An Introduction to *Stochastic Dual Dynamic Programming* (SDDP).

V. Leclère (CERMICS, ENPC)

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Introduction

- Large scale stochastic optimization problems are hard to solve
- Different ways of attacking such problems:
  - decompose the problem and coordinate solutions
  - construct easily solvable approximations (Linear Programming)
  - find approximate value functions or policies
- Behind the name SDDP, *Stochastic Dual Dynamic Programming*, one finds three different things:
  - a class of algorithms, based on specific mathematical assumptions
  - a specific implementation of an algorithm
  - a software implementing this method, and developed by the PSR company

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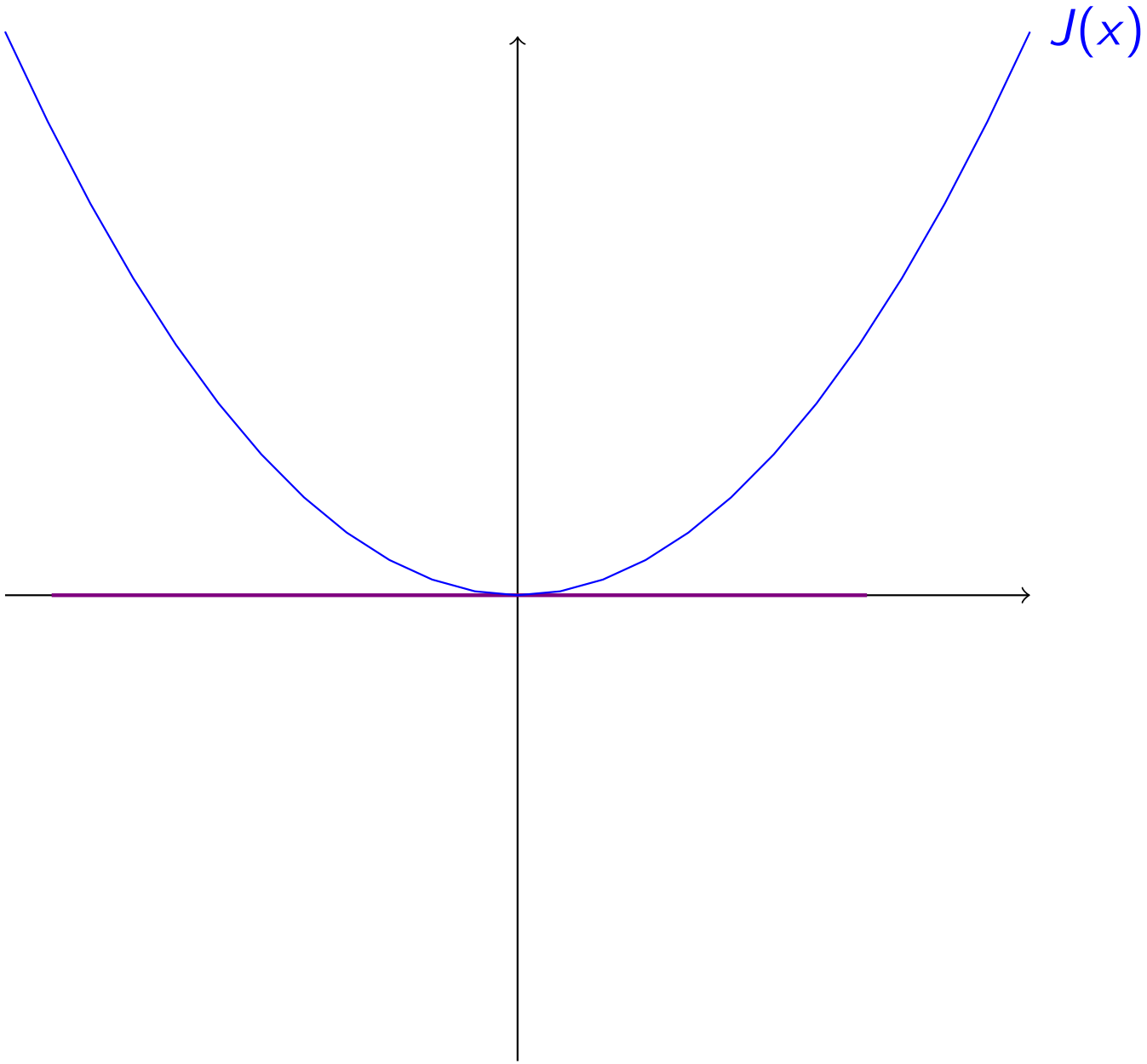
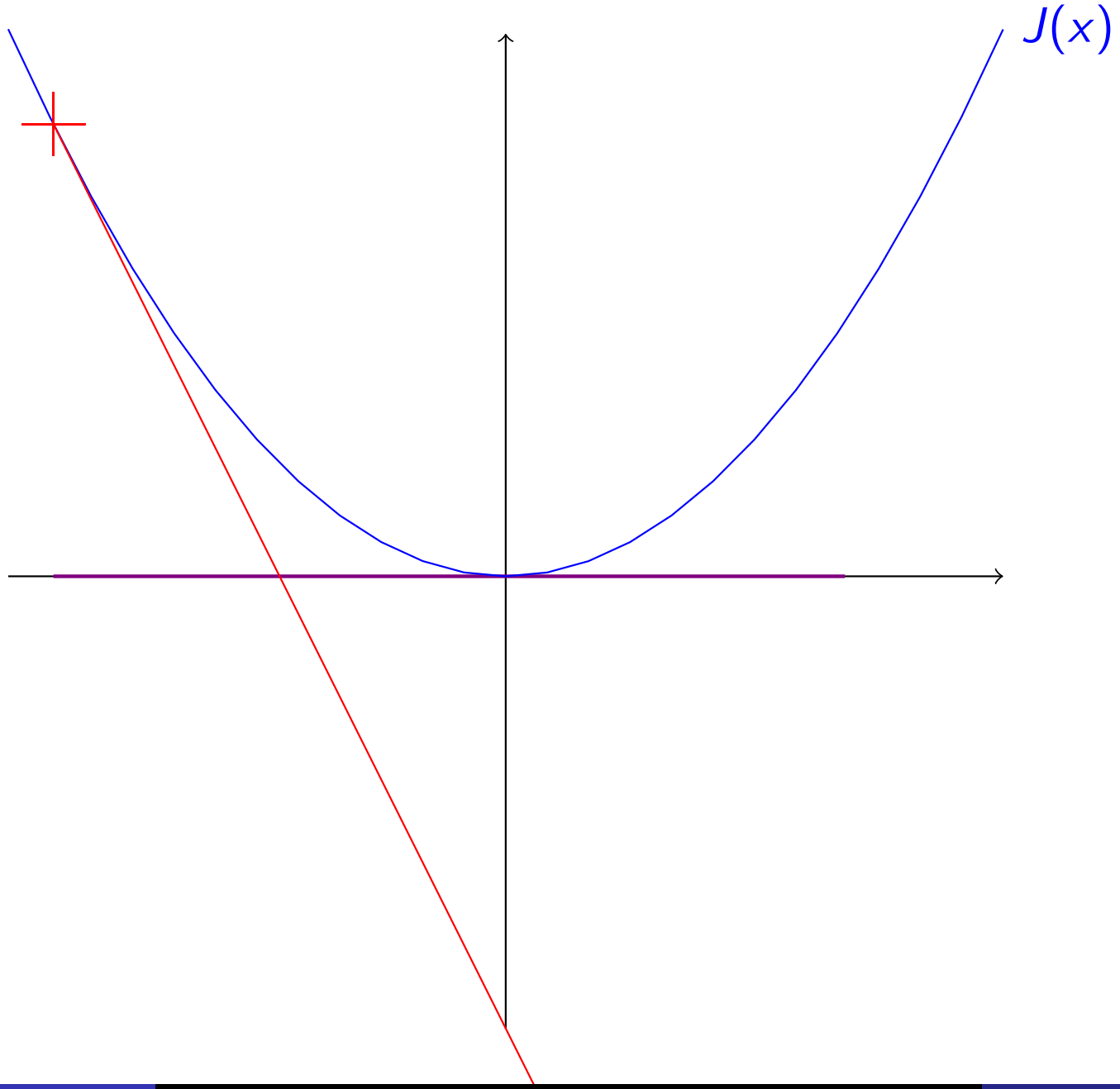
Setting

- Multi-stage stochastic optimization problems with finite horizon.
- Continuous, finite dimensional state and control.
- Convex cost, linear dynamic.
- Discrete, stagewise independent noises.

Contents


$J(x)$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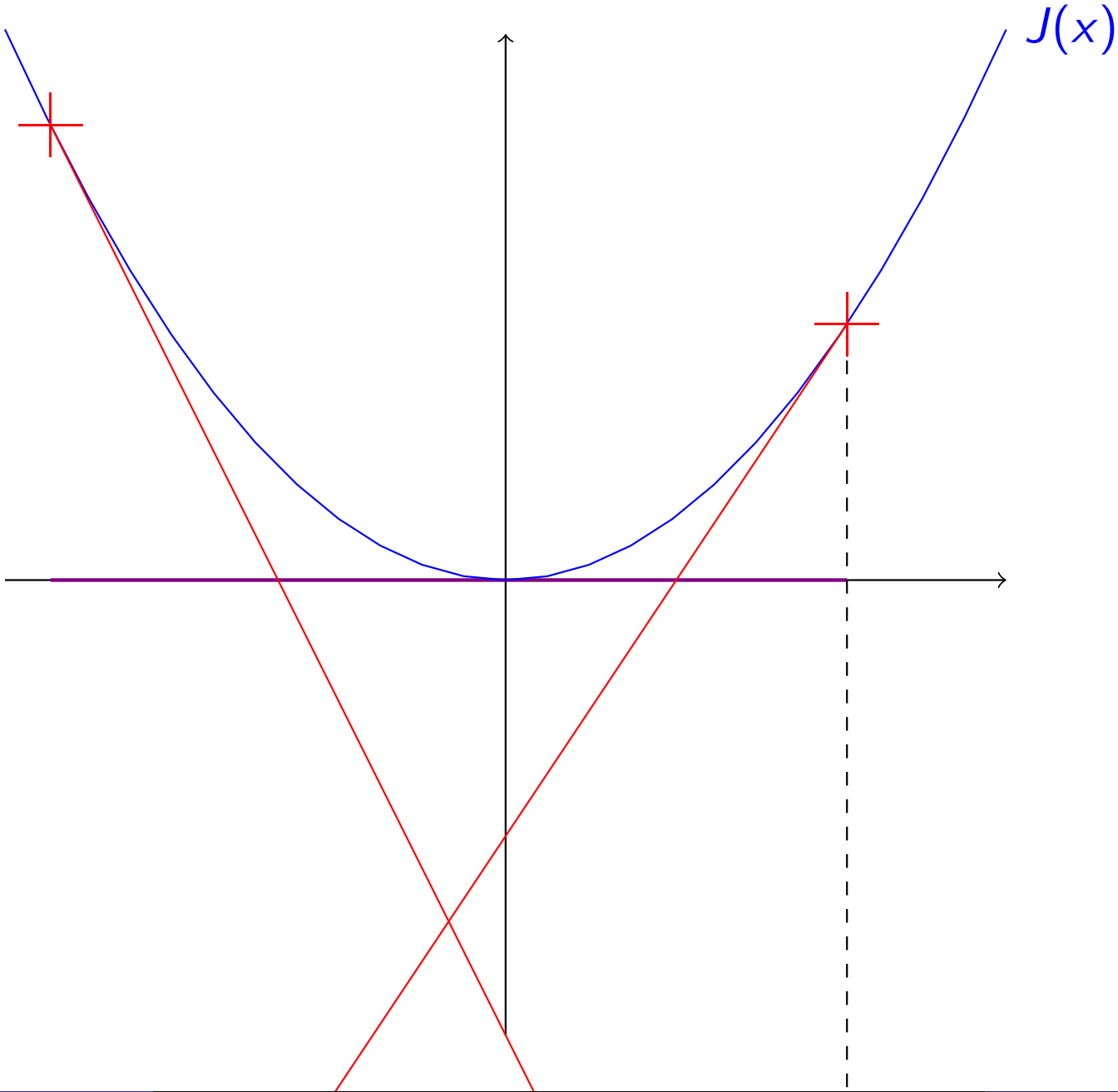
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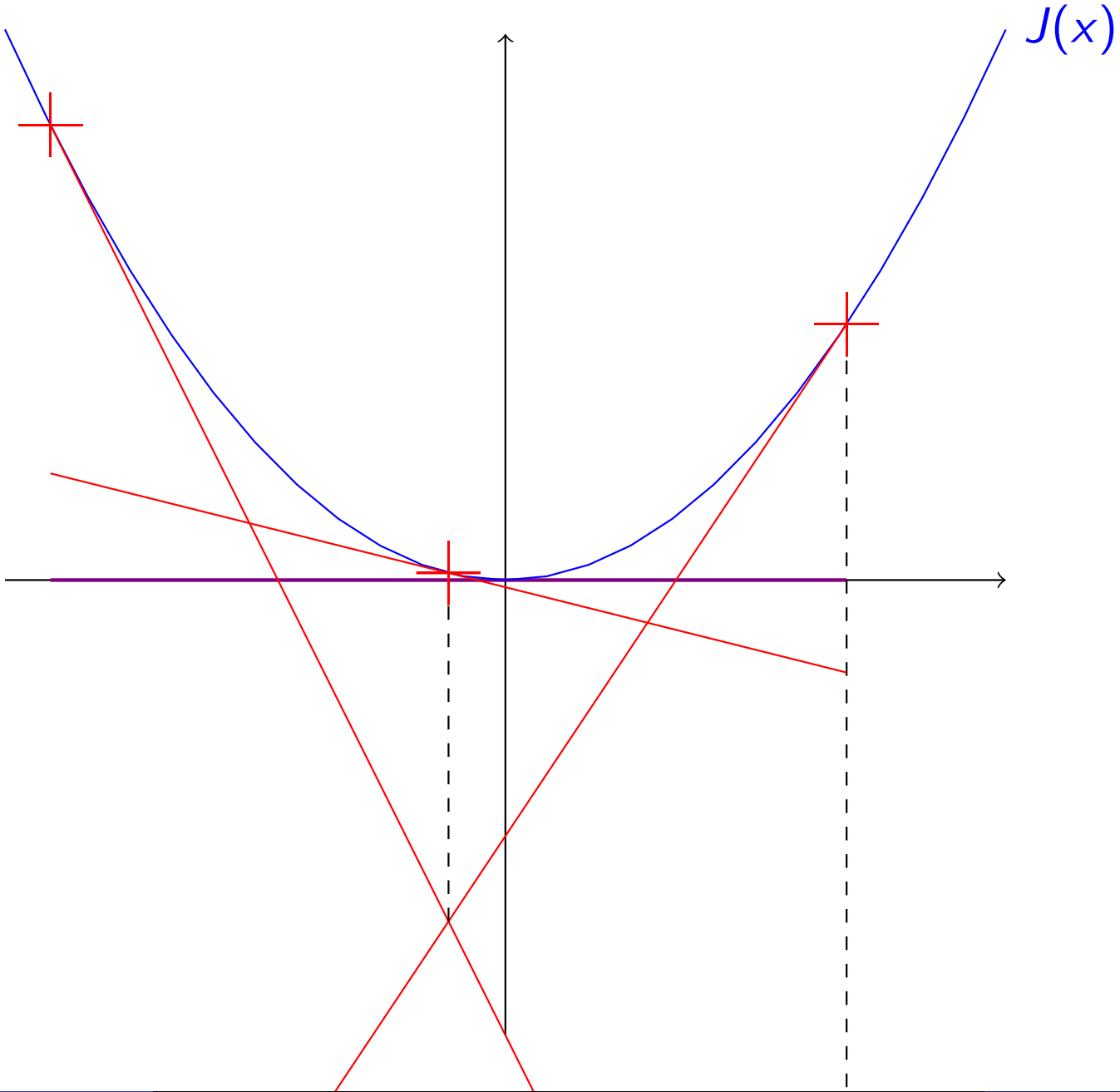
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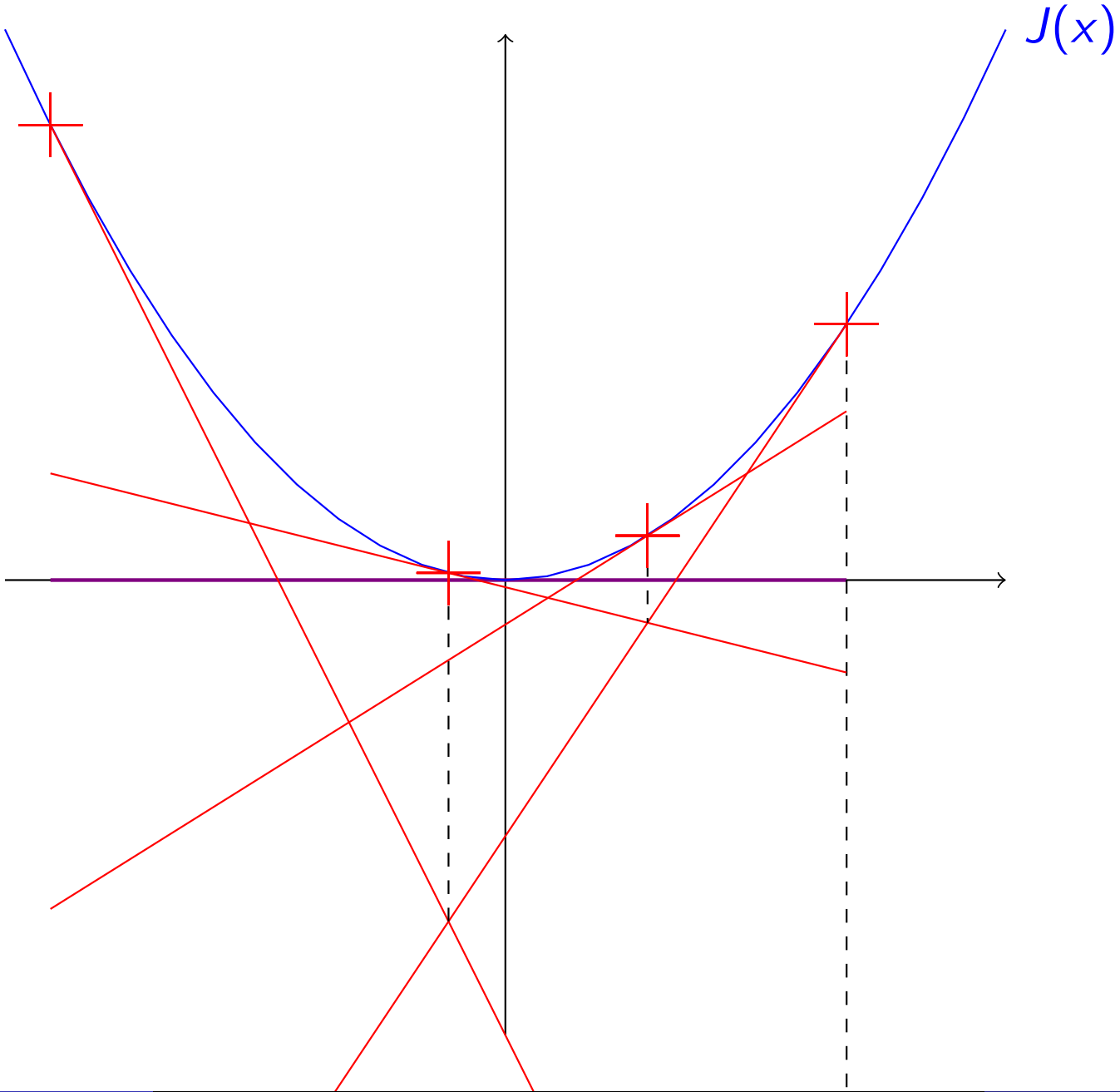
$J(x)$

