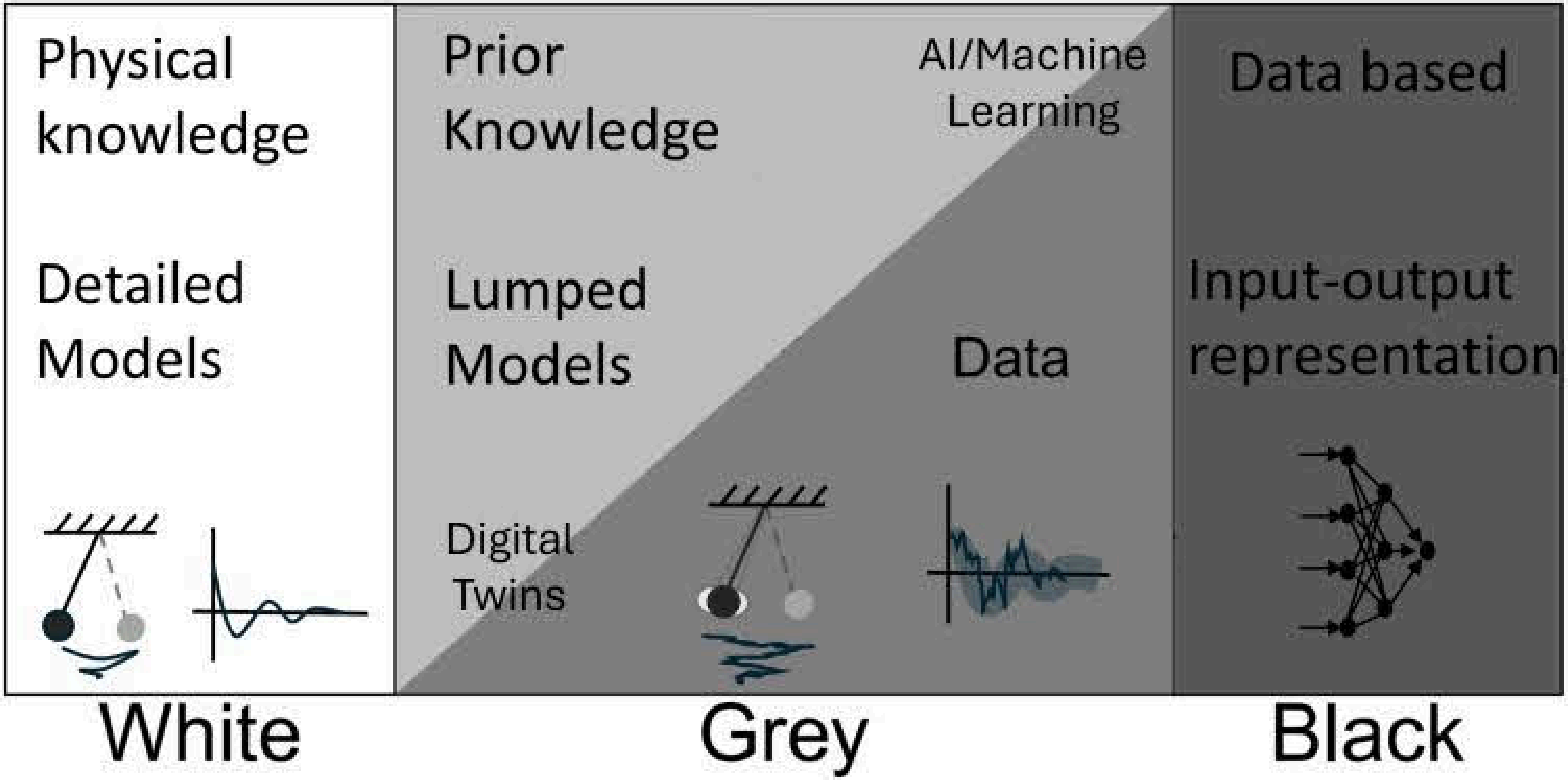
## Grey box models

A **white box** model is a mathematical model of a physical process described by ODE or PDE



# Modeling


Physical knowledge
Detailed Models
Prior Knowledge
AI/Machine Learning
Lumped Models
Data
Digital Twins
Data based
Input-output representation
White
Grey
Black

# Modeling

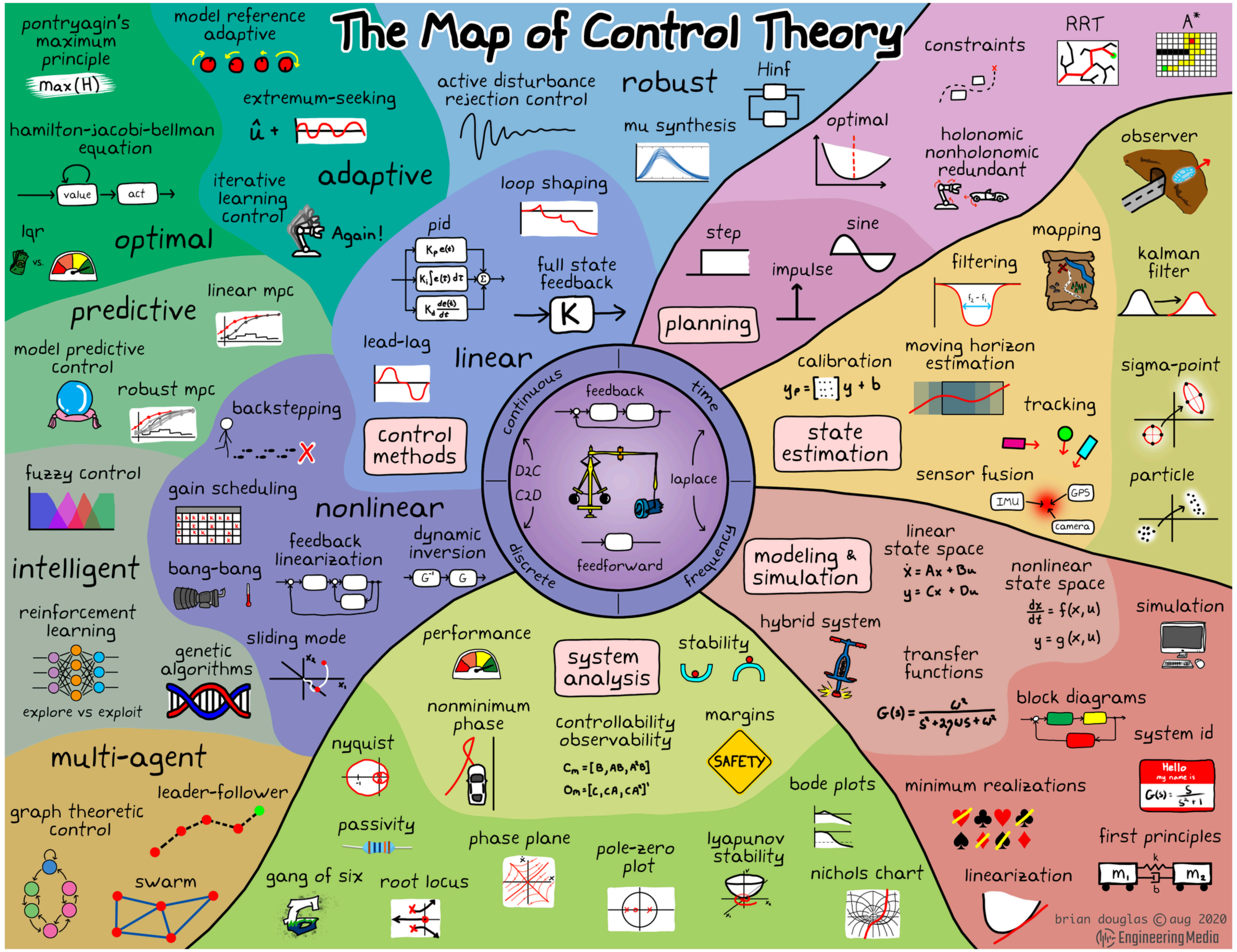
**Models allow simulating and analyzing the system**

**Models are never exact**

**Modeling depends on your goal**

**A single system may have many models**

# Lectures outline

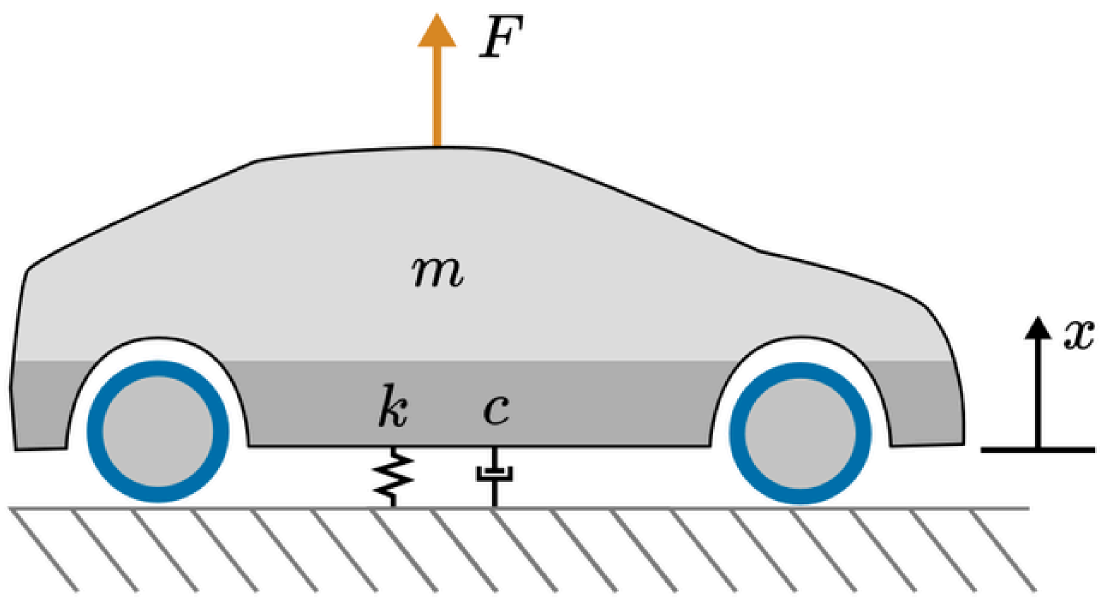



1. Modelling. LTI systems
2. Controllability and Observability
3. Stability and State Observer
4. Control design. PID controller
5. Optimal control design
6. Advanced control techniques
7. Project defence

by Brian Douglas

# Ex.1: . Vehicle Suspension System

https://www.youtube.com/watch?v=lPg695IXbPo




**The suspension system in cars, trucks, and other vehicles uses springs and dampers to absorb shocks from the road and provide a smooth ride.**

# Ex.1: . Vehicle Suspension System

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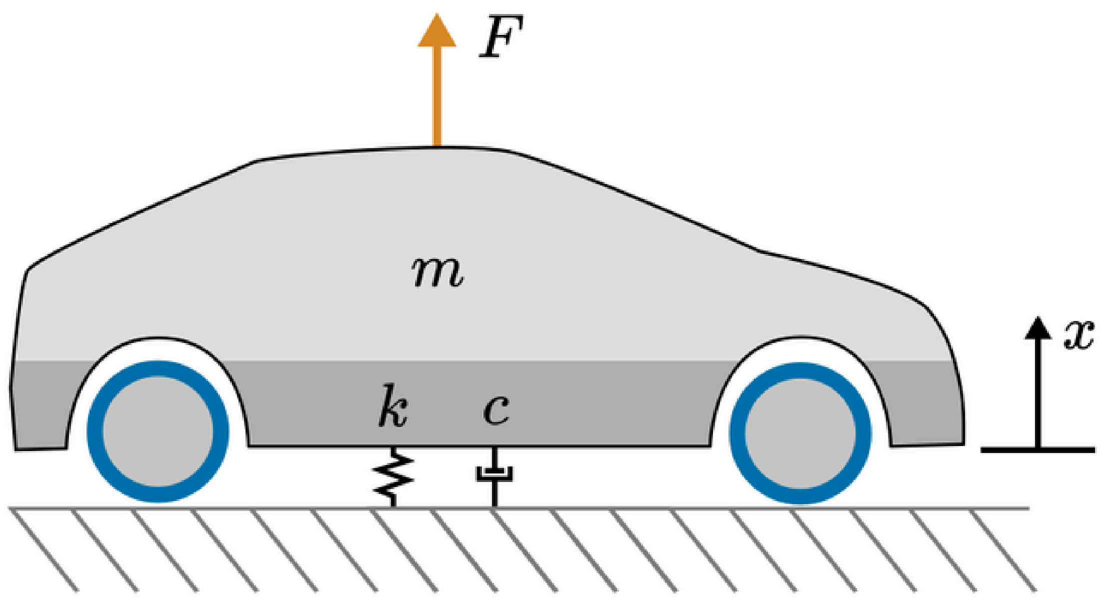


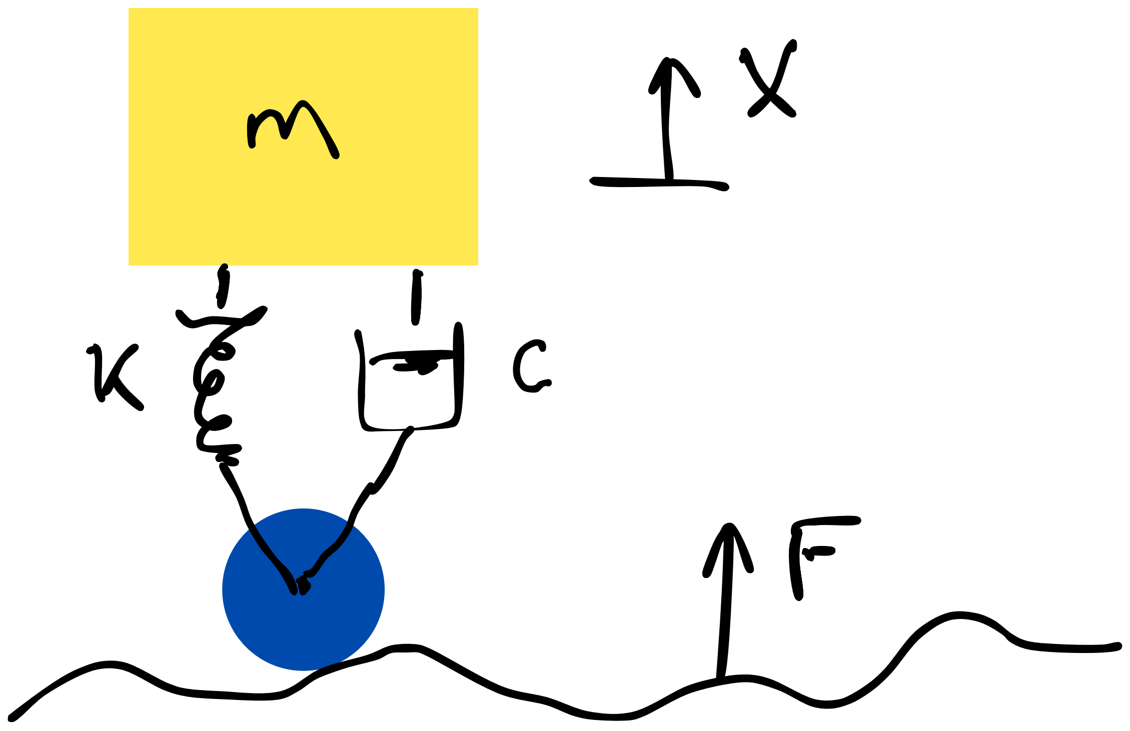

**The suspension system in cars, trucks, and other vehicles uses springs and dampers to absorb shocks from the road and provide a smooth ride.**

**Newton's second law (translational motion):**

$$m\ddot{x} = F_{total} = \underbrace{-kx}_{\text{spring force}} \underbrace{- c\dot{x}}_{\text{friction force}} + \underbrace{F}_{\text{external force}}$$

Hooke's law      Stokes' law

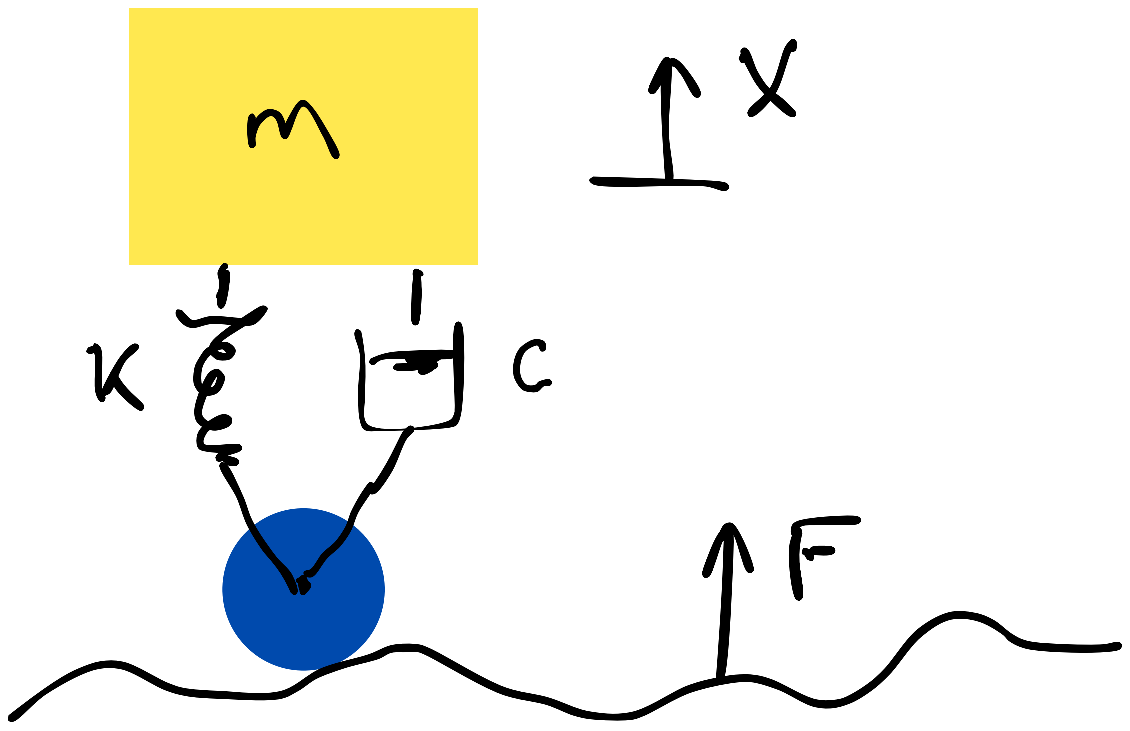
# Ex.1: . Vehicle Suspension System




The suspension system in cars, trucks, and other vehicles uses springs and dampers to absorb shocks from the road and provide a smooth ride.

$$m\ddot{x} = F_{total} = -kx - c\dot{x} + F$$

# Ex.1: . Vehicle Suspension System



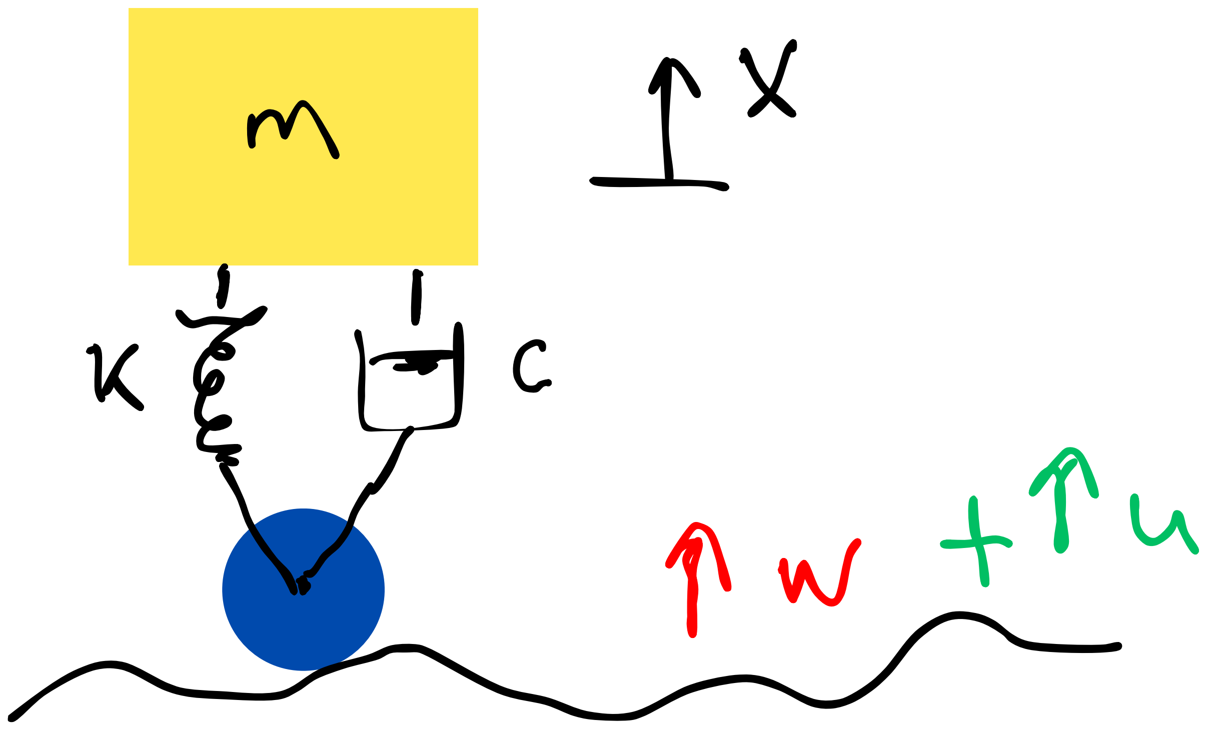

**The suspension system in cars, trucks, and other vehicles uses springs and dampers to absorb shocks from the road and provide a smooth ride.**

$$m\ddot{x} = F_{total} = -kx - c\dot{x} + F$$

Distrubance W

**This disturbance represents real-world conditions that a car suspension system might encounter, such as road irregularities, bumps, or potholes**

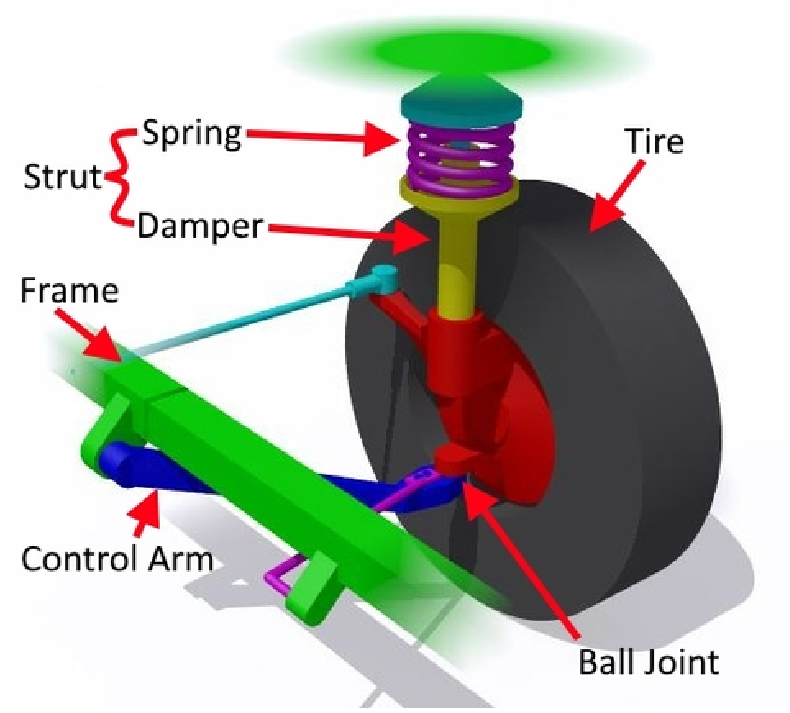
# Ex.1: . Vehicle Suspension System




The suspension system in cars, trucks, and other vehicles uses springs and dampers to absorb shocks from the road and provide a smooth ride.

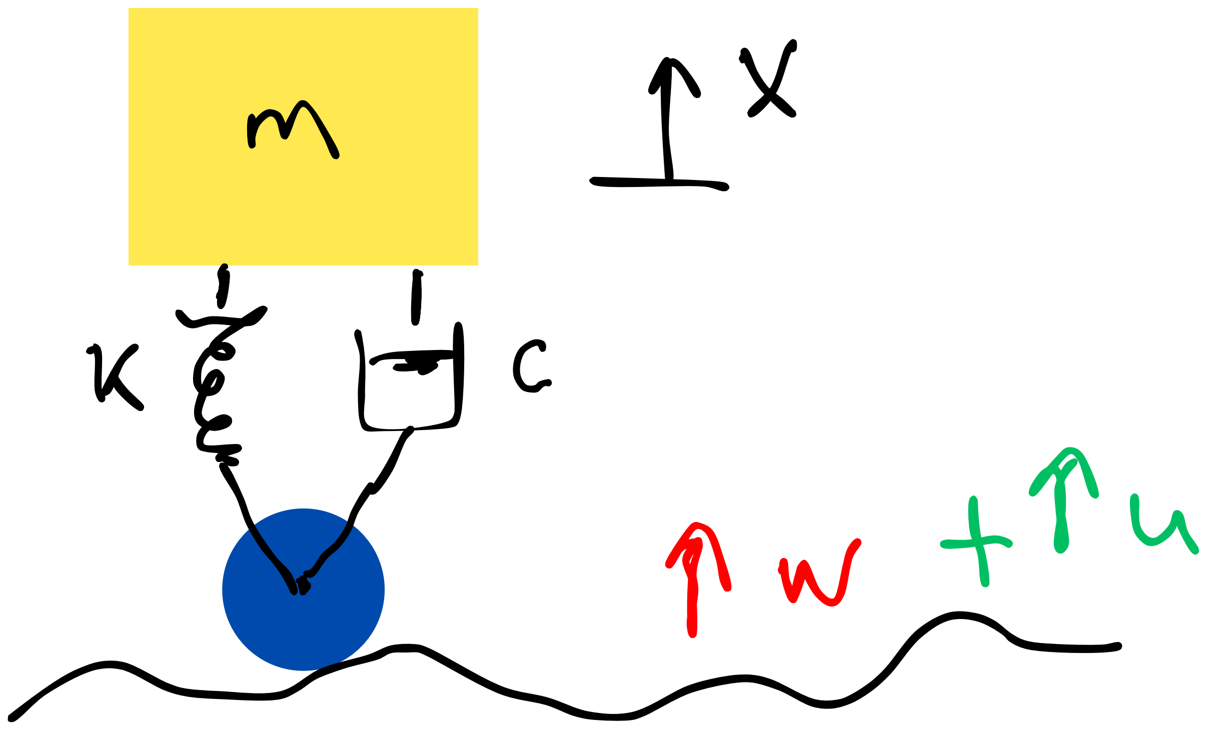
$$m\ddot{x} = F_{total} = -kx + c\dot{x} + F$$

Distrubance $W$ + Control $u$




Design a controller to minimize vibrations and improve ride comfort while maintaining stability.

# Ex.1: . Vehicle Suspension System




**2nd-order linear ODE**

$$\ddot{x} + \frac{k}{m}x + \frac{c}{m}\dot{x} = \frac{1}{m}(u + w)$$

# Canonical form ODE

**For a dynamical system, the canonical form usually involves a set of first-order ODEs. This means that a system of higher-order ODEs is converted into a system of first-order equations.**

$$\dot{x} = f(x, u, w)$$

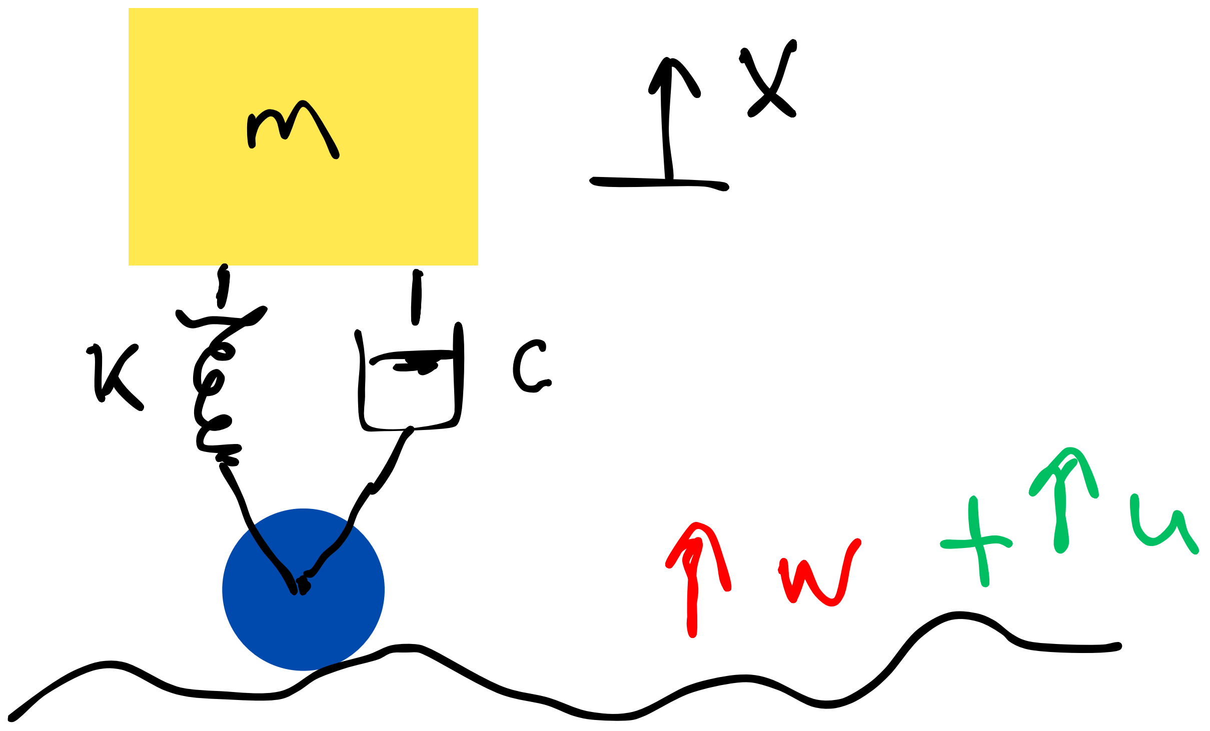
$x \in \mathbb{R}^n$ is a state;

$u \in \mathbb{R}^p$ is a control input;

$w \in \mathbb{R}^k$ is a disturbance input;

**The canonical ODE form essentially refers to expressing a system's dynamics in the simplest, first-order ODE form, which is easier for numerical simulation and analysis.**