Kelley algorithm

Data: Convex objective function J, Compact set X, Initial point $x_0 \in X$

Result: Admissible solution $x^{(k)}$, lower-bound $\underline{v}^{(k)}$

Set $J^{(0)} \equiv -\infty$;

for $k \in \mathbb{N}$ **do**

- Compute a subgradient $\alpha^{(k)} \in \partial J(x^{(k)})$;
- Define a cut $\mathcal{C}^{(k)} : x \mapsto J(x^{(k)}) + \langle \alpha^{(k)}, x - x^{(k)} \rangle$;
- Update the lower approximation $J^{(k+1)} = \max\{J^{(k)}, \mathcal{C}^{(k)}\}$;
- Solve $(P^{(k)}) : \quad \min_{x \in X} J^{(k+1)}(x)$;
- Set $\underline{v}^{(k)} = val(P^{(k)})$;
- Select $x^{(k+1)} \in sol(P^{(k)})$;

end

Algorithm 1: Kelley's cutting plane algorithm

Contents

Problem considered

We consider an optimal control problem in discrete time with finite horizon T

$$\begin{aligned} \min_{x \in \mathbb{R}^{nT}} \quad & \sum_{t=0}^{T-1} c_t(x_t, x_{t+1}) + K(x_T) \\ s.t. \quad & (x_t, x_{t+1}) \in P_t, \quad x_0 \text{ given} \\ & x_t \in X_t \end{aligned}$$

- We assume that $P_t \subset \mathbb{R}^n \times X_{t+1}$ is convex, and X_t convex compact
- the transition costs $c_t(x_t, x_{t+1})$ and the final cost $K(x_T)$ are convex

For example, x_t follow a dynamic $x_{t+1} = f_t(x_t, u_t)$, with

- f_t affine, $u_t \in U_t(x_t)$ is convex compact
- $c_t(x_t, x_{t+1}) = \min \left\{ L_t(x_t, u_t) \mid u_t \in U_t(x_t),\ f_t(x_t, u_t) = x_{t+1} \right\}$, where L_t is a convex instantaneous cost function

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Introducing Bellman's function

We look for solutions as policies, where a policy is a sequence of functions $\pi = (\pi_1, \dots, \pi_{T-1})$ giving for any state x a control u
This problem can be solved by dynamic programming,
thanks to the Bellman function that satisfies

$$\begin{cases} V_T(x) &= K(x), \\ \tilde{V}_t(x) &= \min\limits_{y:(x,y)\in P_t} \left\{ c_t(x,y) + V_{t+1}(y) \right\} \\ V_t &= \tilde{V}_t + \mathbb{I}_{X_t} \end{cases}$$

Indeed, an optimal policy for the original problem is given by

$$\pi_t(x) \in \arg\min_{x_{t+1}} \left\{ c_t(x, x_{t+1}) + V_{t+1}(x_{t+1}) \;\middle|\; (x_t, x_{t+1}) \in P_t \right\}$$

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Introducing Bellman's operator

We define the Bellman operator

$$\mathcal{B}_t(A) : x \mapsto \min_{y:(x,y)\in P_t} \left\{ c_t(x,y) + A(y) \right\}$$

With this notation, the Bellman Equation reads

$$\begin{cases} V_T &= K, \\ V_t &= \mathcal{B}_t(V_{t+1}) + \mathbb{I}_{X_t} \end{cases}$$

Any approximate cost function $\breve{V}_{t+1}$ induce an *admissible* policy

$$\pi_t^{\breve{V}_{t+1}} : x \mapsto \arg\min \mathcal{B}_t\left(\breve{V}_{t+1}\right)(x).$$

By Dynamic Programming, $\pi_t^{V_{t+1}}$ is optimal.

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Properties of the Bellman operator

- Monotonicity:

$$V \leq \overline{V} \quad \Rightarrow \mathcal{B}_t(V) \leq \mathcal{B}_t(\overline{V})$$

- Convexity: if c_t is jointly convex, P and X are closed convex, V is convex then

$$x \mapsto \mathcal{B}_t(V)(x) \quad \text{is convex}$$

- Polyhedrality: for any polyhedral function V, if c_t is also polyhedral, and P_t and X_t are polyhedron, then

$$x \mapsto \mathcal{B}_t(V)(x) \quad \text{is polyhedral}$$

Duality property

- Consider $J : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ jointly convex, and define

$$\varphi(x) = \min_{u \in \mathbb{U}} J(x, u)$$

- Then we can obtain a subgradient $\alpha \in \partial\varphi(x_0)$ as the dual multiplier of

$$\begin{array}{ll} \min\limits_{x,u} & J(x, u), \\ s.t. & x_0 - x = 0 \qquad [\alpha] \end{array}$$

(This is the marginal interpretation of the multiplier)

- In particular, we have that

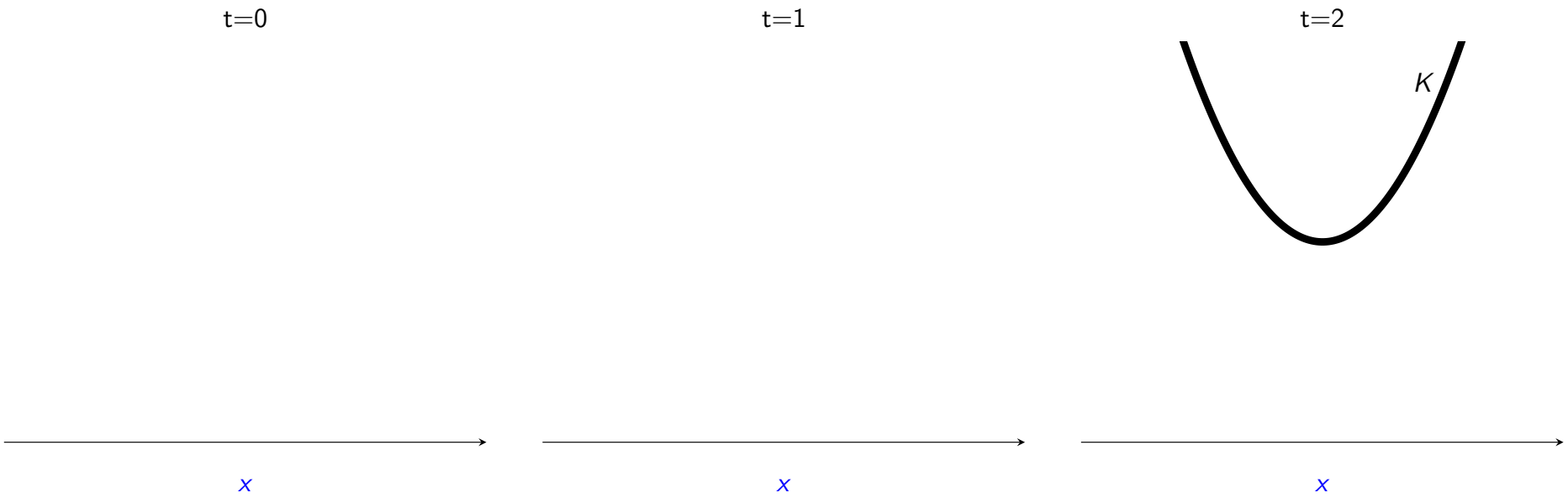
$$\varphi(\cdot) \geq \varphi(x_0) + \langle \alpha, \cdot - x_0 \rangle$$

Contents

General idea

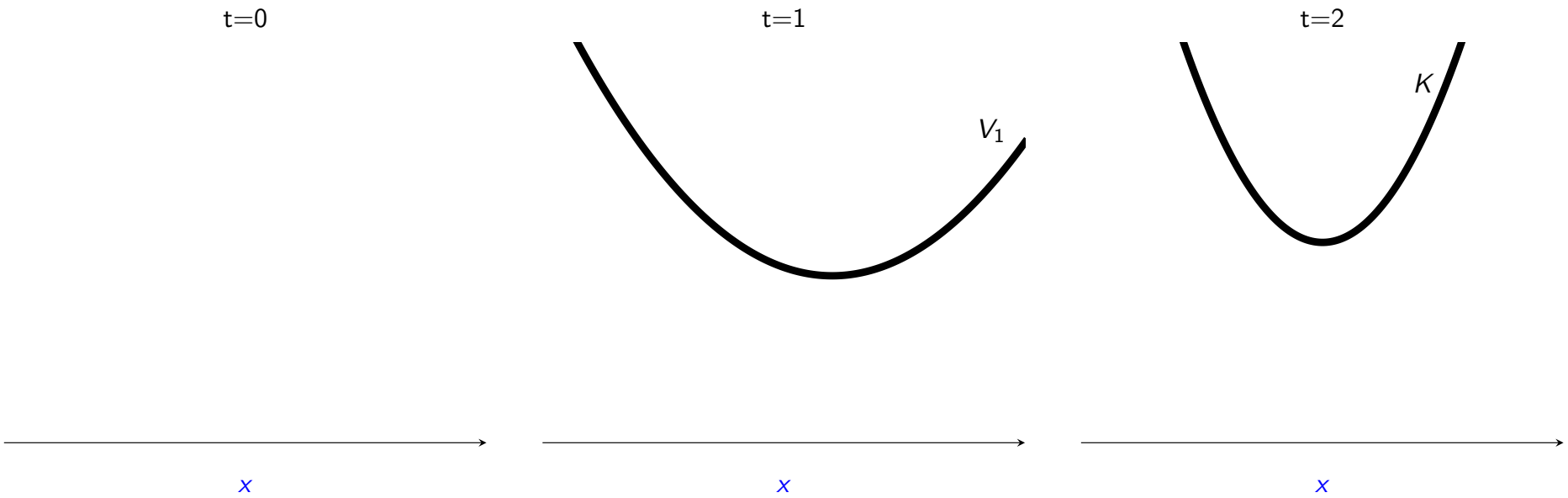
- The SDDP algorithm recursively constructs an approximation of each Bellman function V_t as the supremum of affine functions
- At stage k, we have a lower approximation $\underline{V}_t^{(k)}$ of V_t and we want to construct a better approximation
- We follow an optimal trajectory $(x_t^{(k)})_t$ of the approximated problem, and add a so-called "cut" to improve each Bellman function