# Ex.1: . Vehicle Suspension System




## 2nd-order linear ODE

$$\ddot{x} + \frac{k}{m}x + \frac{c}{m}\dot{x} = \frac{1}{m}(u + w)$$

## Canonical form: 1st-order ODE

$$\dot{x} = v$$ **definition of velocity**

$$\dot{v} = -\frac{k}{m}x - \frac{c}{m}v + \frac{1}{m}(u + w)$$

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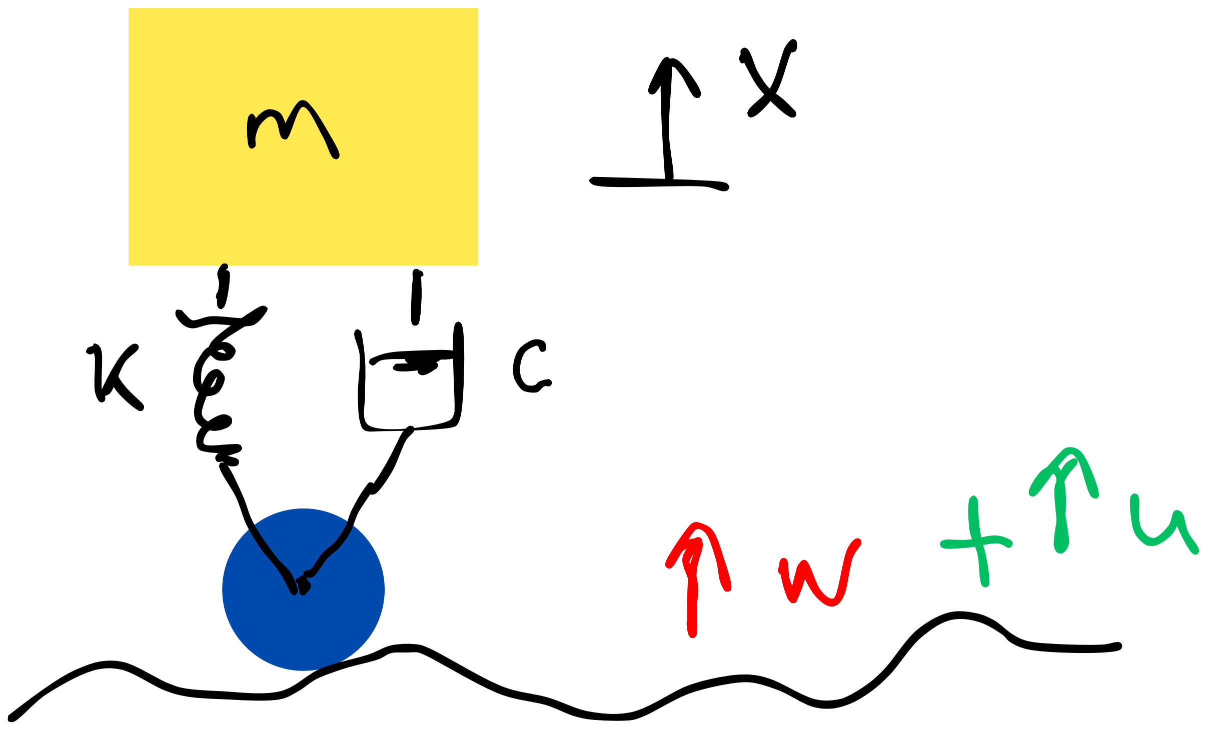


**Linearity:** functions are linear mappings

**Time invariance:** a certain input will always give the same output (up to timing), without regard to when the input was applied to the system.

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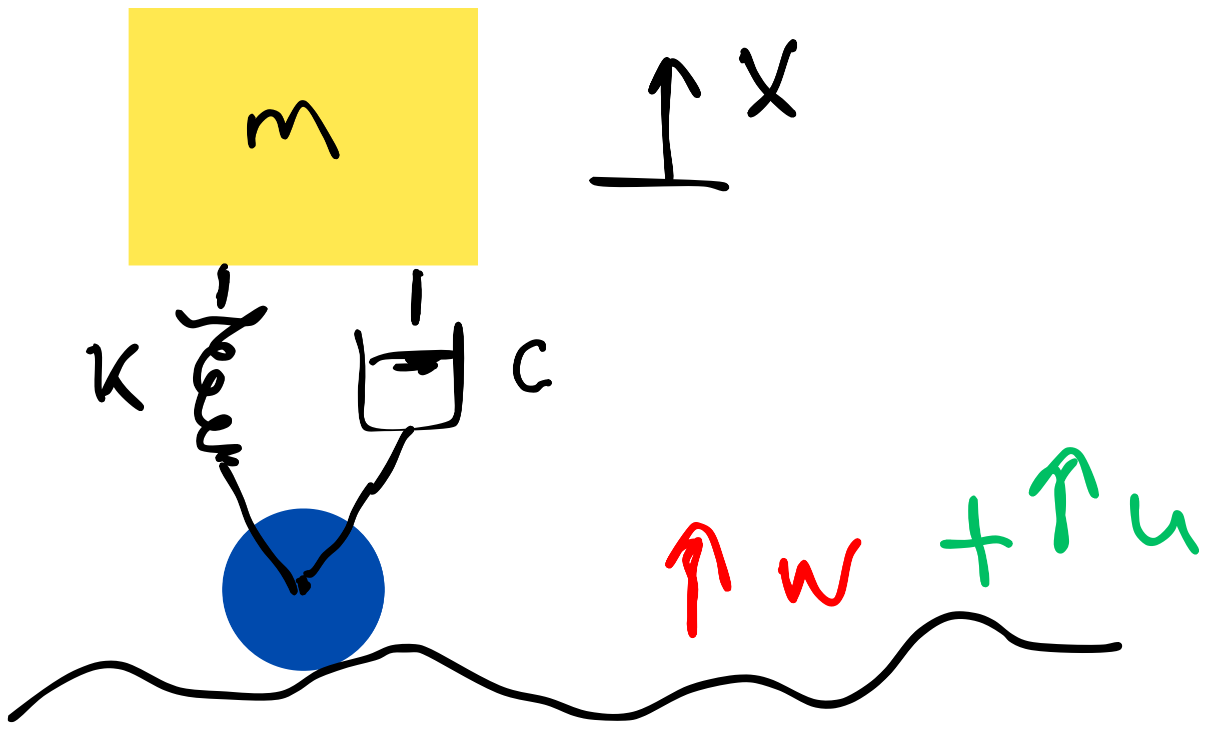


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$$\dot{x} = v$$

$$\dot{v} = -\frac{k}{m}x - \frac{c}{m}v + \frac{1}{m}(u + w)$$

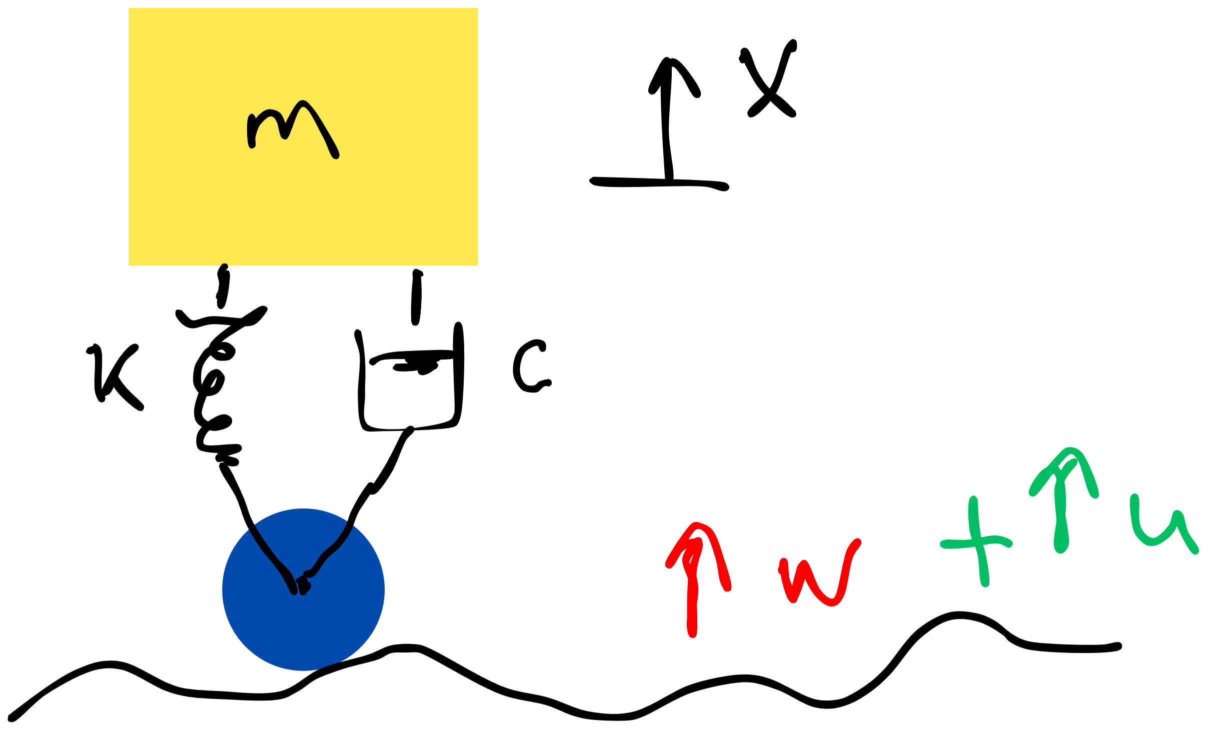
## Measurements

only position $y = x$

or

complete state $y = \begin{pmatrix} x \\ v \end{pmatrix}$

# Ex.1: . Vehicle Suspension System




**Linearity:** functions are linear mappings

**Time invariance:** a certain input will always give the same output (up to timing), without regard to when the input was applied to the system.

## Canonical matrix form of linear time invariant (LTI) systems

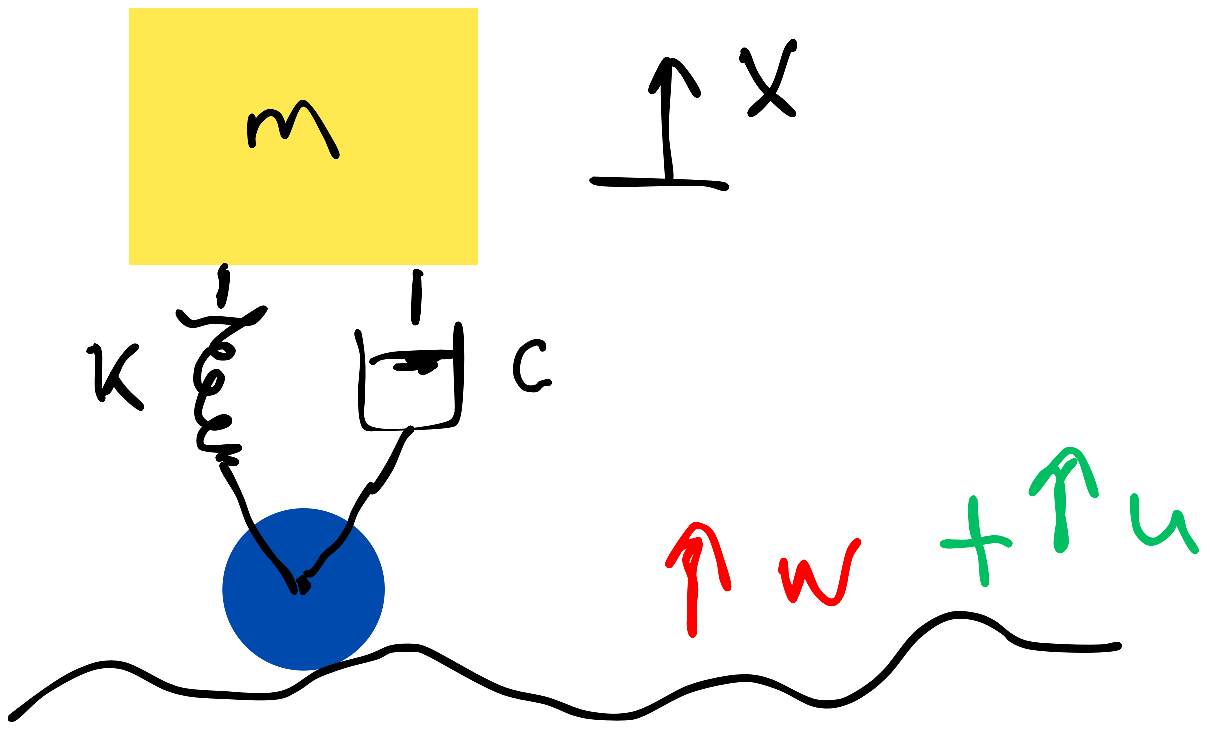
$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} w$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + 0u$$

or

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + 0u$$

# Ex.1: . Vehicle Suspension System

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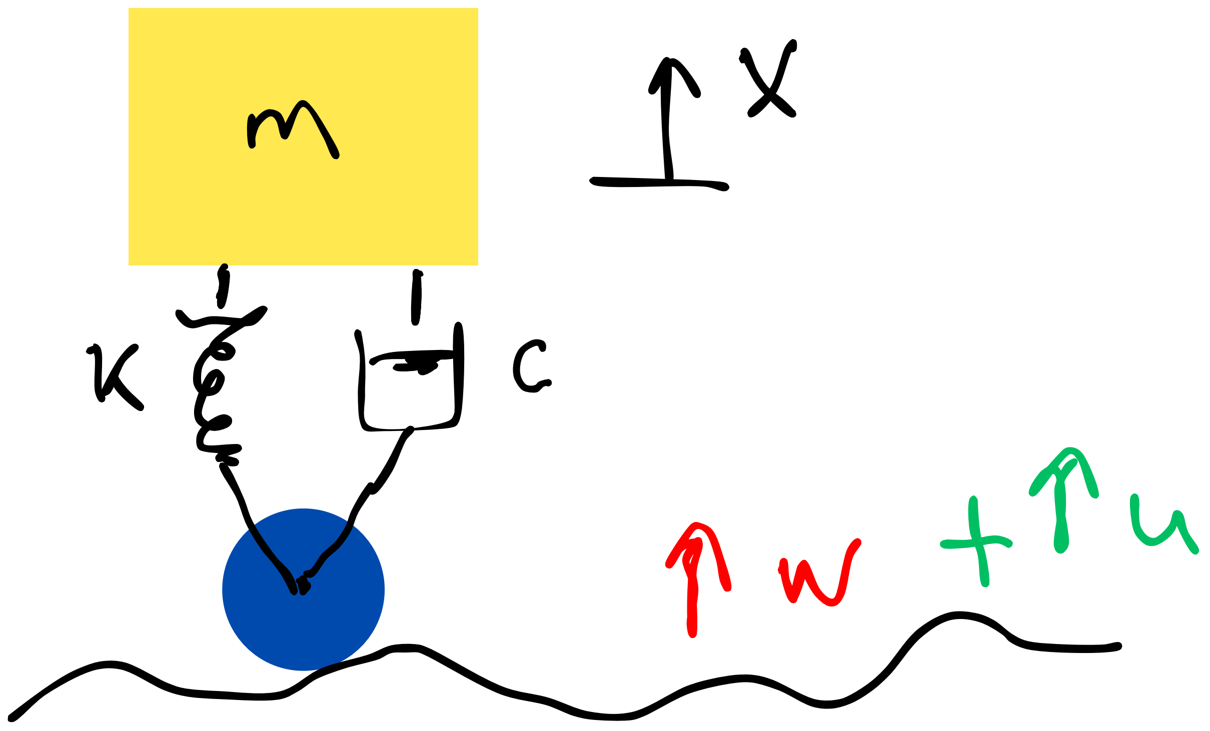
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + 0\,u$$

for example

# Ex.1: . Vehicle Suspension System




**Linearity:** functions are linear mappings

**Time invariance:** a certain input will always give the same output (up to timing), without regard to when the input was applied to the system.