Deterministic SDDP



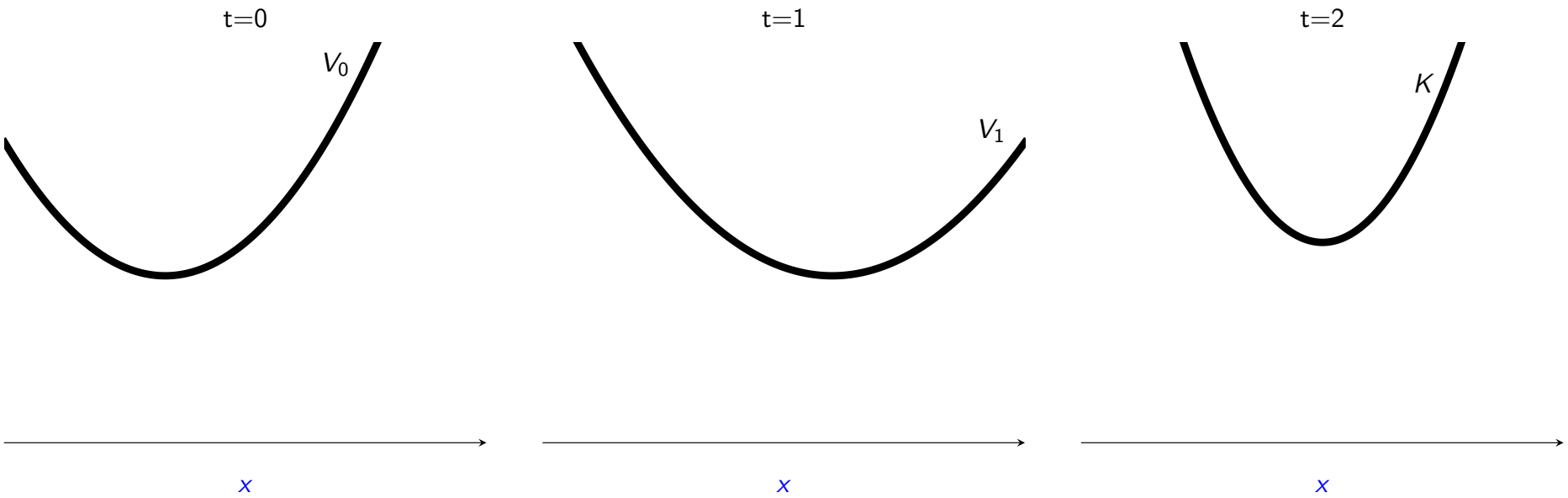
Final Cost $V_2 = K$

Deterministic SDDP



Real Bellman function $V_1 = \mathcal{B}_1(V_2)$

Deterministic SDDP



Real Bellman function $V_0 = \mathcal{B}_0(V_1)$

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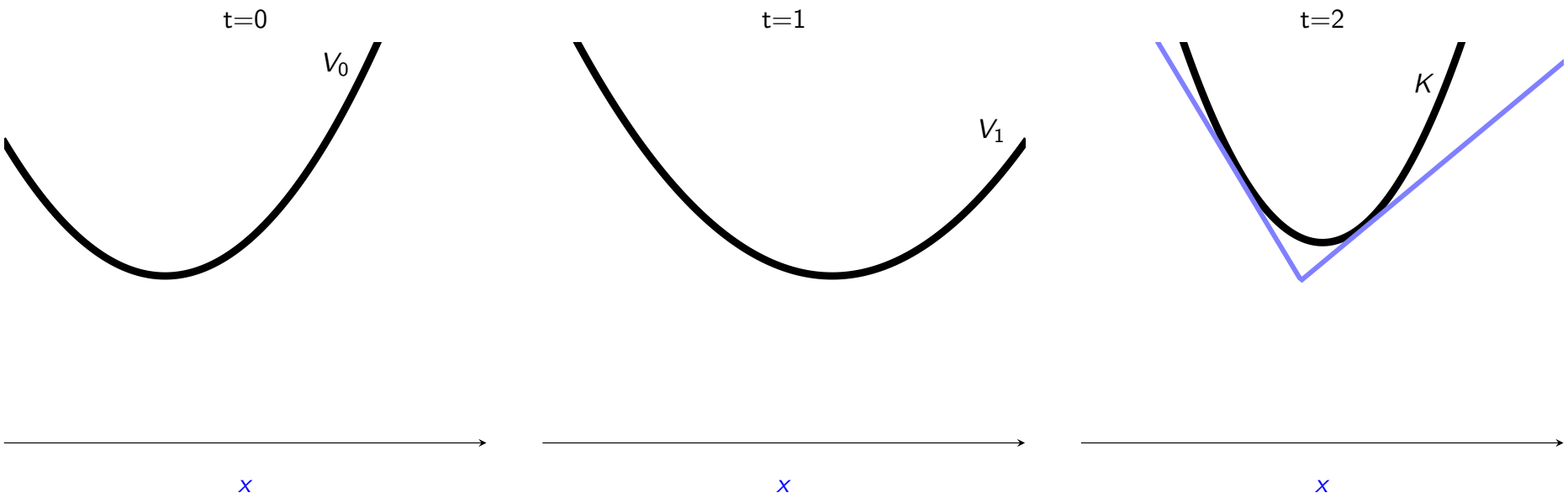
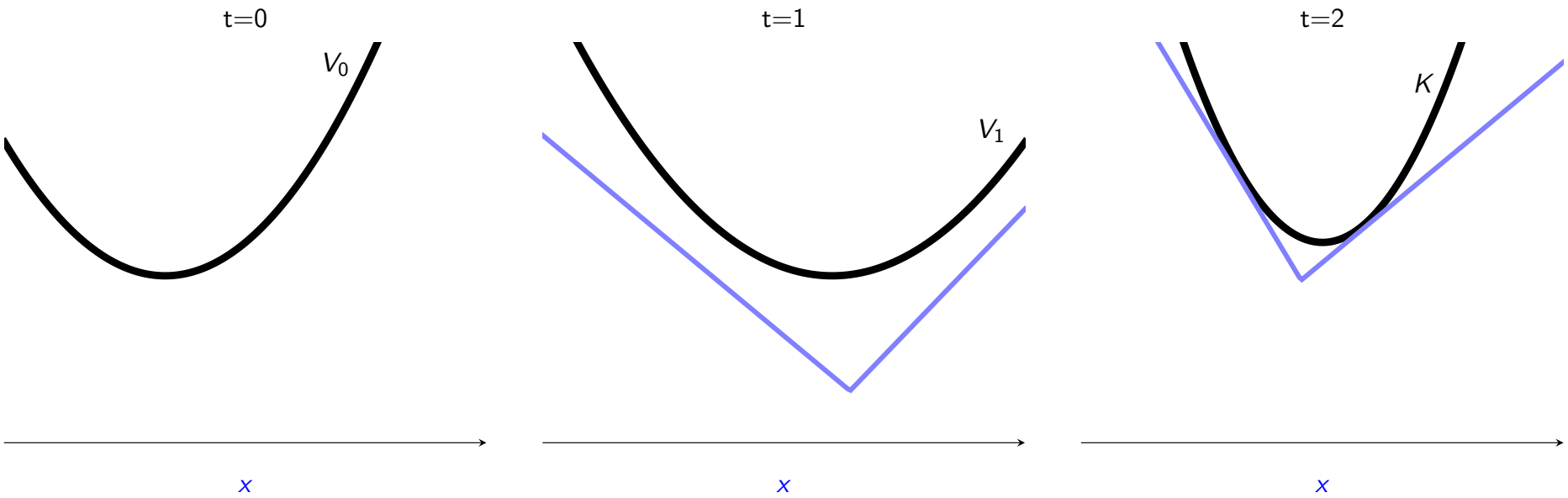


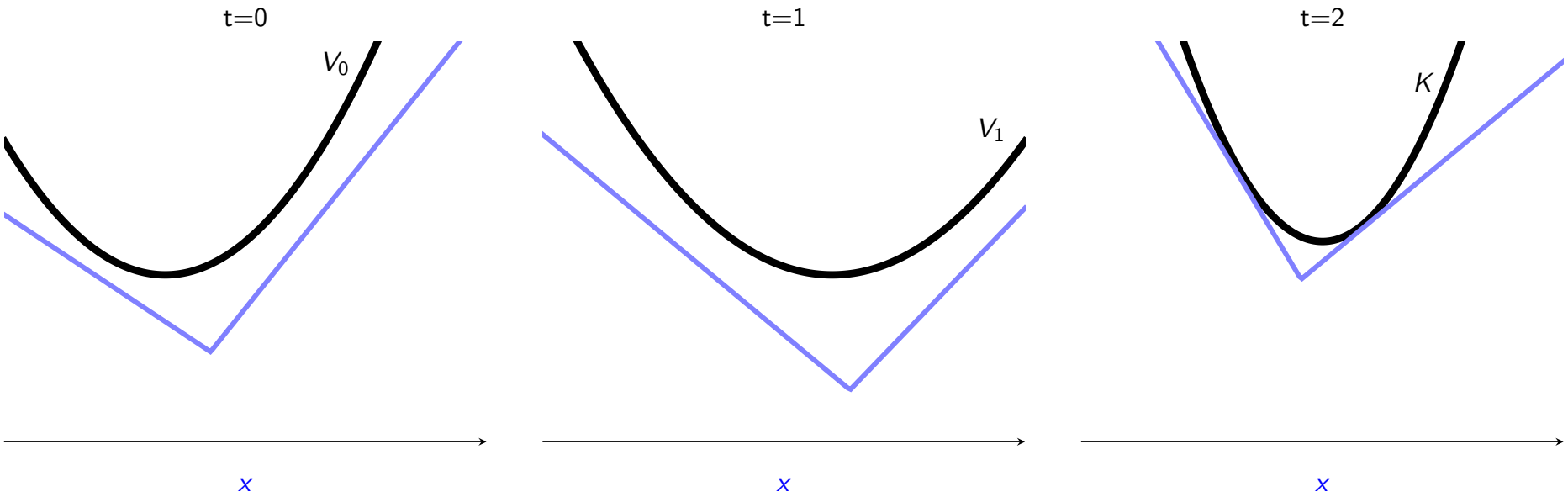
Lower polyhedral approximation $\underline{K}$ of K

Deterministic SDDP



Lower polyhedral approximation $\underline{V}_1 = \mathcal{B}_t(\underline{K})$ of V_1

Deterministic SDDP



Lower polyhedral approximation $\underline{V}_0 = \mathcal{B}_t(\underline{V}_1)$ of V_0

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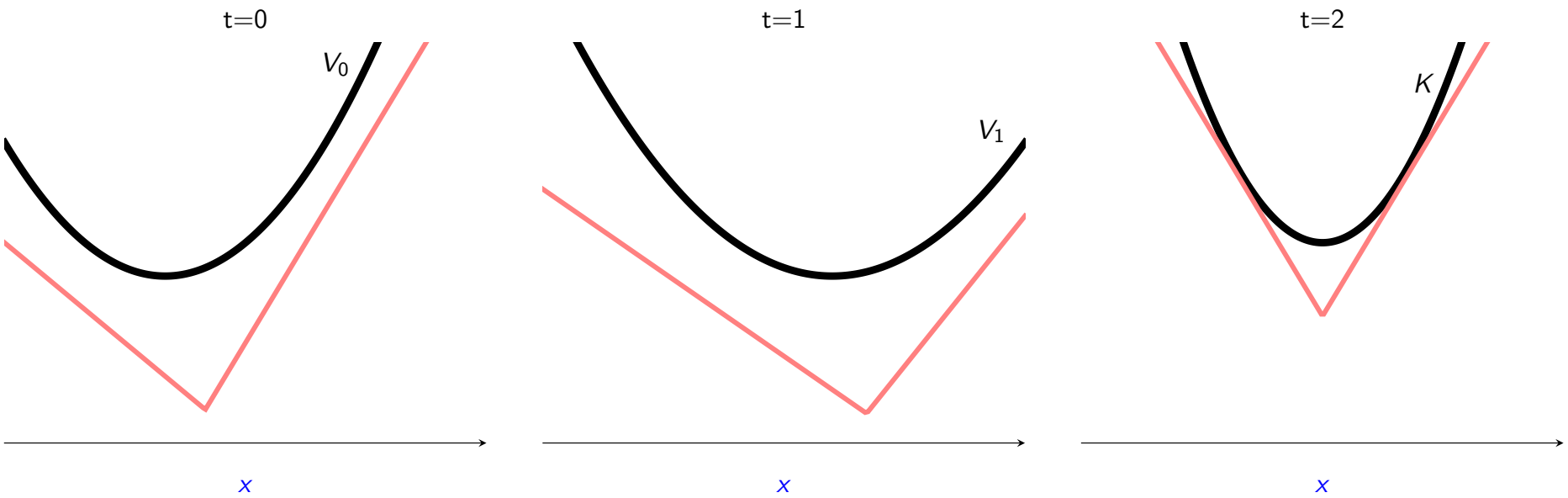
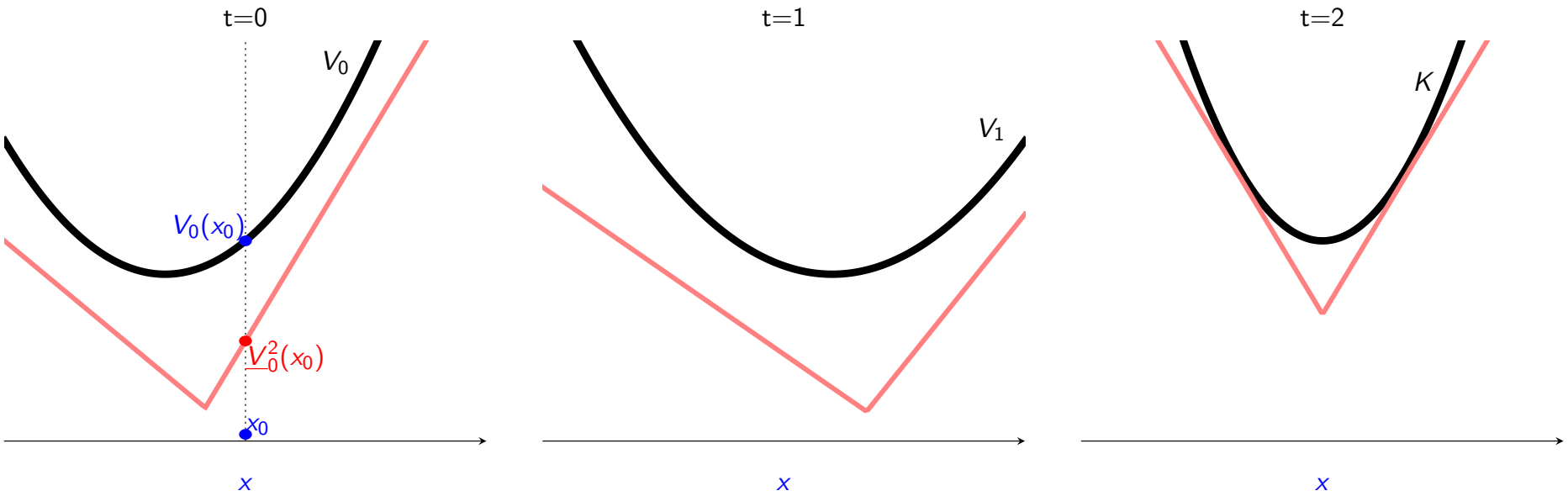


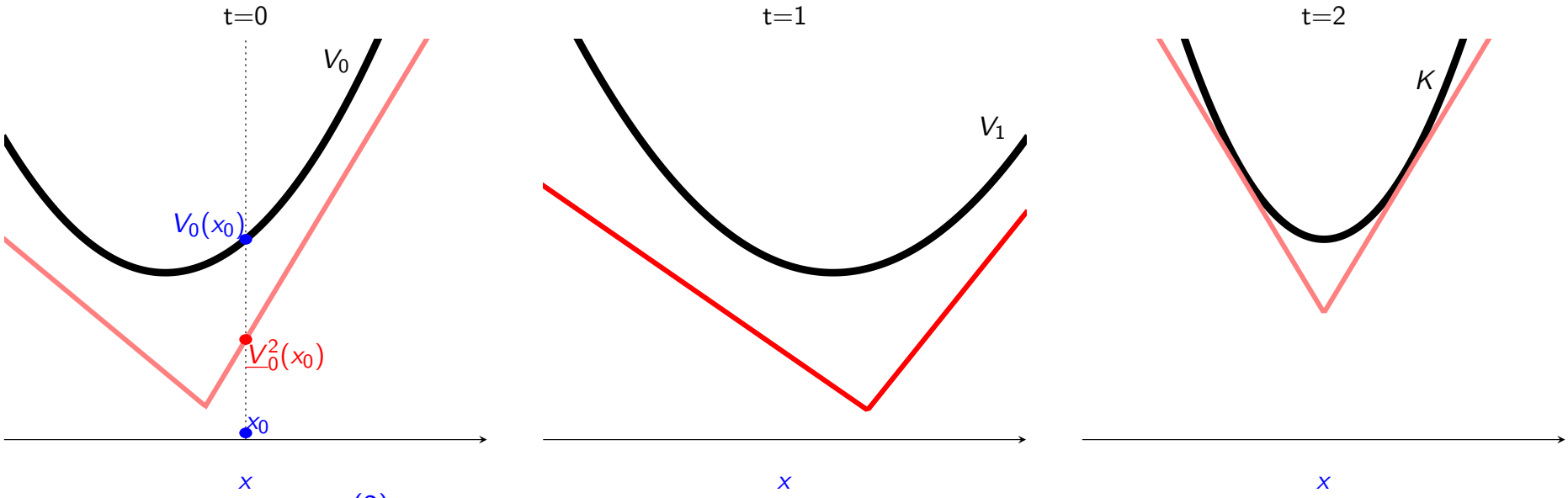
Assume that we have lower polyhedral approximations of V_t