## Canonical matrix form of linear time invariant (LTI) systems

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}}_{\text{dynamic matrix}} \underbrace{x}_{\text{state vector}} + \overbrace{\underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}}^{\text{control matrix}} \underbrace{u}_{\text{control vector}} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{\text{disturbance matrix}} \overset{\text{disturbance}}{\underline{w}}$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\text{sensor matrix}} \begin{pmatrix} x \\ v \end{pmatrix} + \underbrace{0}_{\text{direct matrix}} u$$

# State-space models of LTI systems

## Vehicle Suspension System

$$\dot{x} = \underset{\text{dynamic matrix}}{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}} \underset{\text{state vector}}{x} + \overset{\text{control matrix}}{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}} \underset{\text{control vector}}{u} + \underset{\text{disturbance matrix}}{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}} \overset{\text{disturbance}}{w}$$

$$y = \underset{\text{sensor matrix}}{(1 \quad 0)} \begin{pmatrix} x \\ v \end{pmatrix} + \underset{\text{direct matrix}}{0}\, u$$

## Canonical matrix form of linear time invariant (LTI) control systems

$$\dot{x} = \underset{\text{dynamic matrix}}{A} \overset{\text{state vector}}{x} + \underset{\text{control matrix}}{B} \overset{\text{control vector}}{u} + \underset{\text{disturbance matrix}}{D} \overset{\text{disturbance}}{w}$$

State equation

$$y = \underset{\text{sensor matrix}}{C} \overset{\text{state vector}}{x} + \underset{\text{direct matrix}}{R} \overset{\text{control vector}}{u}$$

Output equation

# State-space models of LTI systems

## State equation

$$\dot{x} = \underset{\text{dynamic matrix}}{A}\overset{\text{state vector}}{x} + \underset{\text{control matrix}}{B}\overset{\text{control vector}}{u} + \underset{\text{disturbance matrix}}{D}\overset{\text{disturbance}}{w}$$

## Output equation

$$y = \underset{\text{sensor matrix}}{C}\overset{\text{state vector}}{x} + \underset{\text{direct matrix}}{R}\overset{\text{control vector}}{u}$$

$x \in \mathbb{R}^n$ is a state;

$u \in \mathbb{R}^p$ is a control input;

$w \in \mathbb{R}^r$ is a disturbance input;

$y \in \mathbb{R}^m$ is a output vector;

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{m\times n}$, $R \in \mathbb{R}^{m\times p}$

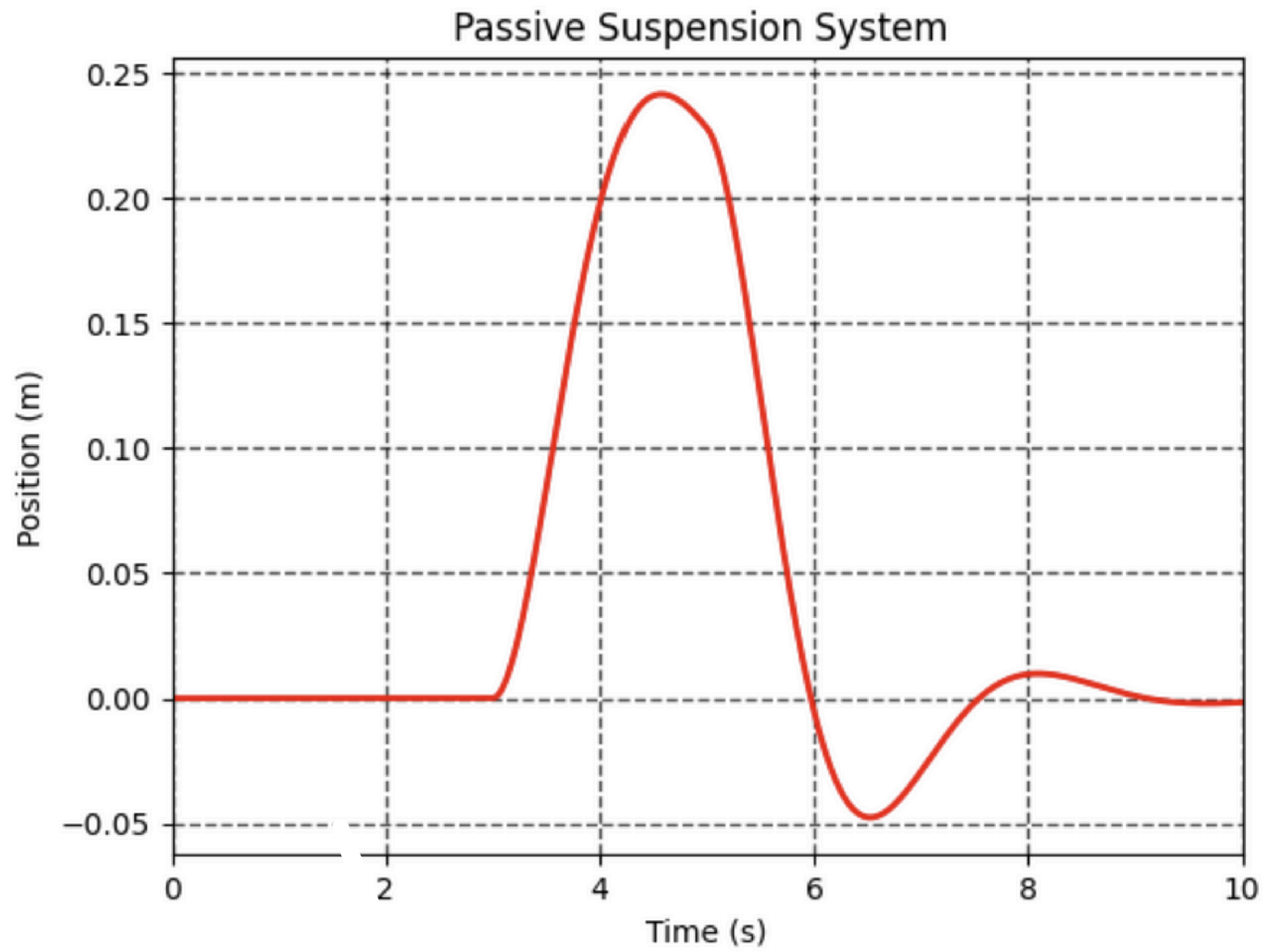
# Passive Suspension System

no control, i.e. u = 0

A step input represents a sudden change in road height, such as driving over a bump or into a pothole.

$$w = \begin{cases} 0, & t \leq 3.0 \\ 1.0, & 3.0 \leq t \leq 7.0 \\ 0, & t \geq 7 \end{cases}$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x + \cancel{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}} u + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} w$$



# Passive Suspension System

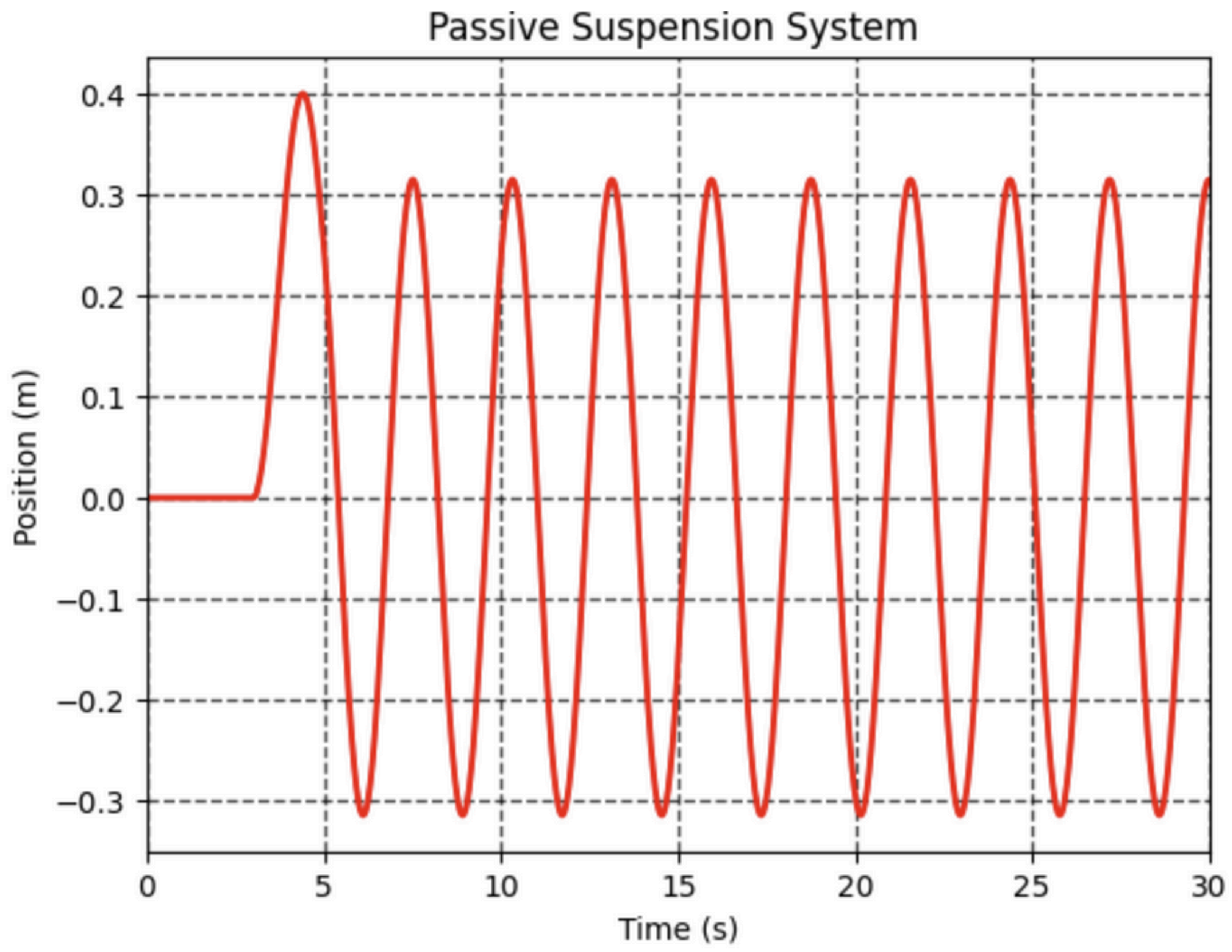
no control, i.e. u = 0

A step input represents a sudden change in road height, such as driving over a bump or into a pothole.

$$w = \begin{cases} 0, & t \leq 3.0 \\ 1.0, & 3.0 \leq t \leq 7.0 \\ 0, & t \geq 7 \end{cases}$$

Why we need **a damper** in the system?

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} x + \cancel{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} w$$



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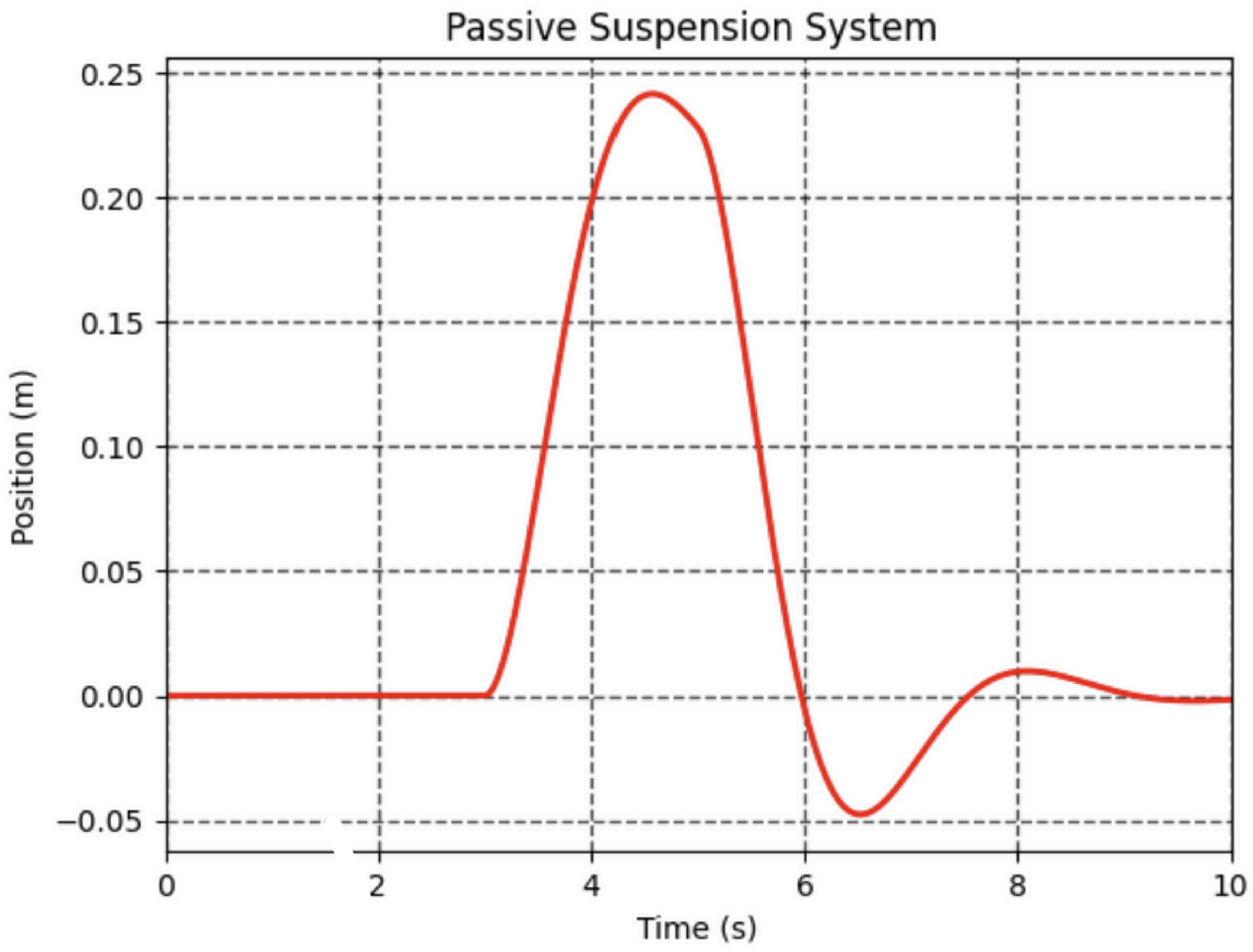


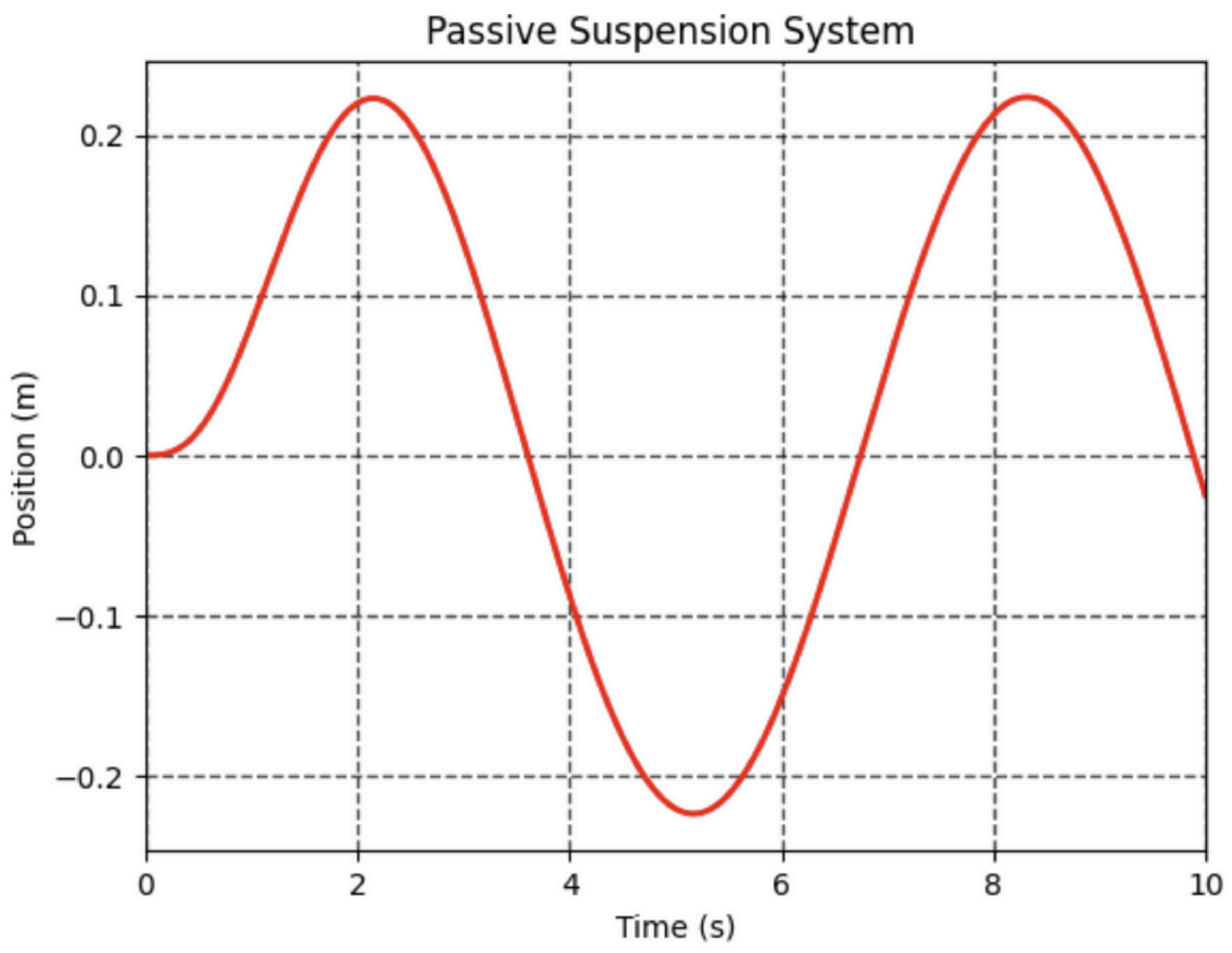
# Passive Suspension System

no control, i.e. u = 0

A sinusoidal input represents a periodic road profile, such as driving on a washboard or corrugated road.

$$w = sin(t)$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x + \cancel{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}} u + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} w$$



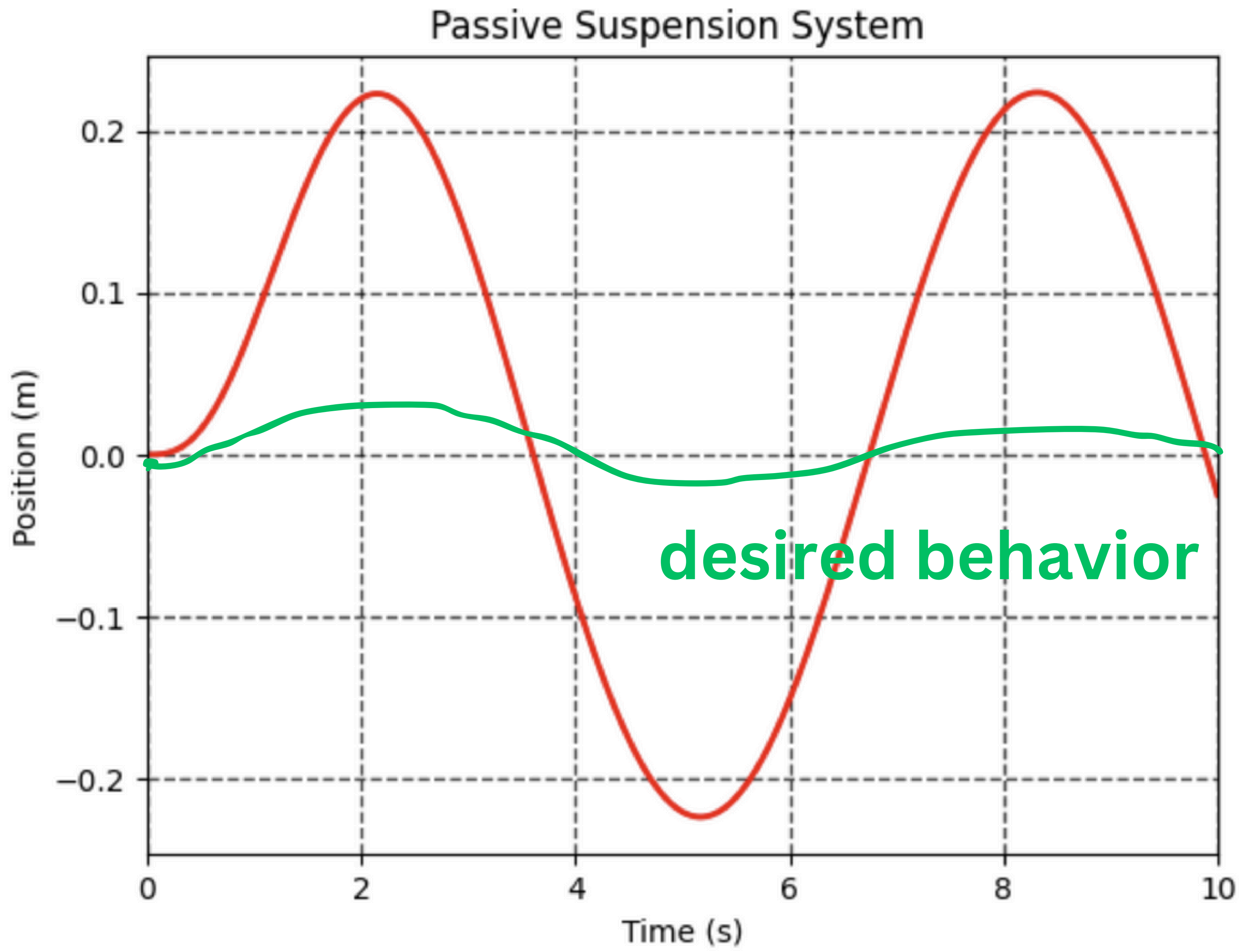
# Active Suspension System

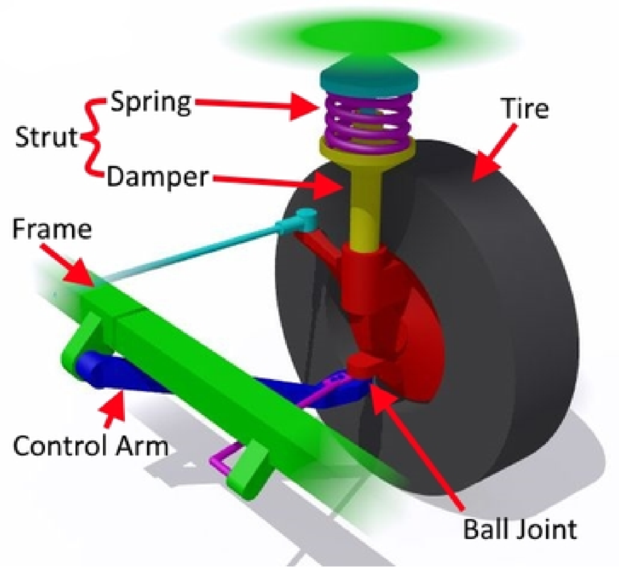
will be considered later in the course

A sinusoidal input represents a periodic road profile, such as driving on a washboard or corrugated road.

$$w = sin(t)$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} w$$



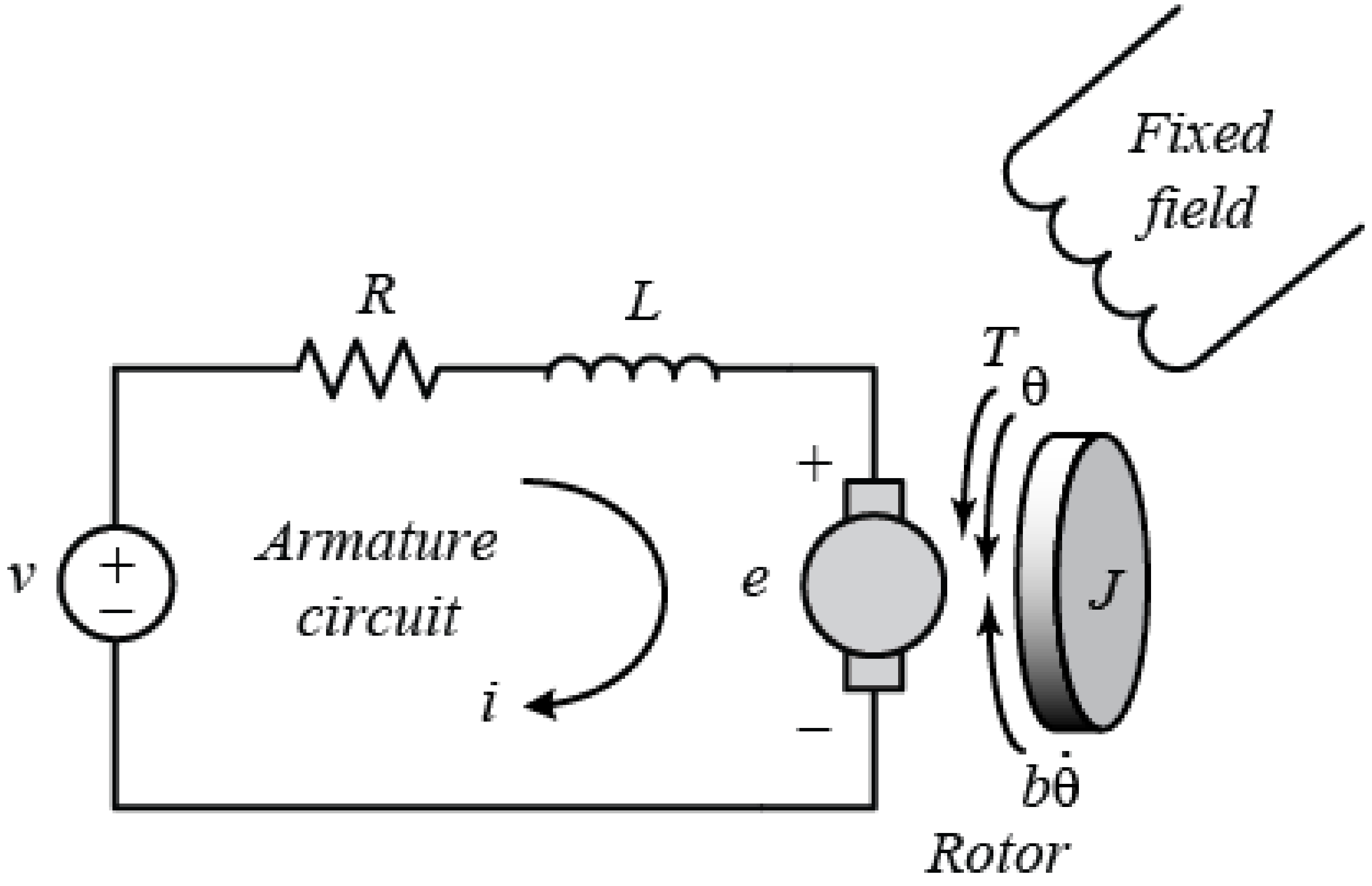

Design a controller to minimize vibrations and improve ride comfort while maintaining stability.

# Ex.2: DC motor


https://www.youtube.com/watch?v=LFvIl68_cOQ

# Ex.2: DC motor

**A common actuator in control systems is the DC motor. A DC motor converts electrical energy into mechanical energy and is widely used in applications like robotics, industrial automation, and electric vehicles.**



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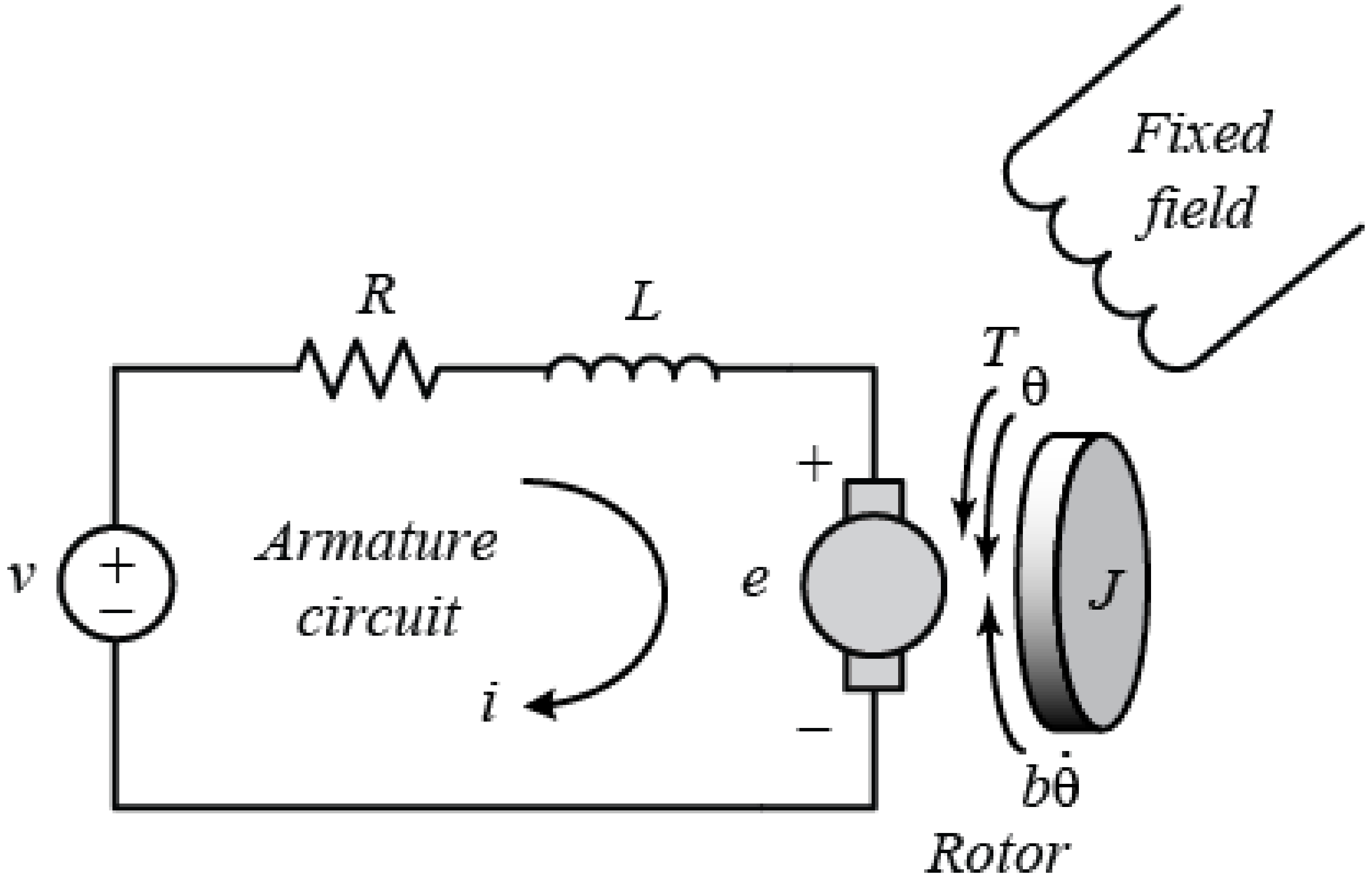