Deterministic SDDP



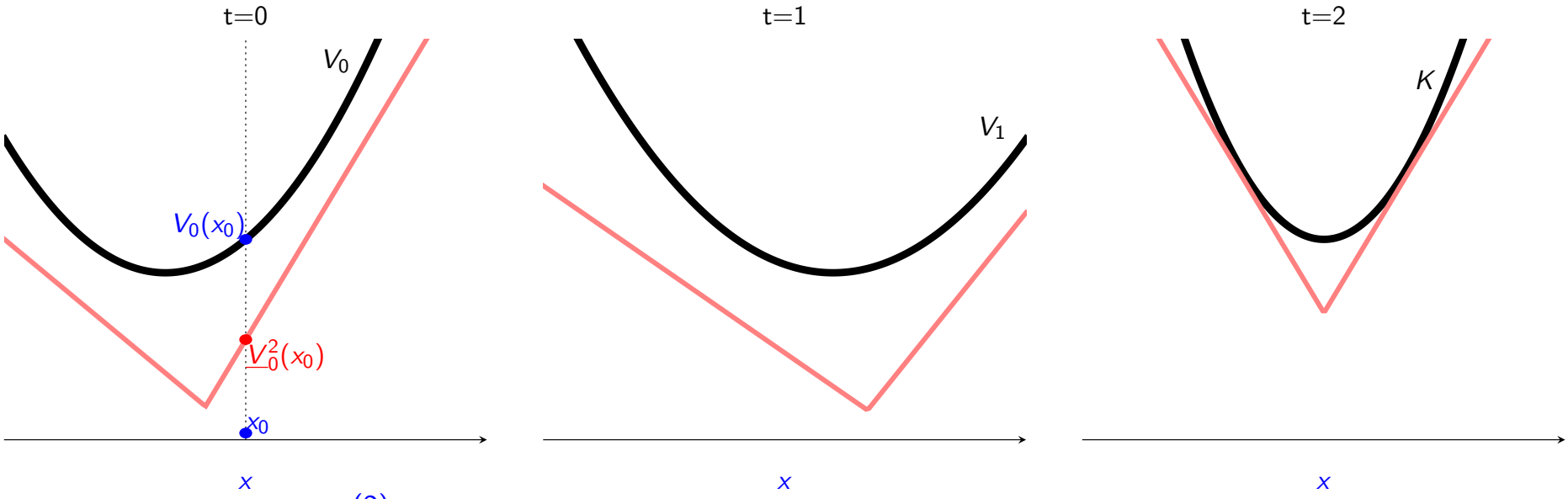
Thus we have a lower bound on the value of our problem

Deterministic SDDP



We apply $\pi_0^{\underline{V}_1^{(2)}}$ to x_0 and obtain $x_1^{(2)}$

Deterministic SDDP



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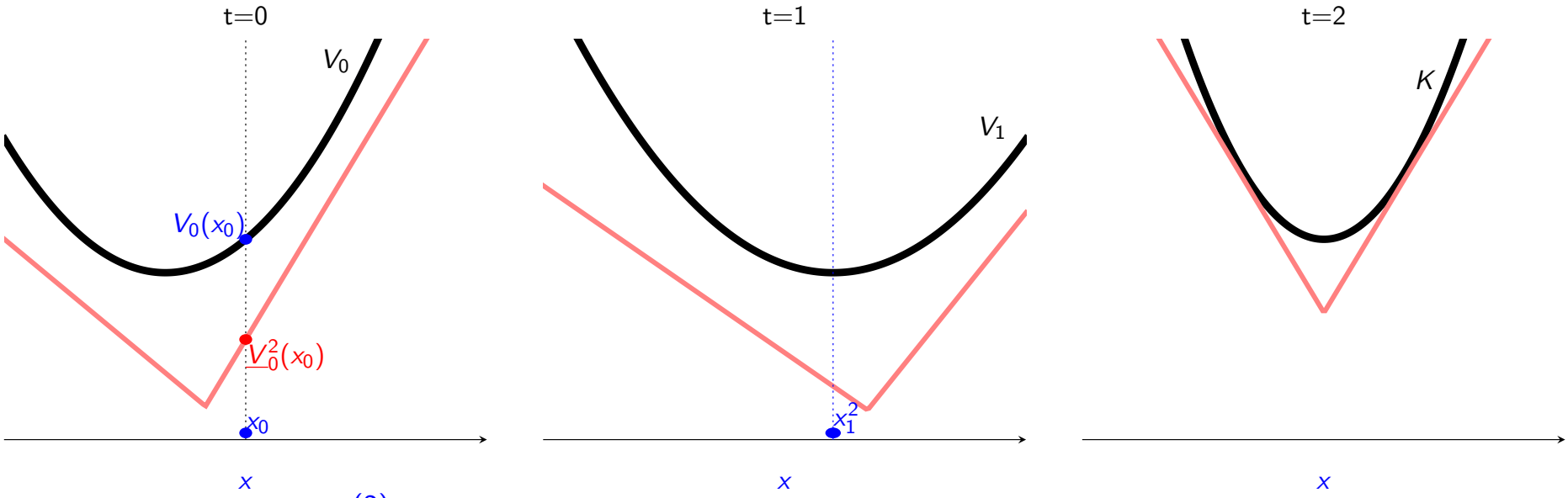

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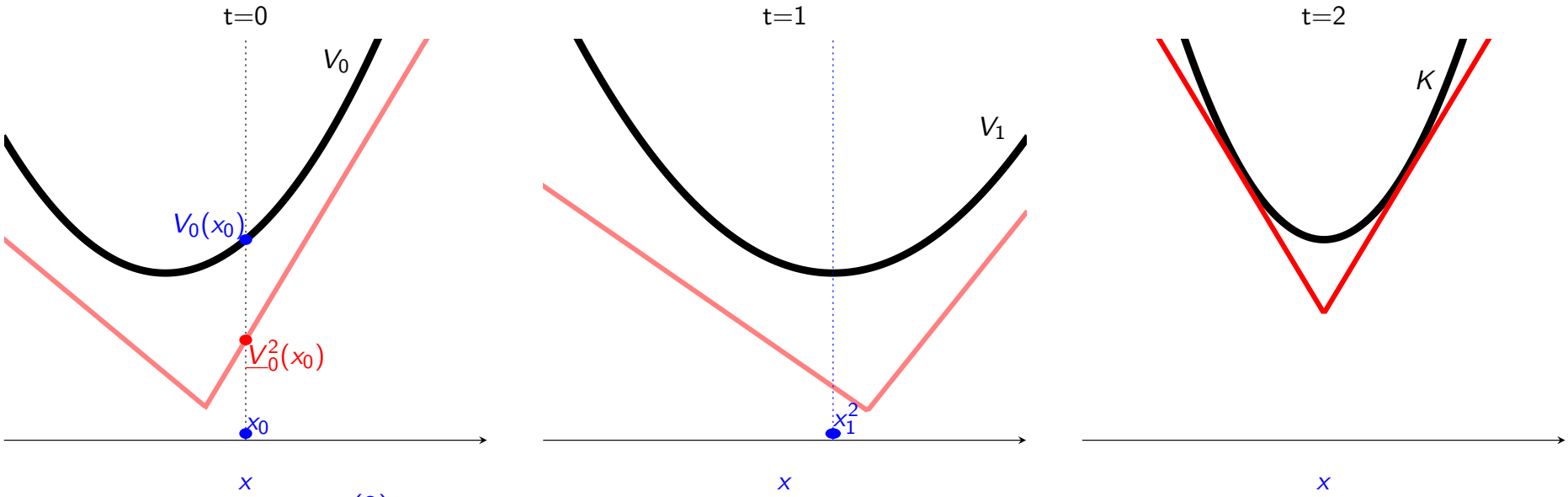

We apply $\pi_1^{\underline{V}_1^{(2)}}$ to $x_1^{(2)}$ and obtain $x_2^{(2)}$

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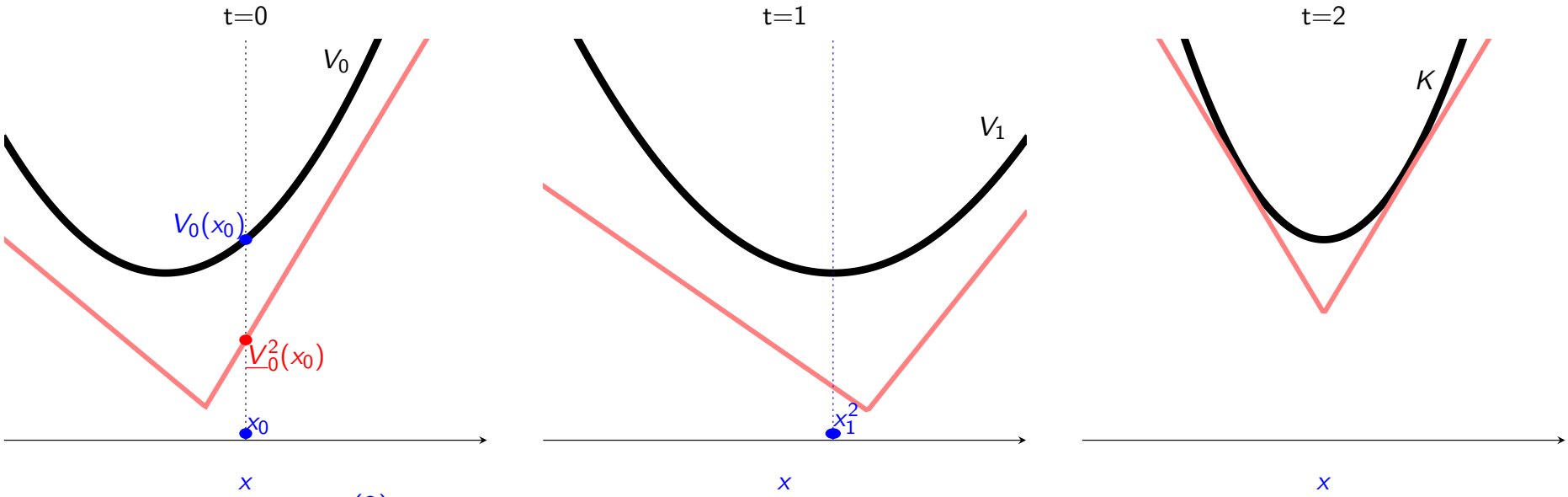

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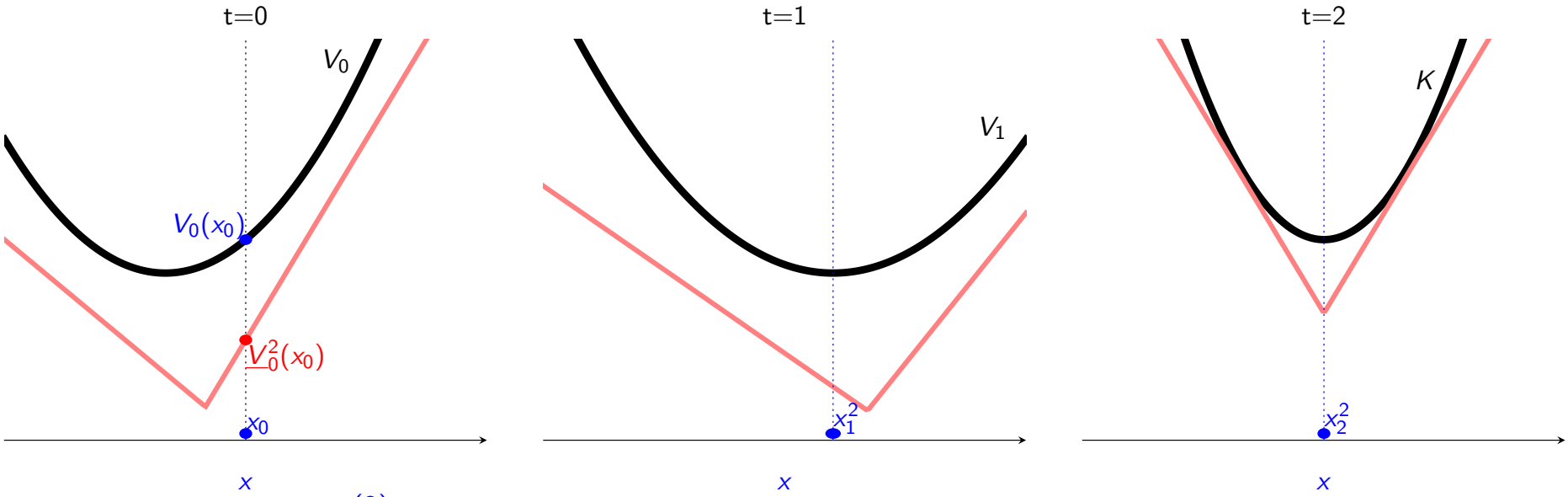

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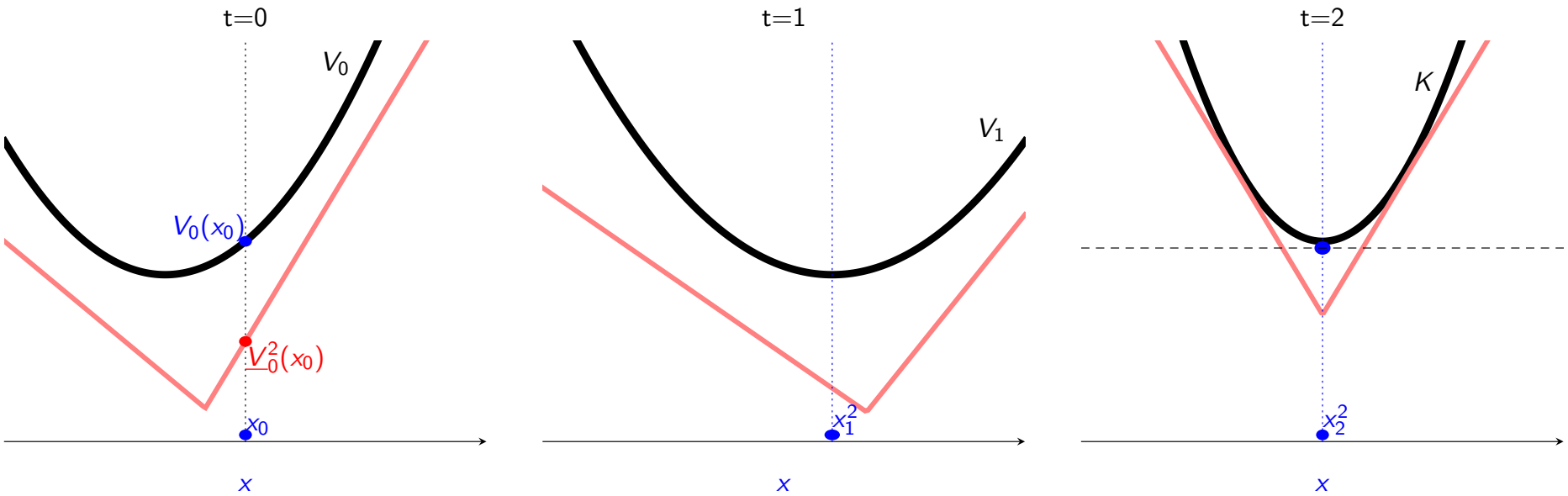
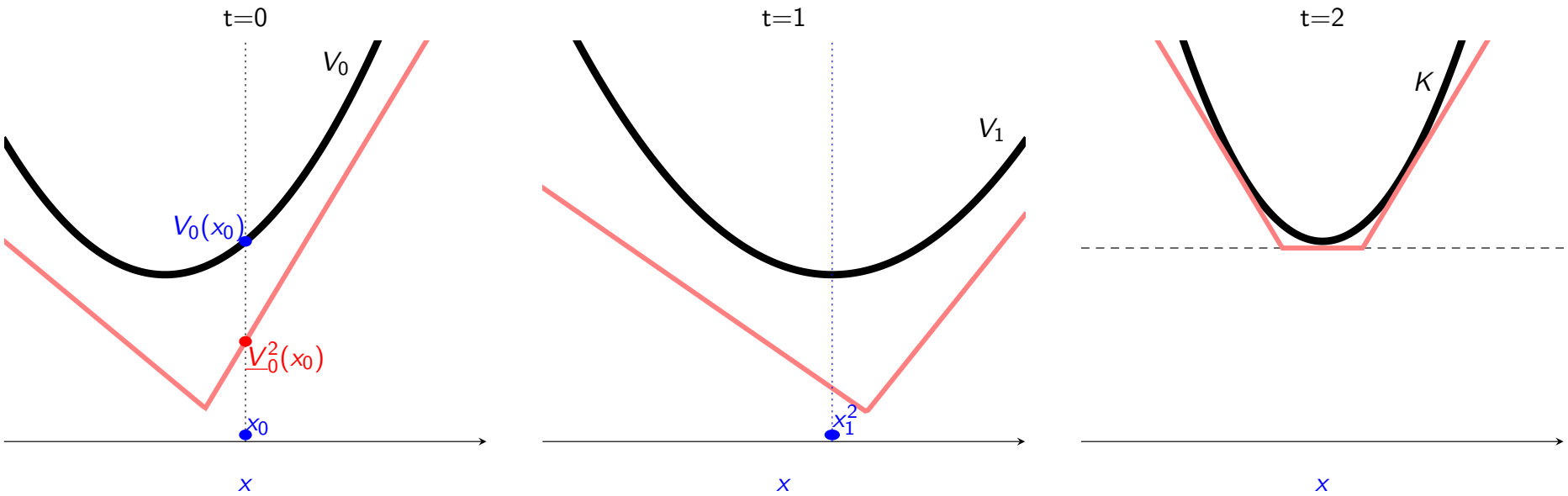


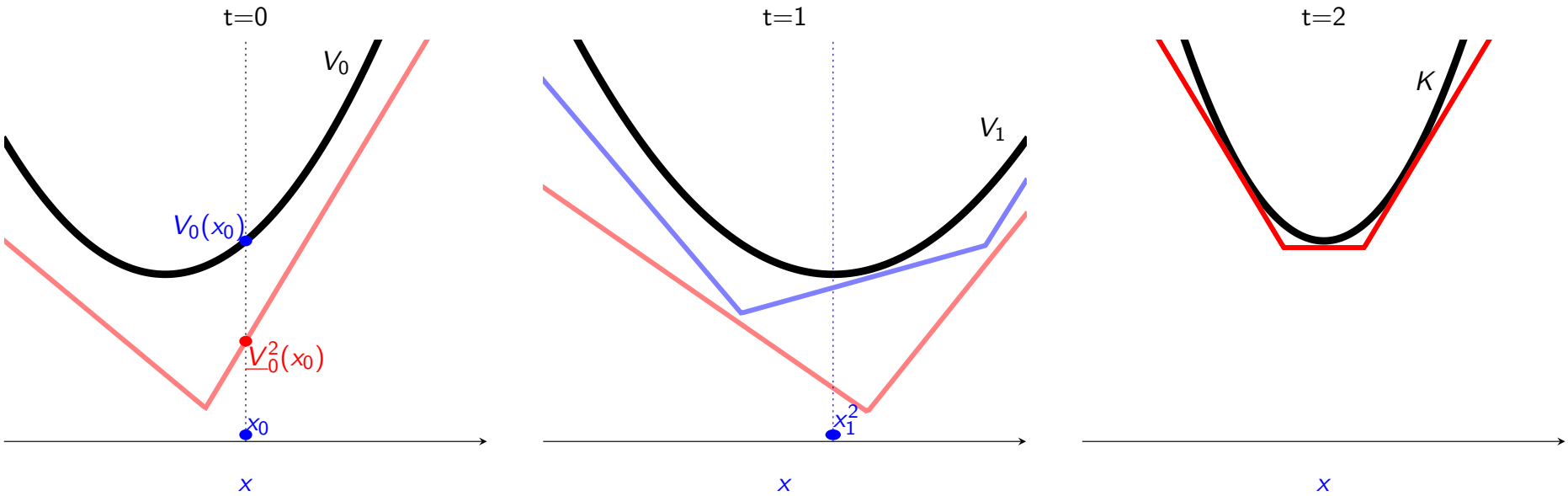
Compute a cut for K at $x_2^{(2)}$

Deterministic SDDP



Add the cut to $\underline{V}_2^{(2)}$ which gives $\underline{V}_2^{(3)}$

Deterministic SDDP



A new lower approximation of V_1 is $\mathcal{B}_1(\underline{V}_2^{(3)})$

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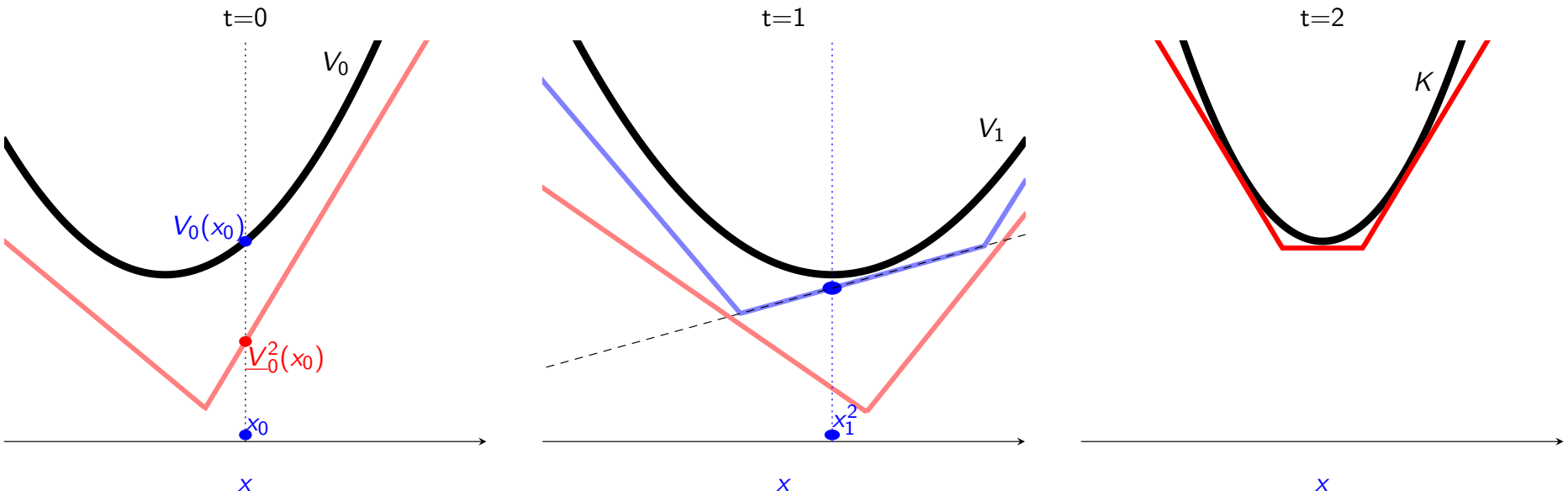
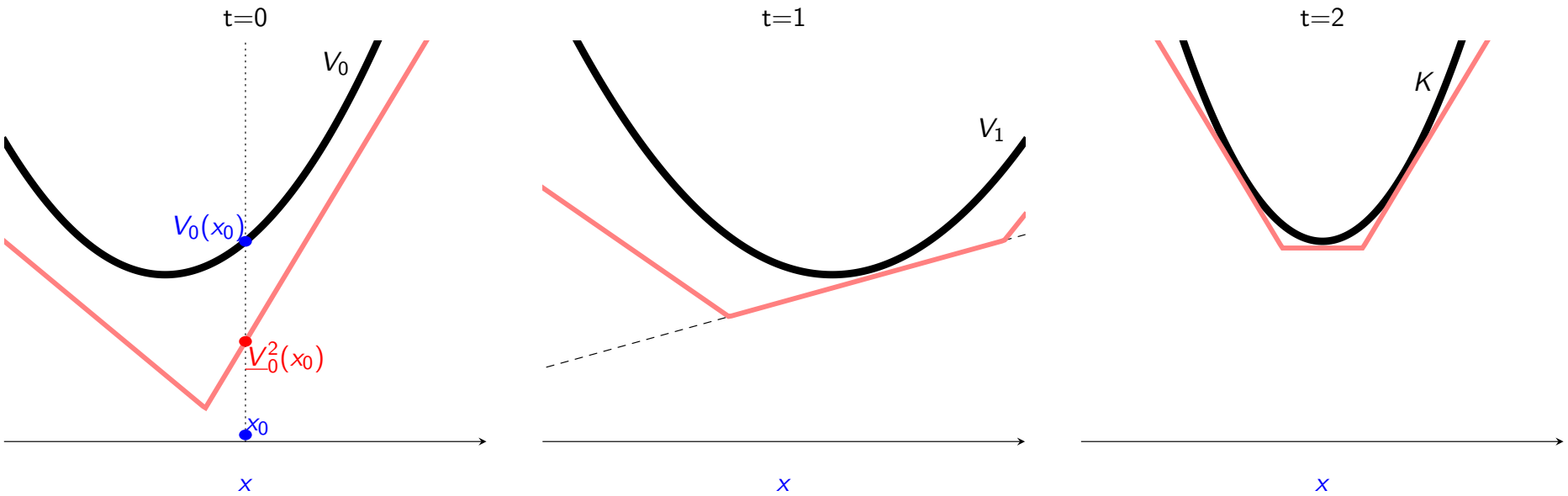


We only compute the face active at $x_1^{(2)}$

Deterministic SDDP



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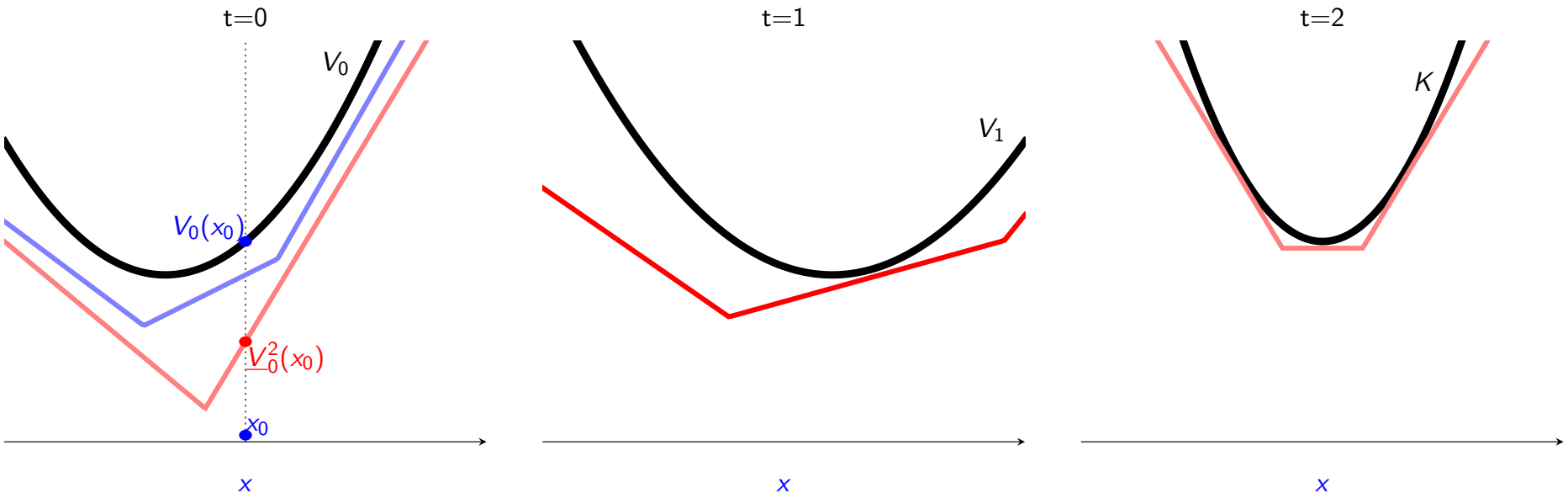