**The electrical behavior of the DC motor is described by the armature circuit equation:**

$$\underbrace{L\frac{di}{dt}}_{v_L} + \underbrace{Ri}_{v_R} = V - \underbrace{K_e\dot{\theta}}_{v_e}$$

**Kirchhoff's voltage law**

**assuming that the magnetic field is constant**

# Ex.2: DC motor

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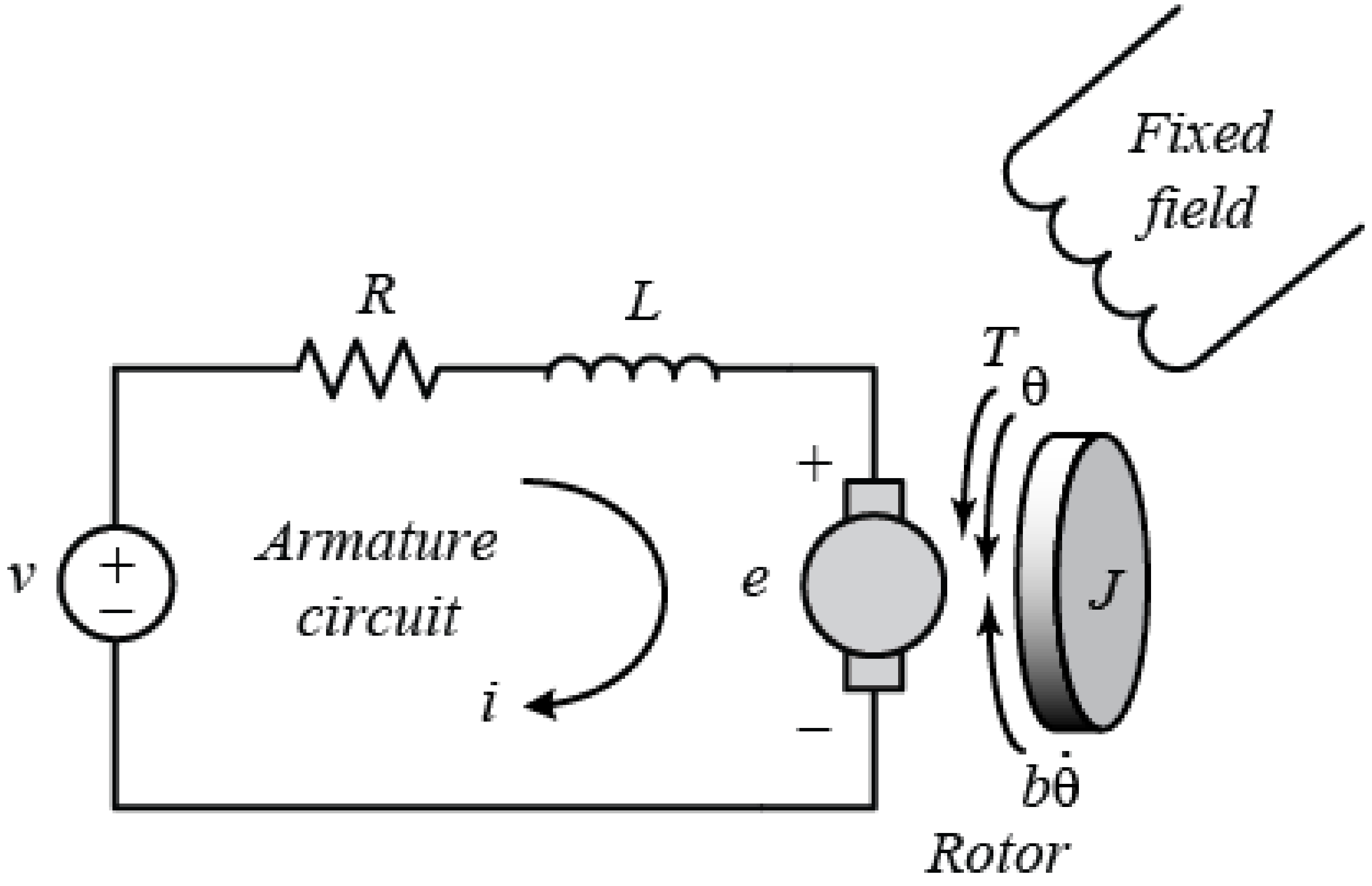



**The electrical behavior of the DC motor is described by the armature circuit equation:**

$$L\frac{di}{dt} + Ri = V - K_e\dot{\theta}$$

electric inductance

electric resistance

Applied armature voltage (input).

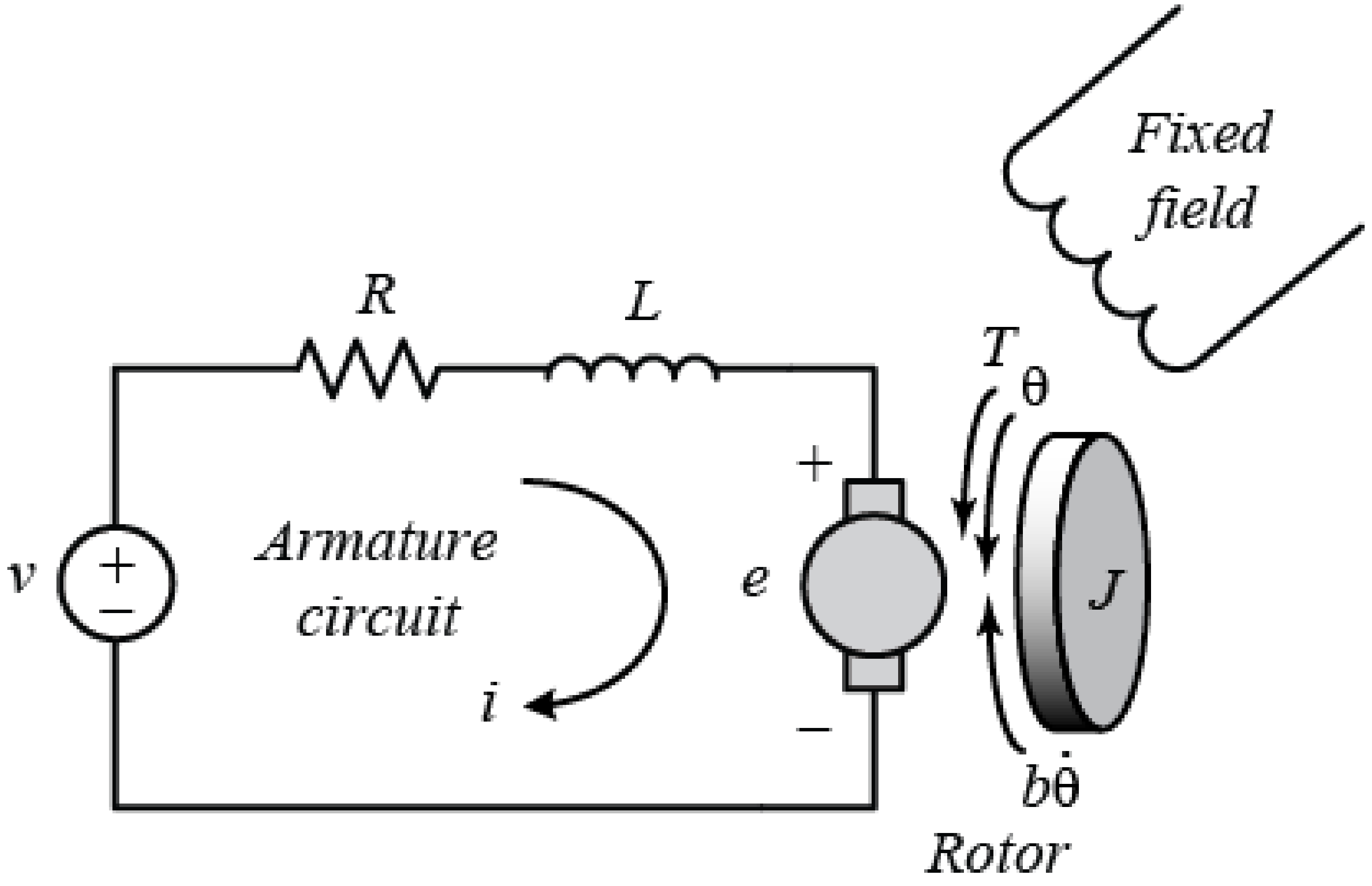
electromotive force constant

Kirchhoff's voltage law

assuming that the magnetic field is constant

# Ex.2: DC motor

**A common actuator in control systems is the DC motor. A DC motor converts electrical energy into mechanical energy and is widely used in applications like robotics, industrial automation, and electric vehicles.**




**The mechanical behavior of the DC motor is described by the torque equation:**

motor torque constant

$$J\ddot{\theta} + b\dot{\theta} = K_t i$$

moment of inertia of the rotor

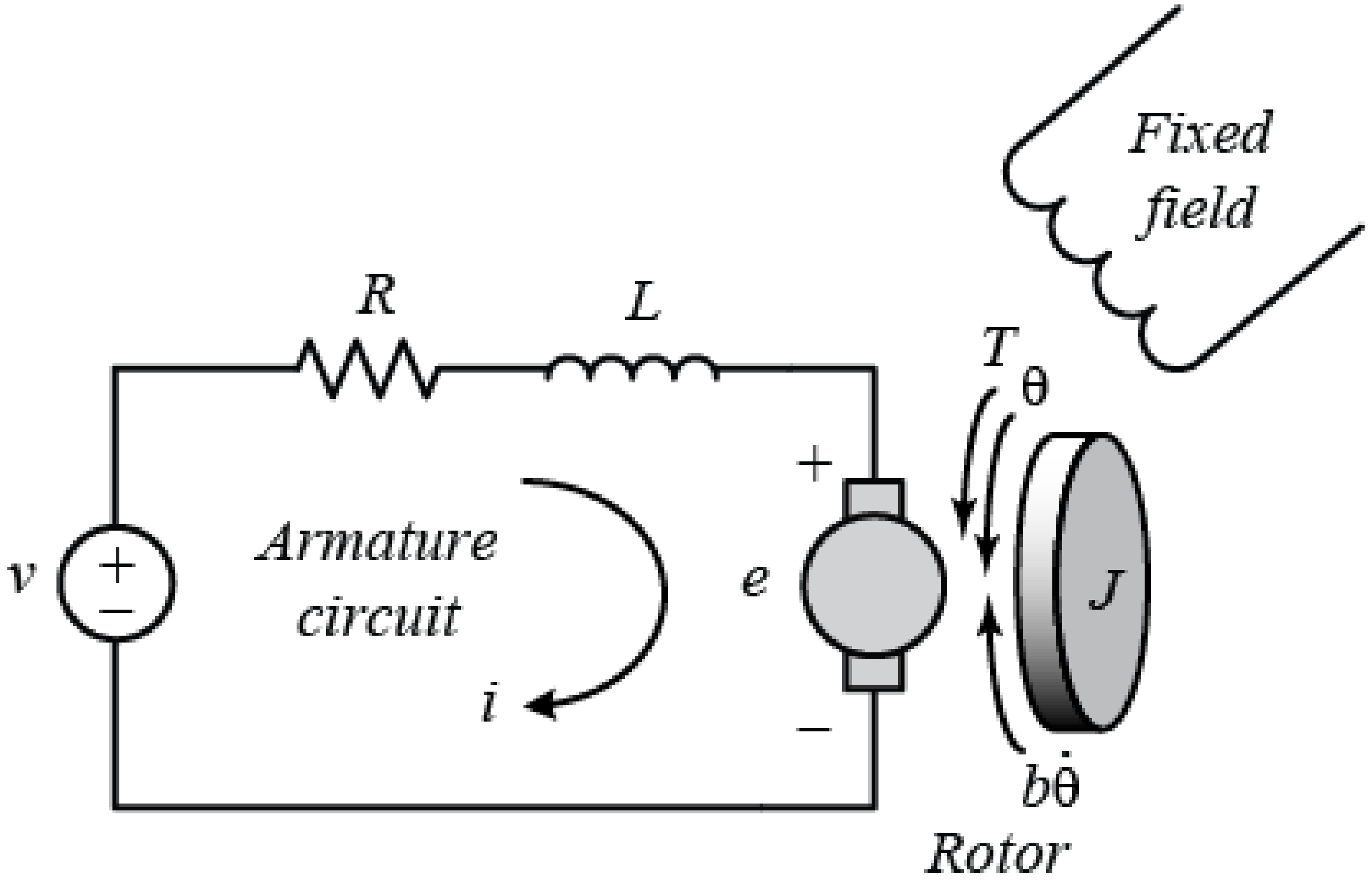
viscous friction constant

motor torque

**Newton's 2nd law**

# Ex.2: DC motor

**A common actuator in control systems is the DC motor. A DC motor converts electrical energy into mechanical energy and is widely used in applications like robotics, industrial automation, and electric vehicles.**