A new lower approximation of V_0 is $\mathcal{B}_0(\underline{V}_1^{(3)})$

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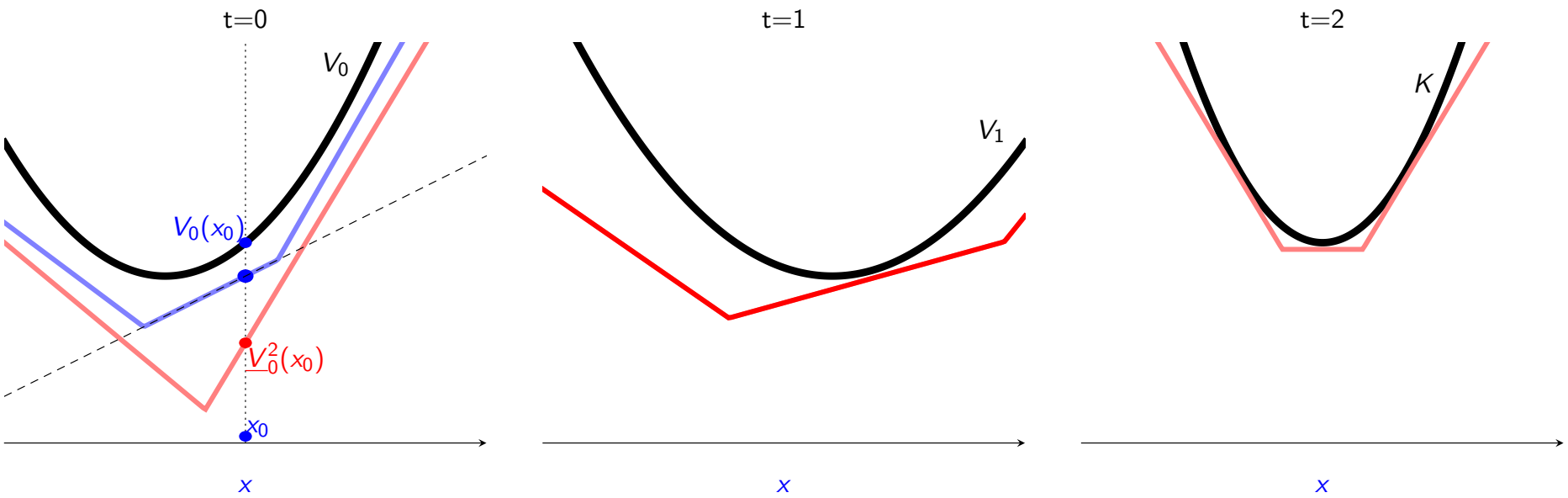
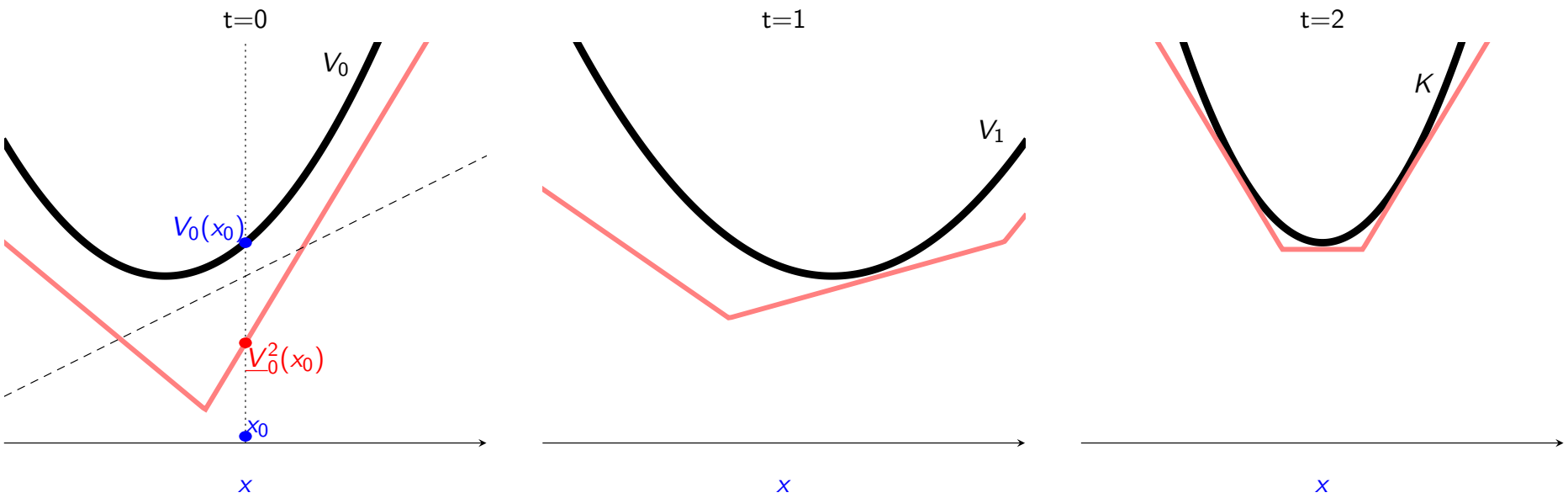


We only compute the face active at x_0

Deterministic SDDP



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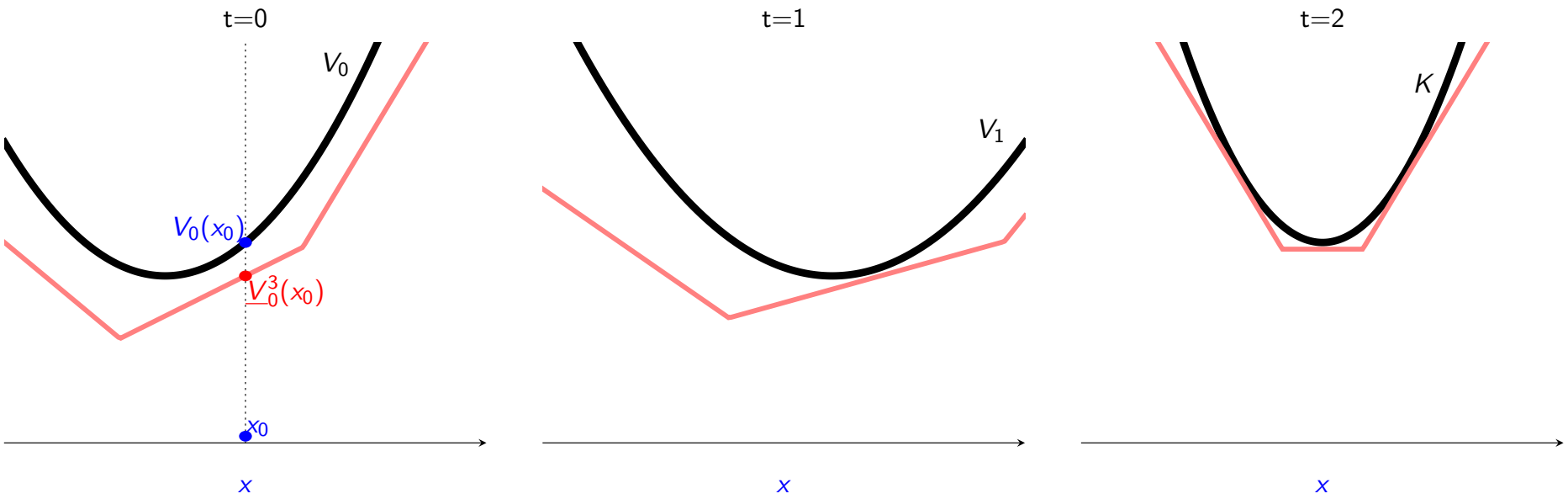

We obtain a new lower bound

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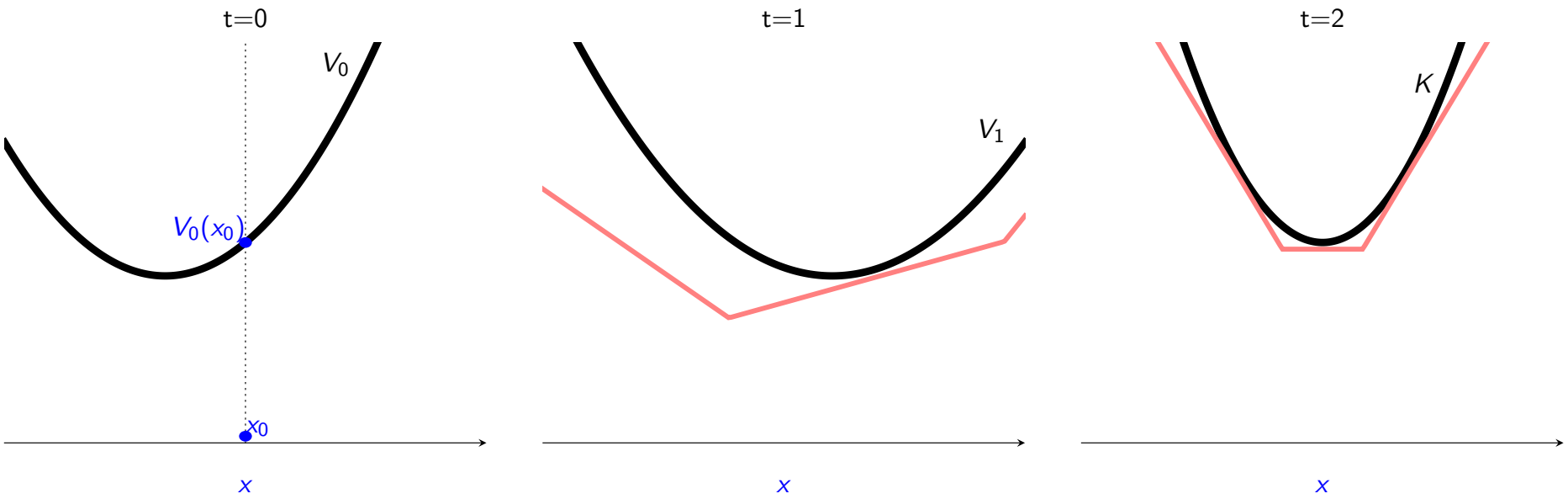

We obtain a new lower bound

DDP description

Data: Starting point, initial lower approximation
Result: optimal trajectory and value function;
$V_T \equiv K$;
for $k = 1, 2, \ldots$ **do**
  set $x_0^{(k)} = x_0$
  /* Forward pass : compute trajectory */
  for $t = 0, \ldots, T-1$ **do**
    find $x_{t+1}^{(k)} \in \arg\min \mathcal{B}_t(\underline{V}_{t+1}^{(k)})(x_t^{(k)})$;
  end
  /* Backward pass : update cuts */
  for $t = T-1, \ldots, 0$ **do**
    Solve $\mathcal{B}_t(\underline{V}_{t+1}^{(k+1)})(x_t^{(k)})$ to compute $\mathcal{C}_t^{(k+1)}$;
    Update lower approximations : $\underline{V}_t^{(k+1)} := \max\{\underline{V}_t^{(k)}, \mathcal{C}_t^{(k+1)}\}$;
  end
end

Algorithm 2: Deterministic Dual Dynamic Programming

Detailing forward pass

- From $t = 0$ to $t = T - 1$ we have to solve T one-stage problem of the form

$$x_{t+1}^{(k)} \in \arg\min_{y} \quad c_t(x_t^{(k)}, y) + \underline{V}_{t+1}^{(k)}(y)$$
$$(x_t^{(k)}, y) \in P_t$$

- We only need to keep the trajectory $(x_t^{(k)})_{t \in [\![0, T]\!]}$.

Detailing Backward pass

- From $t = T - 1$ to $t = 0$ we have to solve T one-stage problem of the form

$$\begin{aligned} \theta_t^{(k+1)} = \min_{x,y} \quad & c_t(x,y) + \underline{V}_{t+1}^{(k+1)}(y) \\ & (x,y) \in P_t \\ & x = x_t^{(k)} \qquad [\alpha_t^{(k+1)}] \end{aligned}$$

- By construction, we have that

$$\theta_t^{(k+1)} = \mathcal{B}_t\left(\underline{V}_{t+1}^{(k+1)}\right)(x_t^{(k)}), \qquad \alpha_t^{(k+1)} \in \partial\mathcal{B}_t\left(\underline{V}_{t+1}^{(k+1)}\right)(x_t^{(k)}).$$

- Which means

$$\mathcal{C}_t^{(k+1)} := \theta_t^{(k+1)} + \langle \alpha_t^{(k+1)}, \cdot - x_t^{(k)} \rangle \leq \mathcal{B}_t\left(\underline{V}_{t+1}^{(k+1)}\right) \leq \mathcal{B}_t\left(V_{t+1}\right) = \tilde{V}_t \leq V_t$$

Contents

Initialization and stopping rule

- To initialize the algorithm, we need a lower bound $\underline{V}_t^{(0)}$ for each value function V_t. This lower bound can be computed backward by arbitrarily choosing a point x_t and using the standard cut computation.
- At any step k we have an admissible, non optimal trajectory $(x_t^{(k)})_t$, with
  - an upper bound

$$\sum_{t=0}^{T-1} c_t\left(x_t^{(k)}, x_{t+1}^{(k)}\right) + K\left(x_T^{(k)}\right)$$

  - a lower bound $\underline{V}_0^{(k)}(x_0)$
- A reasonable stopping rule for the algorithm is given by checking that the (relative) difference between the upper and lower bounds is small enough

Contents

Extended Relatively Complete Recourse

- We say that we are in a relatively complete recourse framework if

$$\forall t, \quad \forall x_t \in X_t, \quad \exists x_{t+1} \in X_{t+1} \quad \text{such that} \quad (x_t, x_{t+1}) \in P_t.$$

- We say that we are in a extended relatively complete recourse framework if there exists $\varepsilon > 0$ such that

$$\forall t, \quad \forall x_t \in X_t + \varepsilon B, \quad \exists x_{t+1} \in X_{t+1} \quad \text{such that} \quad (x_t, x_{t+1}) \in P_t.$$

- RCR is required for the algorithm to run (otherwise we could find non-finite problems, and would require some feasability cuts mechanisms).
- ERCR is required for the convergence proof as the way of ensuring that the multipliers α_t^k remains bounded.

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- ERCR is required for the convergence proof as the way of ensuring that the multipliers α_t^k remains bounded.

Technical lemmas

Lemma

Let $f : X \to \mathbb{R}$ where X is compact. Let $(f^k)_{k\in\mathbb{N}}$ be a sequence of functions such that

- $f^k \leq f^{k+1} \leq f$
- f^k *are Lipschitz continuous uniformly in* k

Consider a sequence $(x^k)_{k\in\mathbb{N}}$ of points of X such that $f(x^k) - f^{k+1}(x^k) \to 0$. Then, we also have $f(x^k) - f^k(x^k) \to 0$.

Lemma

Under convexity assumptions, compactness of X_t, and ERCR the SDDP algorithm is well defined and

i) *for all t, V_t is convex and Lipschitz*

ii) *for all t, k, and $x \in X_t$, $\underline{V}_t^k \leq V_t$*

iii) *There exists $L > 0$ such that $\|\alpha_t^k\| \leq L$, thus $\underline{V}_t^k$ is L-Lipschitz*

Convergence result

Theorem

Let K and c_t be convex functions, X_t and P_t be closed convex sets, and X_t bounded. Assume that we have extended relatively complete recourse. Then, for every t, we have

$$\lim_k \underline{V}_t^{(k)}(x_t^{(k)}) - V_t(x_t^{(k)}) = 0.$$

Further, the cost associated to $\pi^{\underline{V}_t^{(k)}}$ converges toward the optimal value of the problem.

In other words, the upper and lower bounds are both converging.

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Contents

What's new ?

Now we introduce random variables ξ_t in our problem, which complexifies the algorithm in different ways:

- we need some probabilistic assumptions
- for each stage k we need to do a forward phase, for each sequence of realizations of the random variables, that yields a trajectory $(x_t^{(k)})_t$, and a backward phase that gives a new cut
- we cannot compute an exact upper bound for the problem value

Problem statement

We consider the optimization problem

$$\begin{aligned} \min \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} c_t(\boldsymbol{x}_t, \boldsymbol{x}_{t+1}, \boldsymbol{\xi}_{t+1}) + K(\boldsymbol{x}_T)\Big] \\ s.t. \quad & (\boldsymbol{x}_t, \boldsymbol{x}_{t+1}) \in P_t(\boldsymbol{\xi}_{t+1}) \\ & \boldsymbol{x}_t \in X_t, \qquad \boldsymbol{x}_0 = x_0 \\ & \boldsymbol{x}_t \preceq \sigma(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_t) \end{aligned}$$

under the crucial assumption that $(\boldsymbol{\xi}_t)_{t\in\{1,\cdots,T\}}$ is a white noise

$\rightsquigarrow$ we are in an hazard-decision framework.

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Stochastic Dynamic Programming

By the white noise assumption, this problem can be solved by dynamic programming, where the Bellman functions satisfy

$$\begin{cases} V_T & = & K \\ \hat{V}_t(x,\xi) & = & \min\limits_{(x,y)\in P_t(\xi)} c_t(x,y,\xi) + V_{t+1}(y) \\ \tilde{V}_t(x) & = & \mathbb{E}\left[\hat{V}_t(x,\boldsymbol{\xi}_t)\right] \\ V_t & = & \tilde{V}_t + \mathbb{I}_{X_t} \end{cases}$$

Indeed, an optimal policy for this problem is given by

$$\pi_t(x,\xi) \in \operatorname*{arg\,min}_{(x,y)\in P_t(\xi)} \left\{c_t(x,y,\xi) + V_{t+1}(y)\right\}$$

Bellman operator

For any time t, and any function A mapping the set of states and noises $\mathbb{X} \times \Xi$ into $\mathbb{R}$, we define

$$\begin{cases} \hat{\mathcal{B}}_t(A)(x,\xi) & := \min\limits_{(x,y)\in P_t(\xi)} c_t(x,y,\xi) + A(y) \\ \mathcal{B}_t(A)(x) & := \mathbb{E}\left[\hat{\mathcal{B}}_t(A)(x,\boldsymbol{\xi}_t)\right] \end{cases}$$

Thus the Bellman equation simply reads

$$\begin{cases} V_T & = & K \\ V_t & = & \underbrace{\mathcal{B}_t(V_{t+1})}_{\tilde{V}_t} + \mathbb{I}_{X_t} \end{cases}$$

The Bellman operators have the same properties as in the deterministic case

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Contents

Computing cuts (1/2)

Suppose that we have $\underline{V}_{t+1}^{(k+1)} \leq V_{t+1}$

$$\begin{aligned} \hat{\theta}_t^{(k+1)}(\xi) = \min_{x,y} \quad & c_t(x,y,\xi) + \underline{V}_{t+1}^{(k+1)}(y) \\ s.t \quad & x = x_t^{(k)} \qquad [\hat{\alpha}_t^{(k+1)}(\xi)] \\ & (x,y) \in P_t(\xi) \end{aligned}$$

This can also be written as

$$\hat{\theta}_t^{(k+1)}(\xi) = \hat{\mathcal{B}}_t\left[\underline{V}_{t+1}^{(k+1)}\right](x,\xi)$$

$$\hat{\alpha}_t^{(k+1)}(\xi) \in \partial_x \hat{\mathcal{B}}_t\left[\underline{V}_{t+1}^{(k+1)}\right](x,\xi)$$

Thus, for all ξ, $\hat{\mathcal{C}}_t^{(k+1),\xi} : x \mapsto \hat{\theta}_t^{(k+1)}(\xi) + \left\langle \hat{\alpha}_t^{(k+1)}(\xi), x - x_t^{(k)} \right\rangle$ satisfy

$$\hat{\mathcal{C}}_t^{(k+1),\xi}(x) \leq \hat{\mathcal{B}}_t\left[\underline{V}_{t+1}^{(k+1)}\right](x,\xi) \leq \hat{\mathcal{B}}_t\left[V_{t+1}\right](x,\xi) = \hat{V}_t(x,\xi)$$

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Computing cuts (2/2)

Thus, we have an affine minorant of $\hat{V}_t(x, \boldsymbol{\xi}_t)$ for each realization of $\boldsymbol{\xi}_t$
Replacing ξ by the random variable $\boldsymbol{\xi}_t$ and taking the expectation yields the following affine minorant

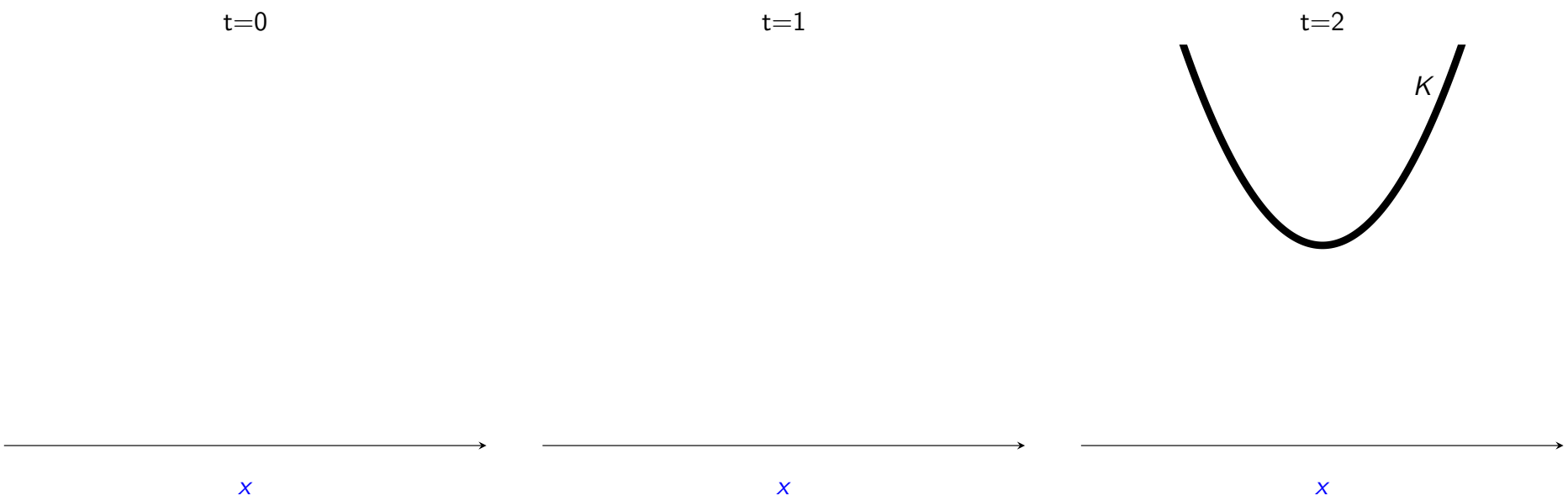
$$\mathcal{C}^{(k+1)} := \theta_t^{(k+1)} + \left\langle \alpha_t^{(k+1)}, \cdot - x_t^{(k)} \right\rangle \leq V_t$$

where

$$\begin{cases} \theta_t^{(k+1)} & := & \mathbb{E}\left[\hat{\theta}_t^{(k+1)}(\boldsymbol{\xi}_t)\right] = \mathcal{B}_t\left[\underline{V}_{t+1}^{(k)}\right](x) \\ \alpha_t^{(k+1)} & := & \mathbb{E}\left[\hat{\alpha}_t^{(k+1)}(\boldsymbol{\xi}_t)\right] \in \partial\mathcal{B}_t\left[\underline{V}_{t+1}^{(k)}\right](x) \end{cases}$$

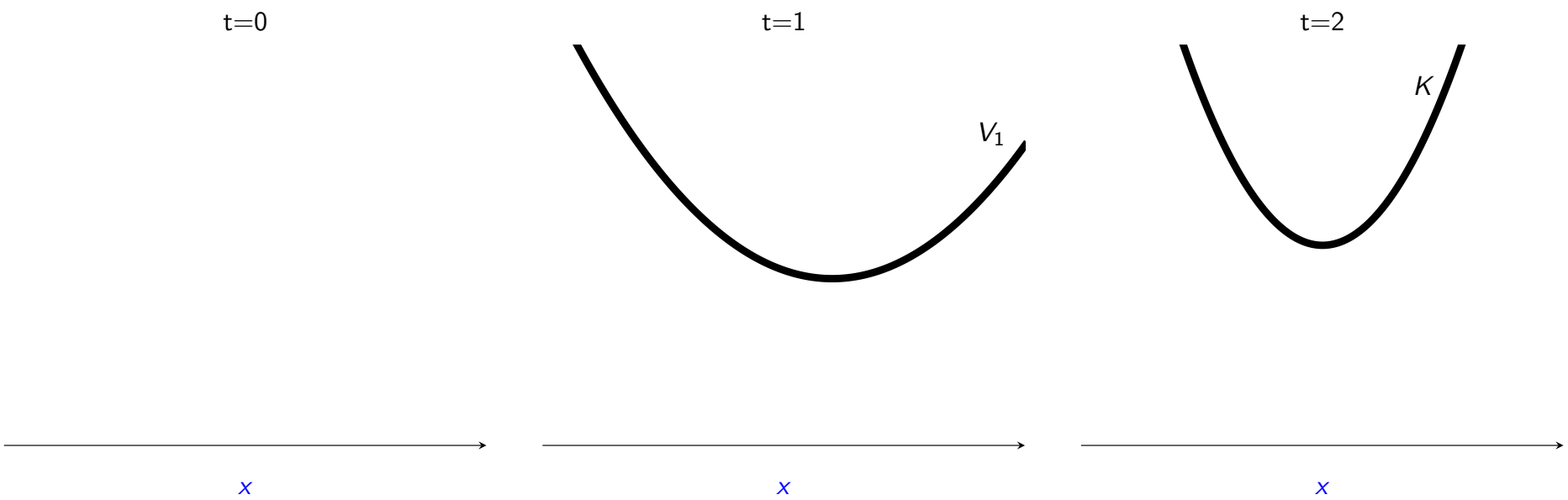
Contents

Abstract SDDP



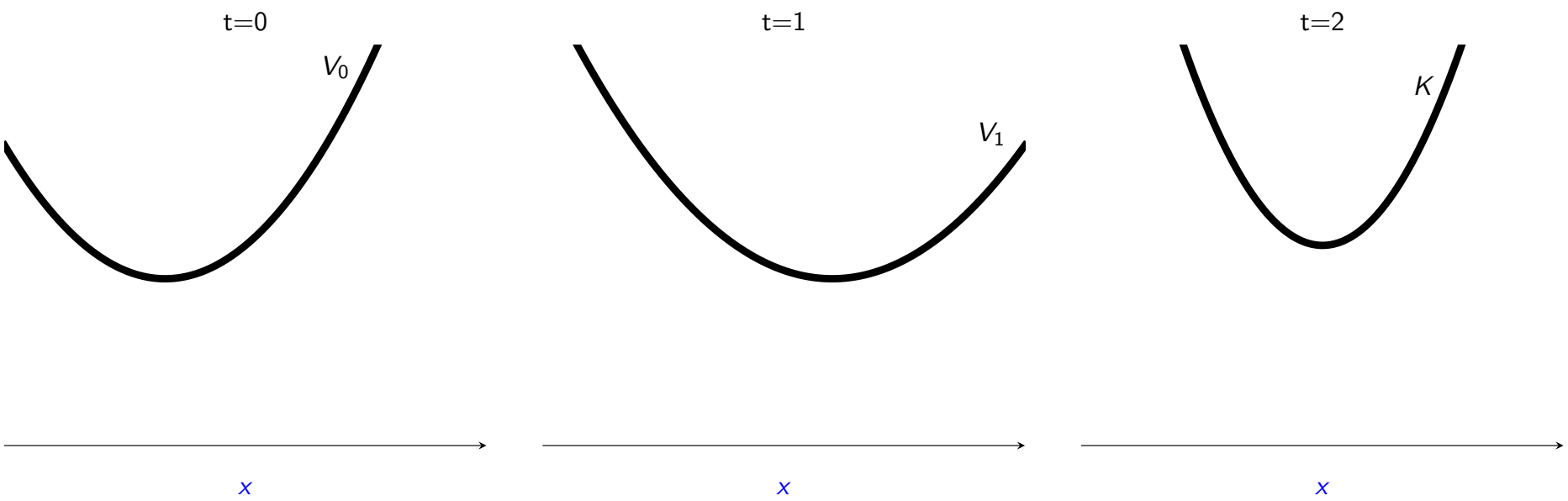
Final Cost $V_2 = K$

Abstract SDDP



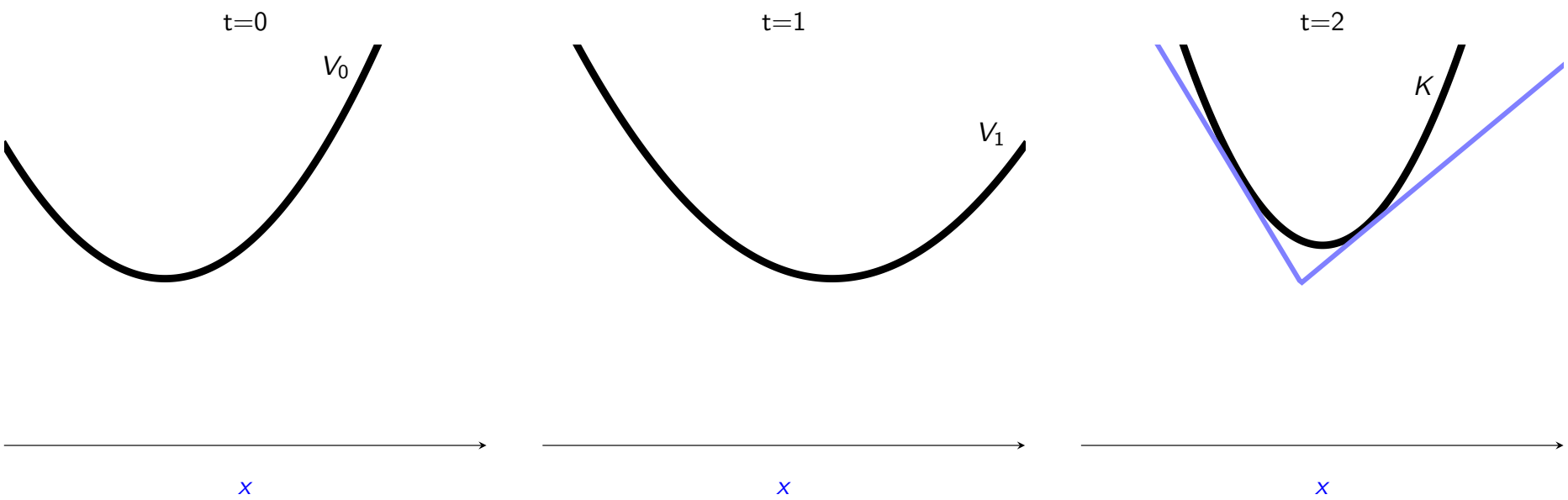
Real Bellman function $V_1 = \mathcal{B}_1(V_2)$

Abstract SDDP



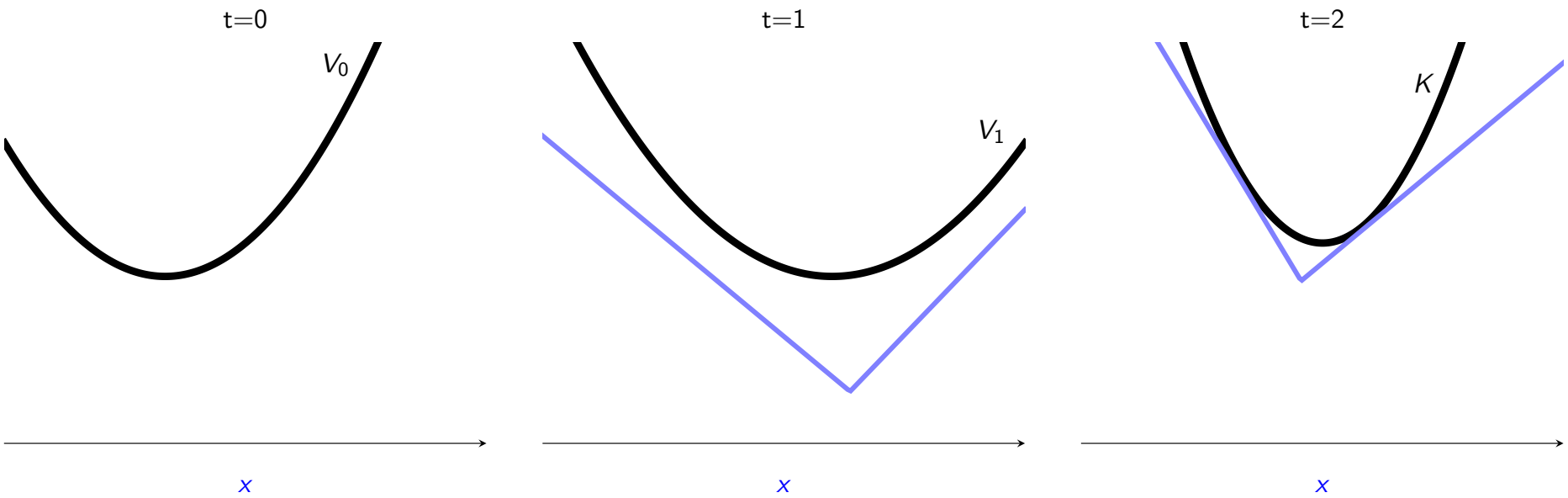
Real Bellman function $V_0 = \mathcal{B}_0(V_1)$