**Combined System**

$$J\ddot{\theta} + b\dot{\theta} = K_t i$$

$$L\frac{di}{dt} + Ri = V - K_e\dot{\theta}$$

**Exercise: Derive canonical state space model.**
**Is system linear?**

# Nonlinear Control Systems

# Ex.3: Pendulum

## Newton's 2nd law (rotation motion):

moment of inertia

pendulum torque

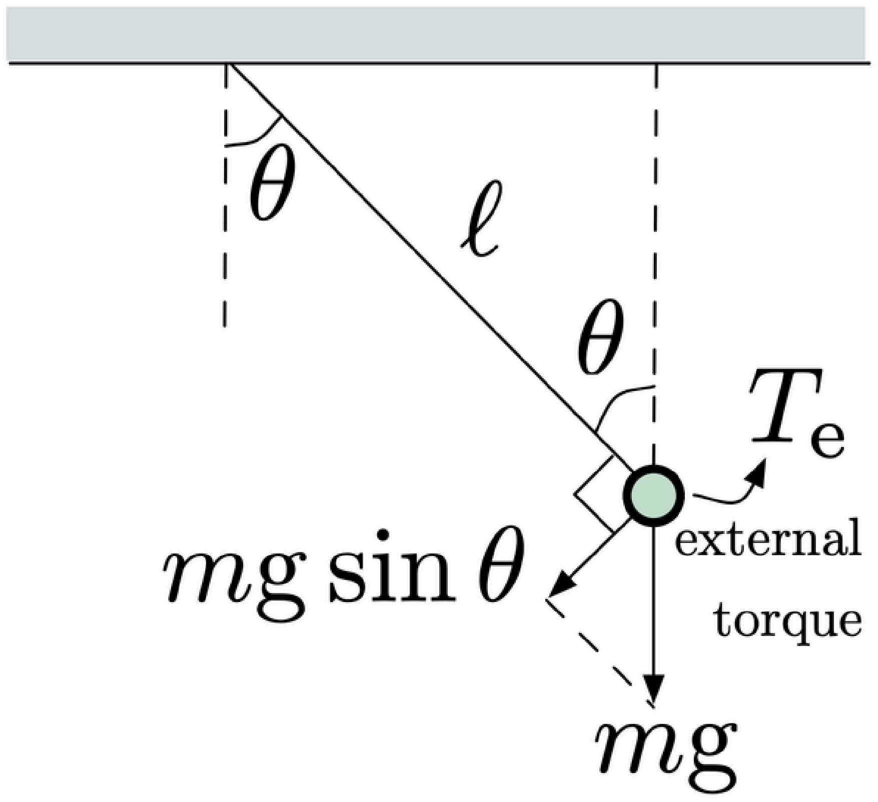
$$\underbrace{ml^2}_{\text{moment of inertia}} \cdot \underbrace{\ddot{\theta}}_{\text{angular acceleration}} = \underbrace{-mg\sin(\theta)}_{\text{force}} \cdot \underbrace{l}_{\text{lever arm}} + \underbrace{T_e}_{\text{external torque}}$$

angular acceleration

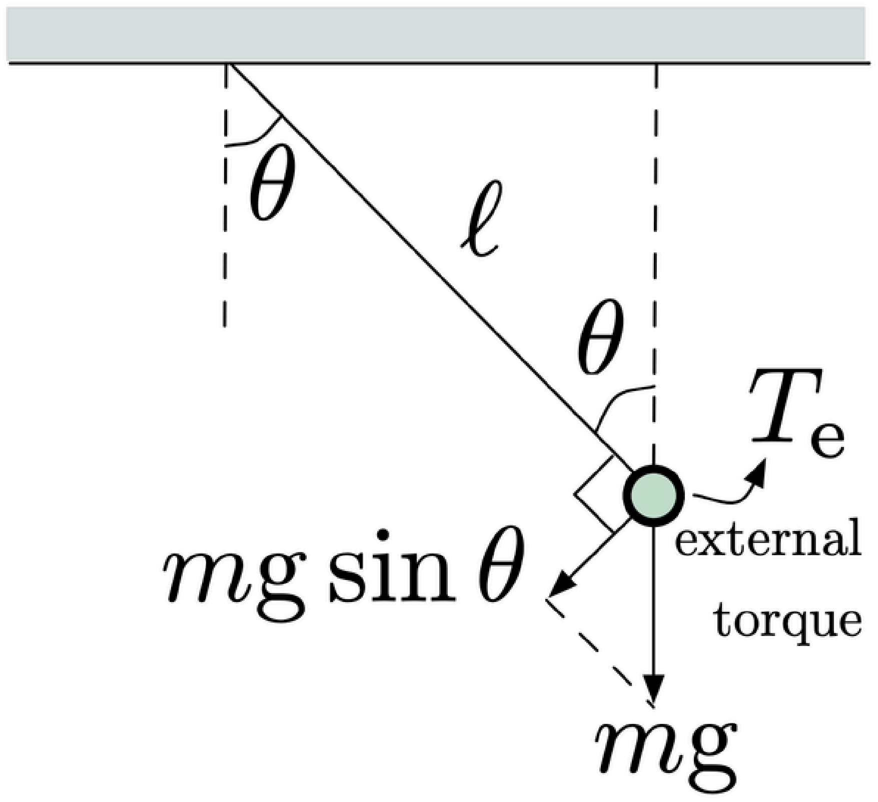
force

lever arm

external torque



# Ex.3: Pendulum




## Newton's 2nd law (rotation motion):

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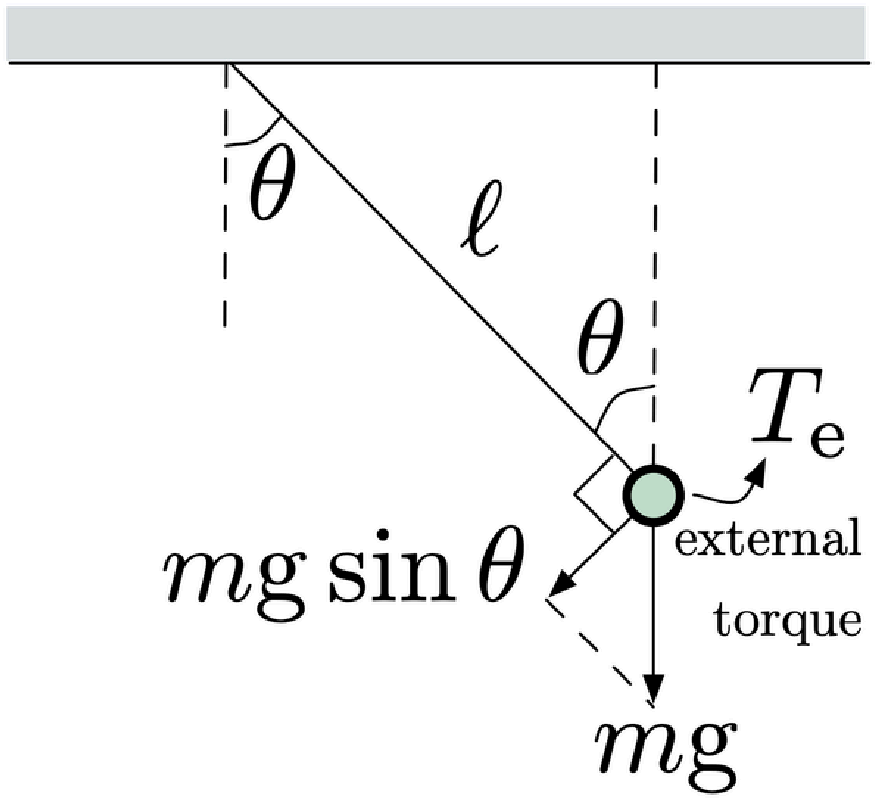
$$\underbrace{ml^2}_{\text{moment of inertia}} \cdot \underbrace{\ddot{\theta}}_{\text{angular acceleration}} = \underbrace{-mg\sin(\theta)}_{\text{force}} \cdot \underbrace{l}_{\text{lever arm}} + \underbrace{T_e}_{\text{external torque}}$$

## Nonlinear 2nd order equation

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{1}{ml^2}T_e$$

# Ex.3: Pendulum

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## Nonlinear 2nd order equation

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{1}{ml^2}T_e$$

**Let** $\theta_1 = \theta, \;\; \theta_2 = \dot{\theta}$

$$\boxed{\begin{gathered}\dot{x} = f(x, u) \\ x = \begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} \quad u = T_e\end{gathered}}$$

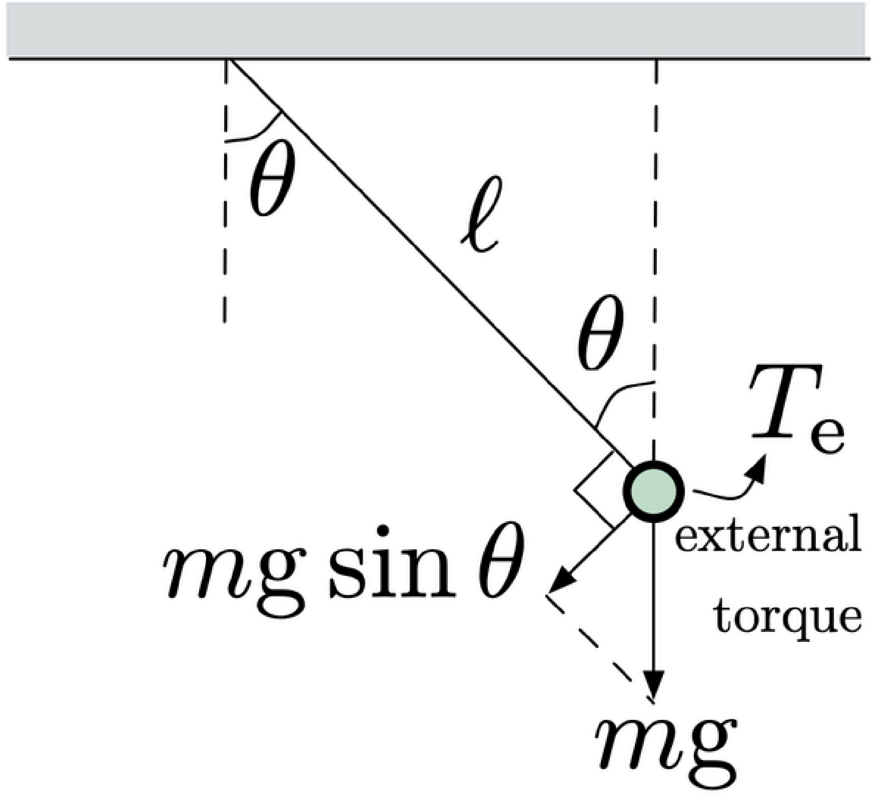
## Canonical Nonlinear State Space Model

# Ex.3: Pendulum

## Newton's 2nd law (rotation motion):

pendulum torque

$$\underbrace{ml^2}_{\text{moment of inertia}} \cdot \underbrace{\ddot{\theta}}_{\text{angular acceleration}} = \underbrace{-mg\sin(\theta)}_{\text{force}} \cdot \underbrace{l}_{\text{lever arm}} + \underbrace{T_e}_{\text{external torque}}$$




$$\boxed{\begin{array}{c} \dot{x} = f(x, u) \\ x = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad u = T_e \end{array}}$$

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

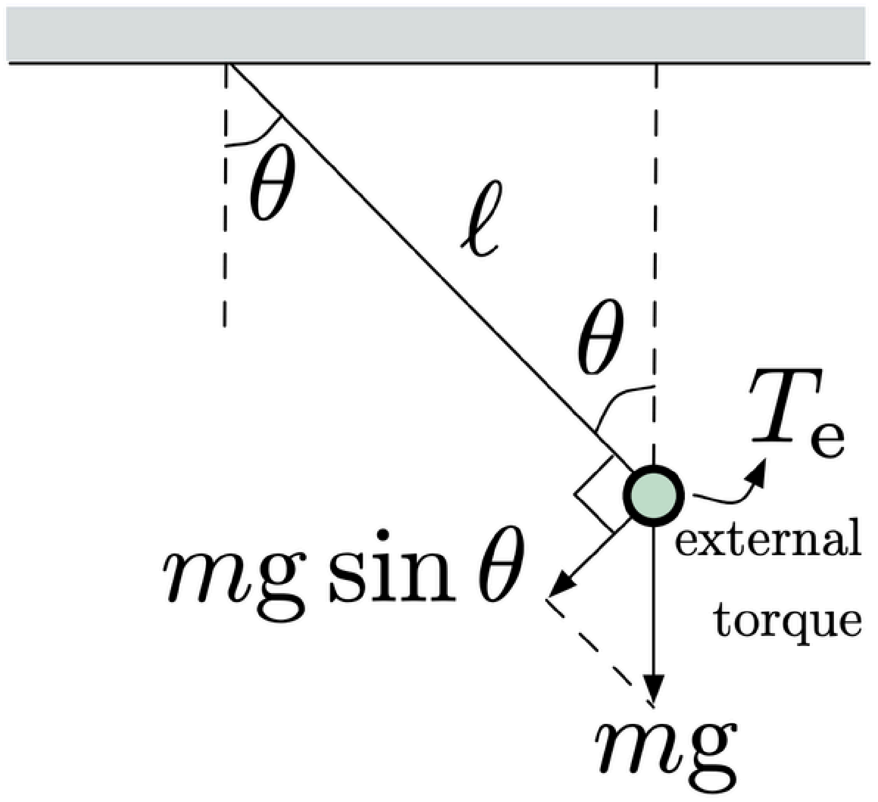
**Canonical Nonlinear State Space Model**

# Ex.3: Pendulum

## Newton's 2nd law (rotation motion):

pendulum torque

$$\underbrace{ml^2}_{\text{moment of inertia}} \cdot \underbrace{\ddot{\theta}}_{\text{angular acceleration}} = \underbrace{-mg\sin(\theta)}_{\text{force}} \cdot \underbrace{l}_{\text{lever arm}} + \underbrace{T_e}_{\text{external torque}}$$




## LTI State Space model

$$\dot{x} = Ax + Bu + Dw$$

How to get it?

$$\boxed{\begin{gathered} \dot{x} = f(x,u) \\ x = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad u = T_e \end{gathered}}$$

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

## Canonical Nonlinear State Space Model

# General linearisation procedure

- Start from nonlinear state-space model $\dot{x} = f(x, u)$
- Find equilibrium point $(x_0, u_0)$ such that $f(x_0, u_0) = 0$

  *Note:* different systems may have different equilibria, not necessarily $(0, 0)$, so we need to shift variables:

$$\underline{x} = x - x_0 \qquad \underline{u} = u - u_0$$
$$\underline{f}(\underline{x}, \underline{u}) = f(\underline{x} + x_0, \underline{u} + u_0) = f(x, u)$$

  Note that the transformation is *invertible*:

$$x = \underline{x} + x_0, \qquad u = \underline{u} + u_0$$

# General linearisation procedure

- Pass to shifted variables $\underline{x} = x - x_0$, $\underline{u} = u - u_0$

$$
\begin{aligned}
\dot{\underline{x}} &= \dot{x} && (x_0 \text{ does not depend on } t) \\
&= f(x, u) \\
&= \underline{f}(\underline{x}, \underline{u})
\end{aligned}
$$

— equivalent to original system

- The transformed system is in equilibrium at $(0, 0)$:

$$\underline{f}(0, 0) = f(x_0, u_0) = 0$$

# General linearisation procedure

- Now linearize:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}, \qquad \text{where } A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=x_0\\u=u_0}}, \; B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=x_0\\u=u_0}}$$

# General linearisation procedure

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- Why do we require that $f(x_0, u_0) = 0$ in equilibrium?

# General linearisation procedure

- This requires some thought. Indeed, we may talk about a *linear approximation* of any smooth function $f$ at any point $x_0$:

  $$f(x) \approx f(x_0)+f'(x_0)(x-x_0) \qquad \text{— } f(x_0) \text{ does not have to be } 0$$

- The key is that we want to approximate a given nonlinear system $\dot{x} = f(x, u)$ by a *linear* system $\dot{x} = Ax + Bu$ (may have to shift coordinates: $x \mapsto x - x_0$, $u \mapsto u - u_0$)

  Any linear system *must* have an equilibrium point at $(x, u) = (0, 0)$:

  $$f(x, u) = Ax + Bu \qquad f(0, 0) = A0 + B0 = 0.$$

# General linearisation procedure

- Start from nonlinear state-space model $\dot{x} = f(x, u)$

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

# General linearisation procedure

- Start from nonlinear state-space model $\dot{x} = f(x, u)$

$$\dot{\theta}_1 = \theta_2$$
$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

- Find equilibrium point $(x_0, u_0)$ such that $f(x_0, u_0) = 0$

$\theta_1^0 = 0$, $\theta_2^0 = 0$, $T_e^0 = 0$ is an equilibrium point

# General linearisation procedure

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Is it the only one?

# General linearisation procedure

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well, it is not, but let's chose this one

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Is it the only one?

- Pass to shifted variables $\underline{x} = x - x_0$, $\underline{u} = u - u_0$

- The transformed system is in equilibrium at $(0, 0)$:

well, it is not, but let's chose this one

$$\underline{f}(0, 0) = f(x_0, u_0) = 0$$

# General linearisation procedure

- Start from nonlinear state-space model $\dot{x} = f(x, u)$

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- Pass to shifted variables $\underline{x} = x - x_0$, $\underline{u} = u - u_0$
- The transformed system is in equilibrium at $(0, 0)$:

well, it is not, but let's chose this one

$$\underline{f}(0, 0) = f(x_0, u_0) = 0$$

Done ✓

# General linearisation procedure

- Now linearize:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \qquad \text{where } A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=x_0\\u=u_0}}, \quad B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=x_0\\u=u_0}}$$

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

in $(\theta_1, \theta_2, T_e) = (0, 0, 0)$

- matrix A

$$\left.\frac{\partial f_1}{\partial \theta_1}\right|_{\substack{\theta=(0,0)\\T_e=0}} = 0 \; ; \; \left.\frac{\partial f_1}{\partial \theta_2}\right|_{\substack{\theta=(0,0)\\T_e=0}} = 1$$

$$\left.\frac{\partial f_2}{\partial \theta_1}\right|_{\substack{\theta=(0,0)\\T_e=0}} = -\frac{g}{l}\cos(\theta_1)\Big|_{\substack{\theta=(0,0)\\T_e=0}} = -\frac{g}{l} \; ; \; \left.\frac{\partial f_2}{\partial \theta_2}\right|_{\substack{\theta=(0,0)\\T_e=0}} = 0$$

$$\Rightarrow A = \begin{pmatrix} 0, & 1 \\ -\frac{g}{l}, & 0 \end{pmatrix}$$

# General linearisation procedure

- Now linearize:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \qquad \text{where } A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=x_0\\u=u_0}}, \quad B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=x_0\\u=u_0}}$$

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$

in $(\theta_1, \theta_2, T_e) = (0, 0, 0)$

- matrix B

$$\left.\frac{\partial f_1}{\partial T_e}\right|_{\substack{T_e=0\\ \theta=(0,0)}} = 0$$

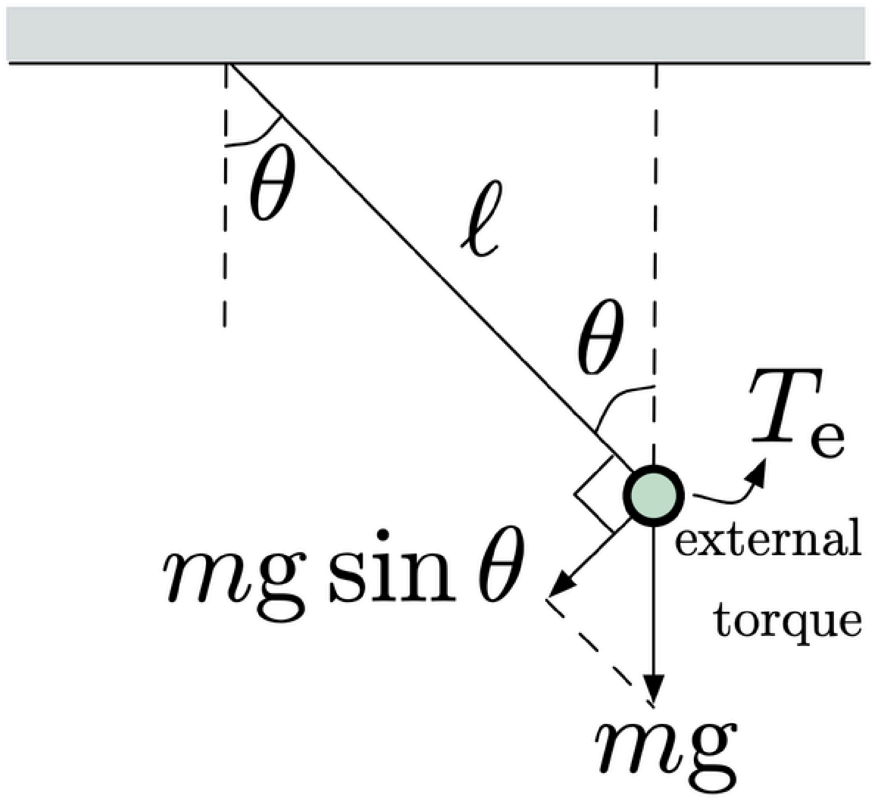
$$\left.\frac{\partial f_2}{\partial T_e}\right|_{\substack{T_e=0\\ \theta=(0,0)}} = \frac{1}{ml^2}$$

$$\Rightarrow B = \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}$$

# Ex.3: Pendulum

## Newton's 2nd law (rotation motion):

$$\underbrace{ml^2}_{\text{moment of inertia}} \cdot \underbrace{\ddot{\theta}}_{\text{angular acceleration}} = \overbrace{\underbrace{-mg\sin(\theta)}_{\text{force}} \cdot \underbrace{l}_{\text{lever arm}}}^{\text{pendulum torque}} + \underbrace{T_e}_{\text{external torque}}$$



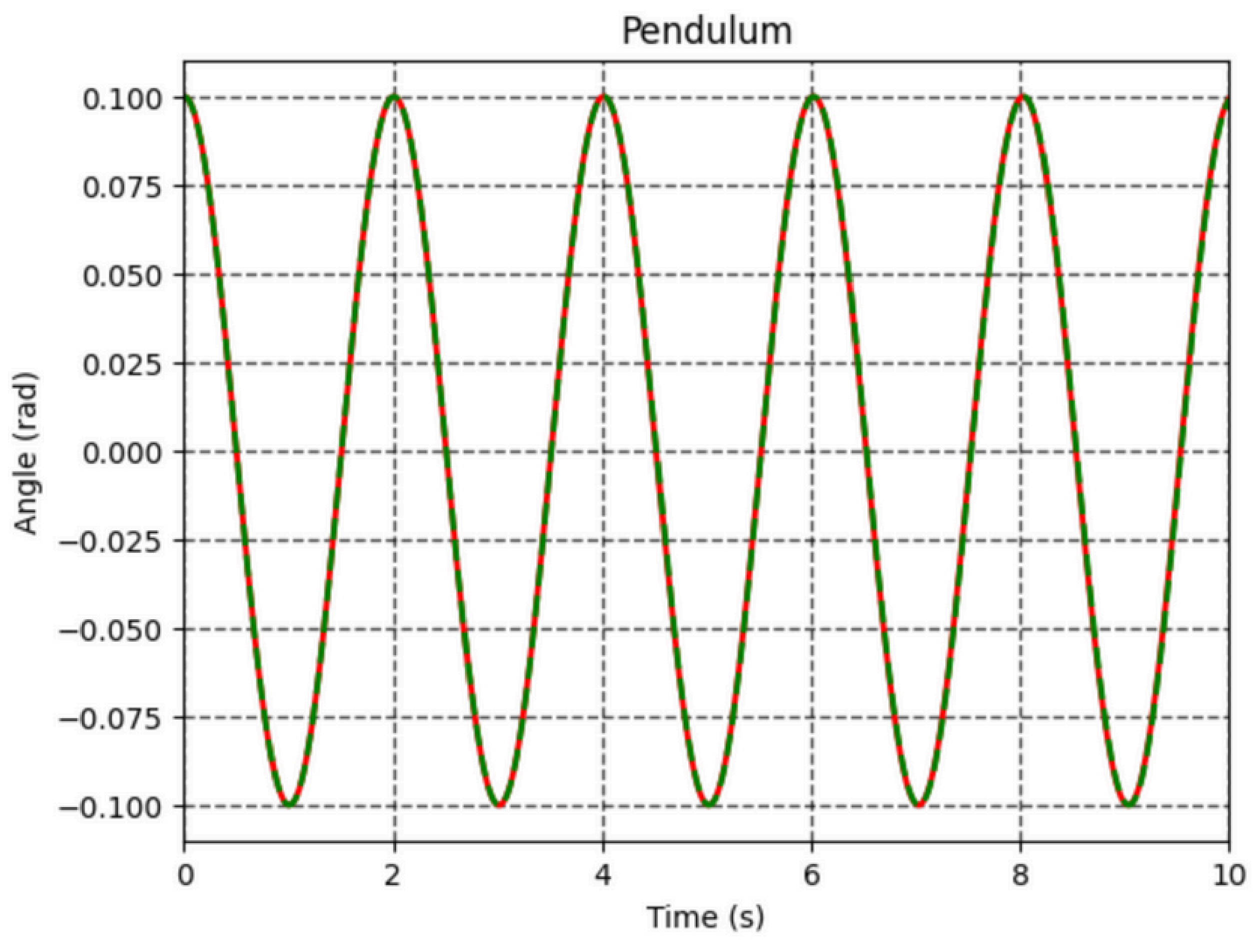

## LTI state space model

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} T_e$$

# Ex.3: Pendulum

**Linear System -**

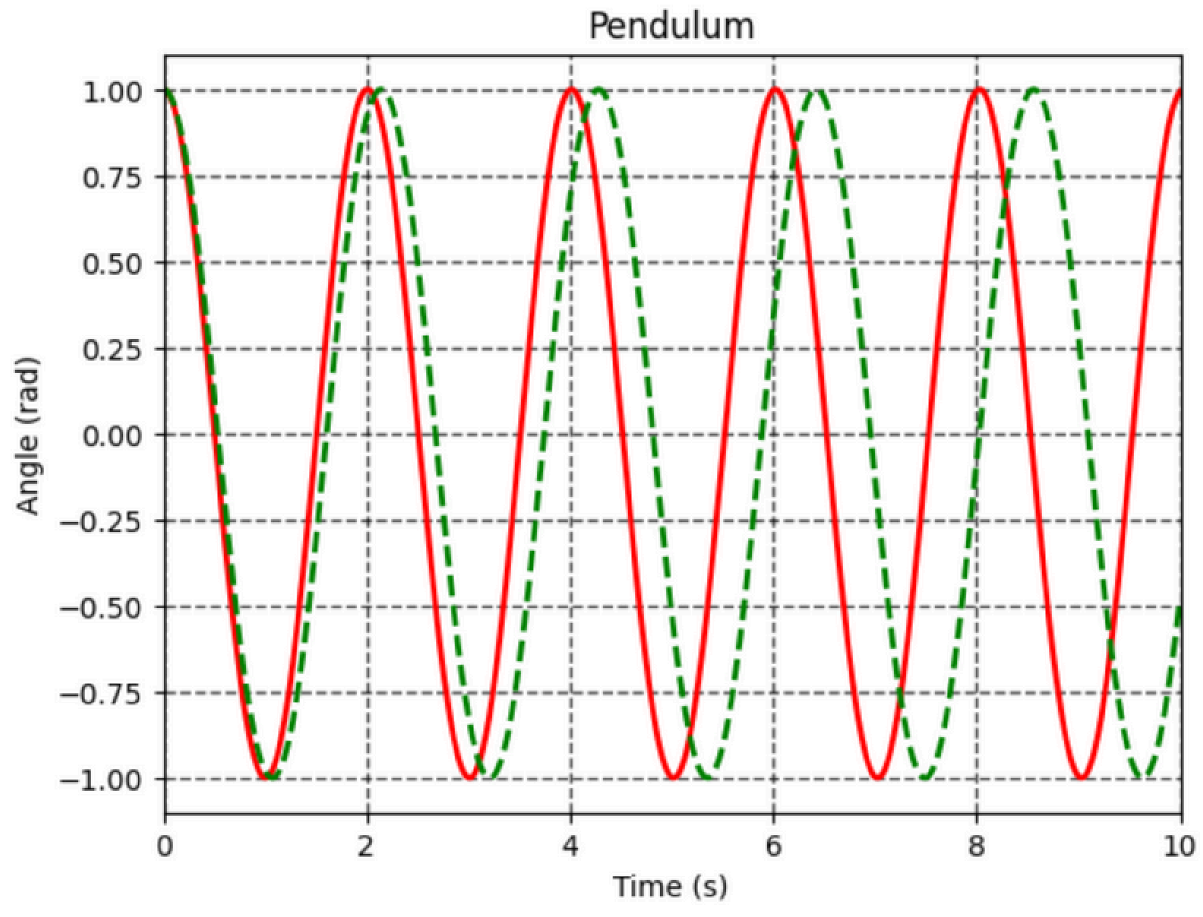
$$\begin{pmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -\frac{g}{l} & 0\end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} + \begin{pmatrix}0 \\ \frac{1}{ml^2}\end{pmatrix} T_e$$




**Non linear system --**

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin(\theta_1) + \frac{1}{ml^2}T_e$$



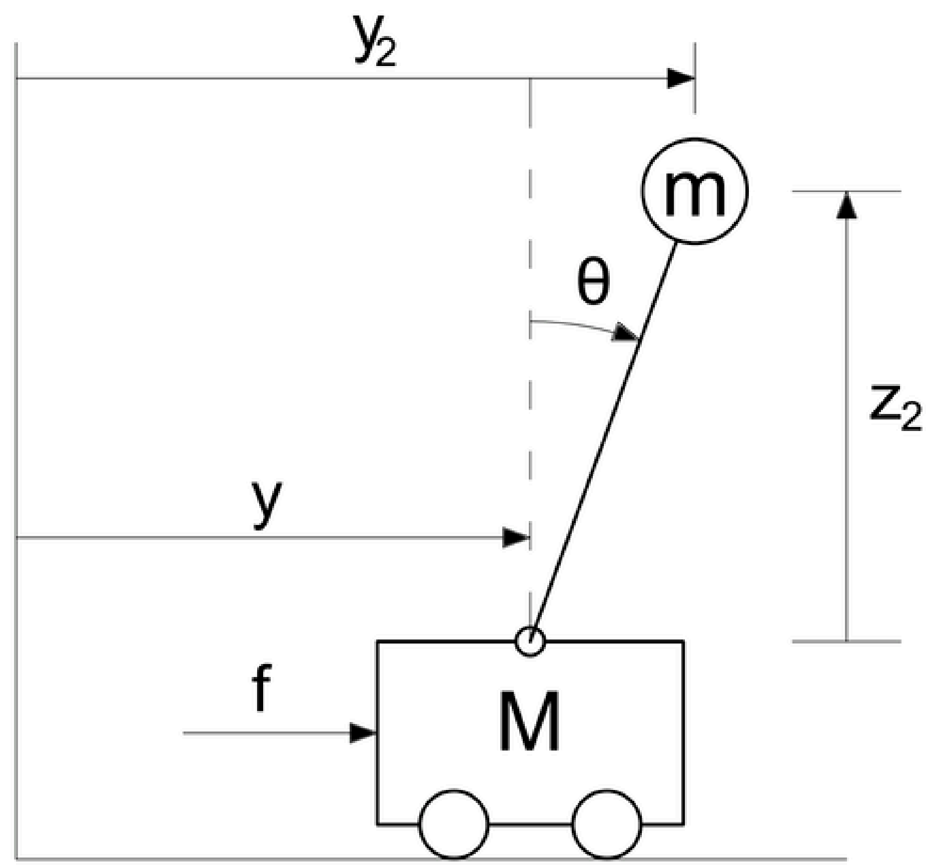

As angle increases, linear and nonlinear system trajectories diverge.

# Ex. 4: Modeling a balance system


# Ex. 4: Modeling a balance system


**Real-world examples modeled as an inverted pendulum on the cart.**

# Ex. 4: Modeling a balance system




**and other exercises for today session...**

**Please, install Jupyter Notebook**

https://jupyter.org/install

**and work on the notebook you can find at**

**https://perso.ensta-paris.fr/~manzaner/Cours/AUT202/**

**The completed notebook should be sent to your tutor before the beginning of the next session.**

**Please add [ APM_4AUT2_TA] to the topic of e-mail.**