Abstract SDDP



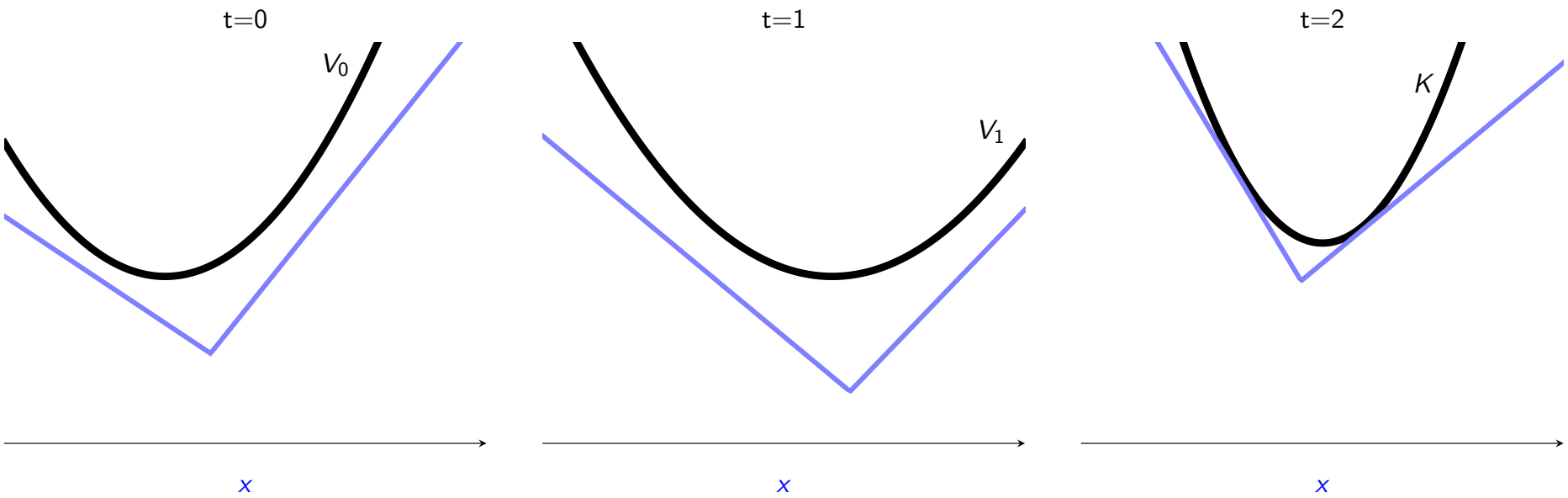
Lower polyhedral approximation $\underline{K}$ of K

Abstract SDDP



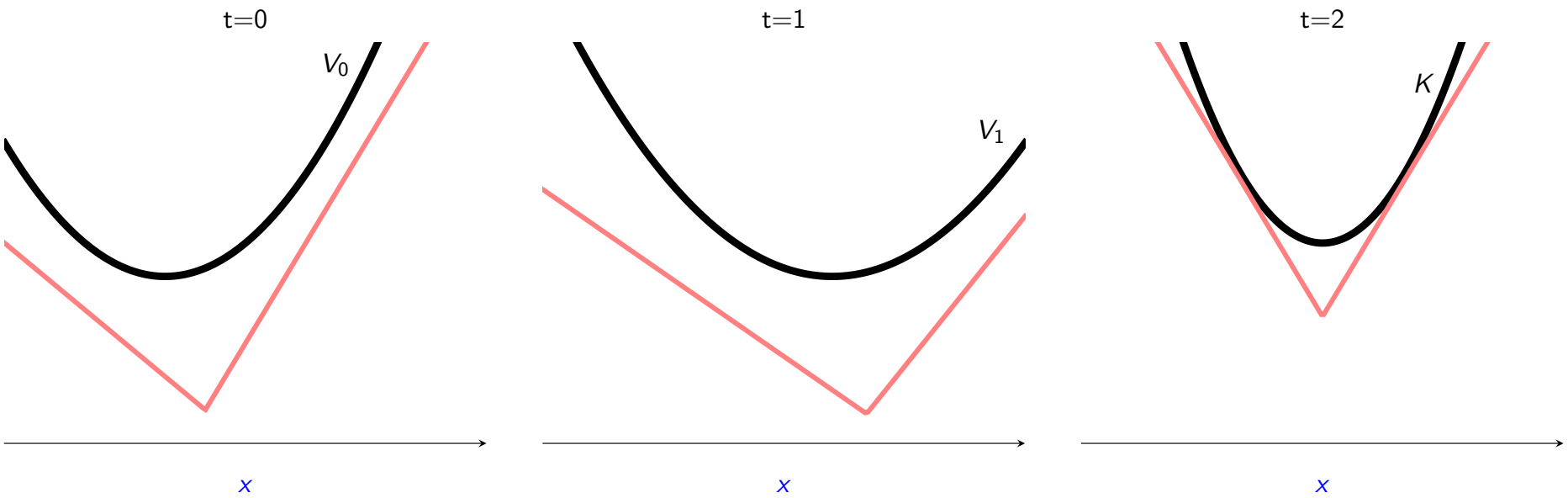
Lower polyhedral approximation $\underline{V}_1 = \mathcal{B}_t(\underline{K})$ of V_1

Abstract SDDP



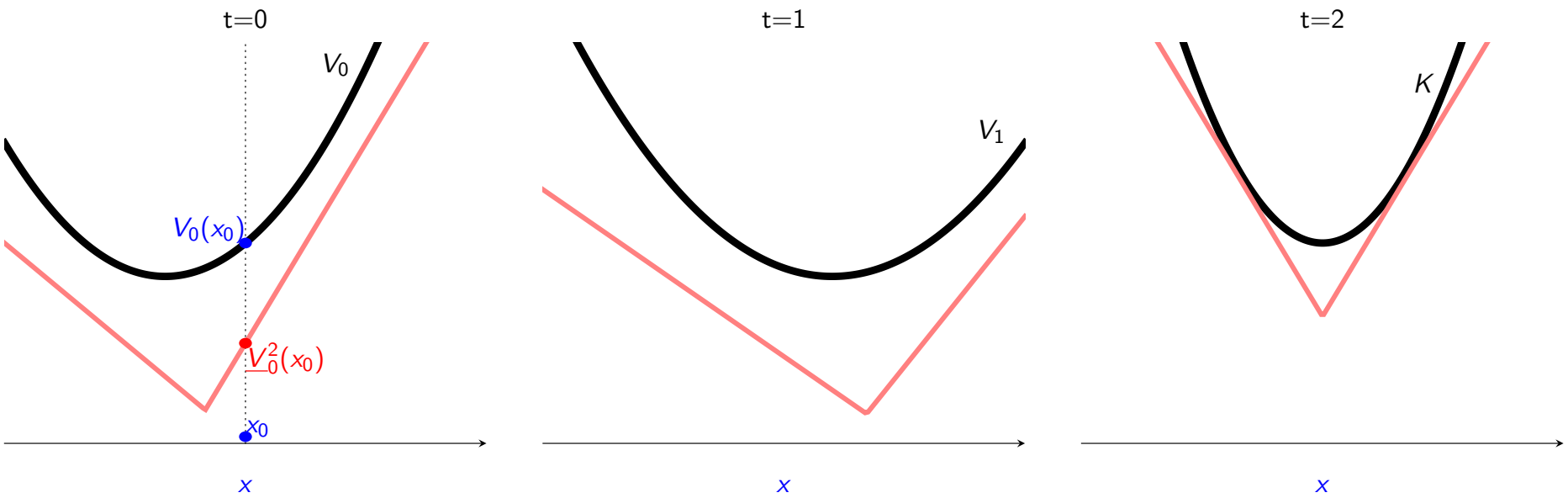
Lower polyhedral approximation $\underline{V}_0 = \mathcal{B}_t(\underline{V}_1)$ of V_0

Abstract SDDP



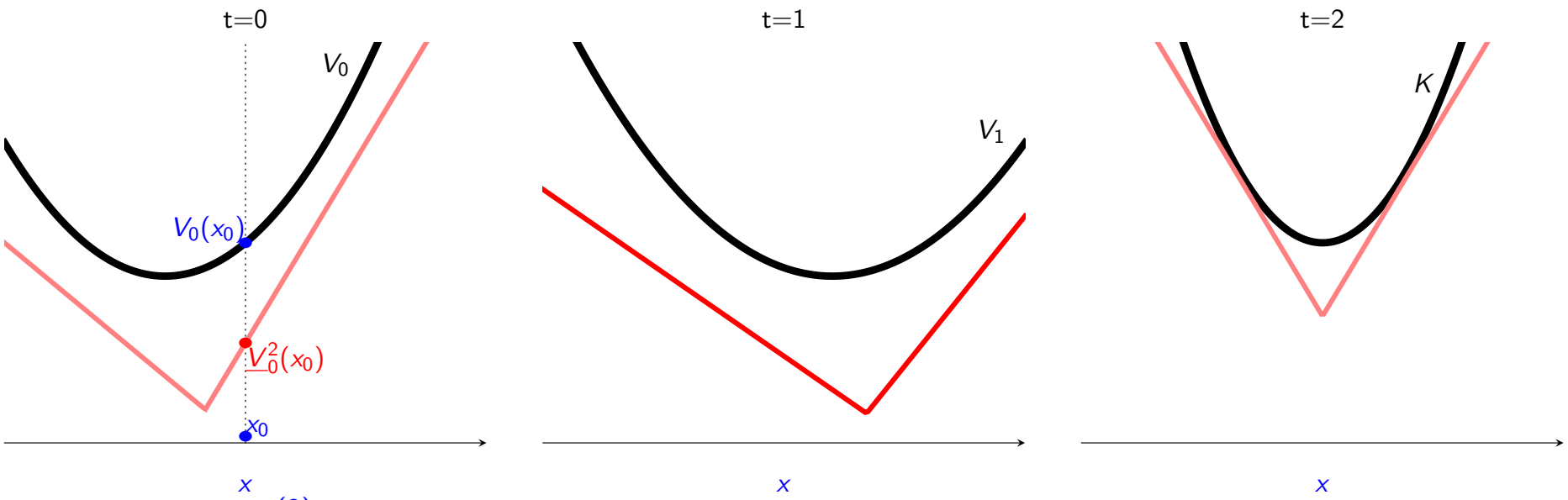
Assume that we have lower polyhedral approximations of V_t

Abstract SDDP



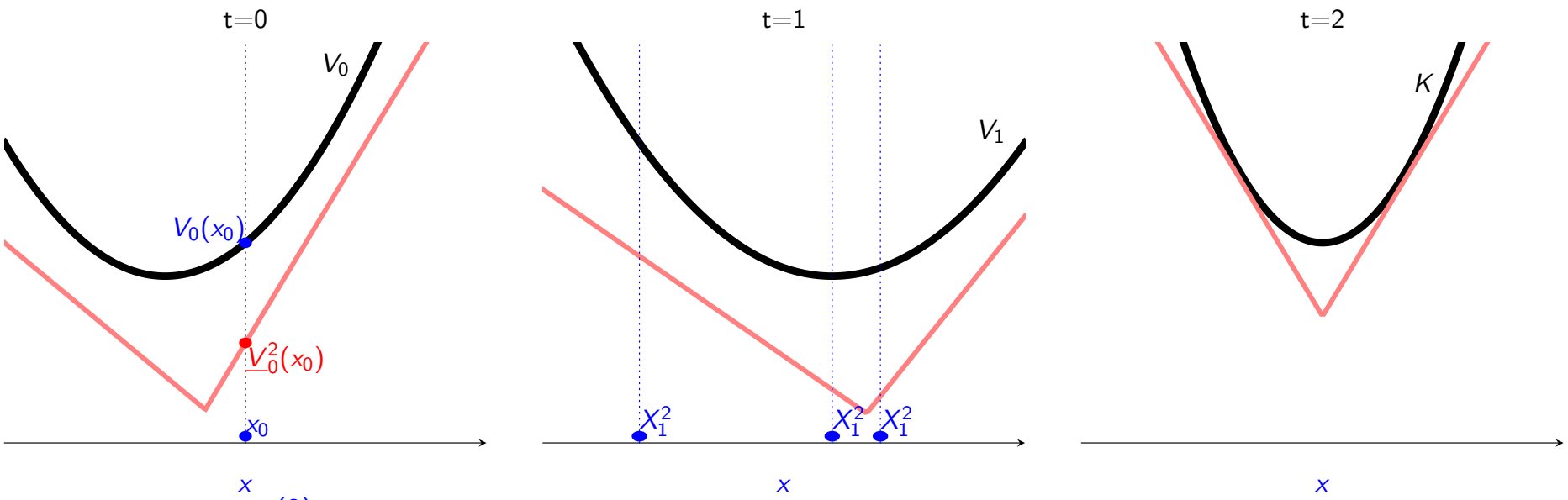
Obtain a lower bound on the value of our problem

Abstract SDDP



Apply $\pi_0^{\underline{V}_1^{(2)}}$ to x_0 and obtain $\boldsymbol{X}_1^{(2)}$

Abstract SDDP



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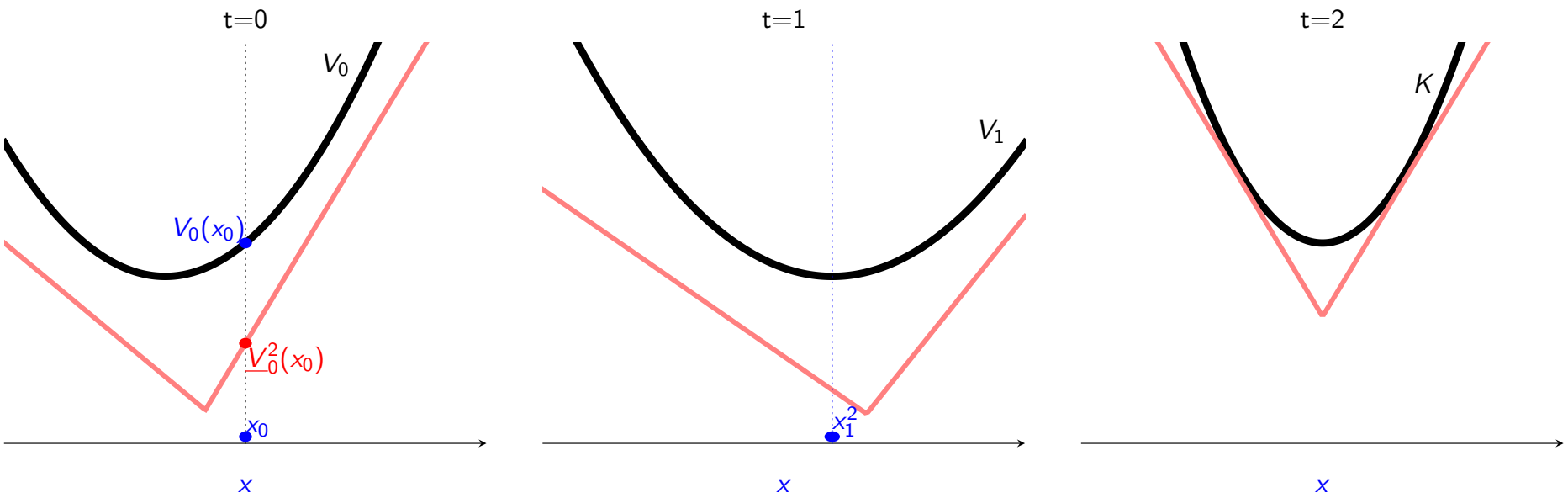
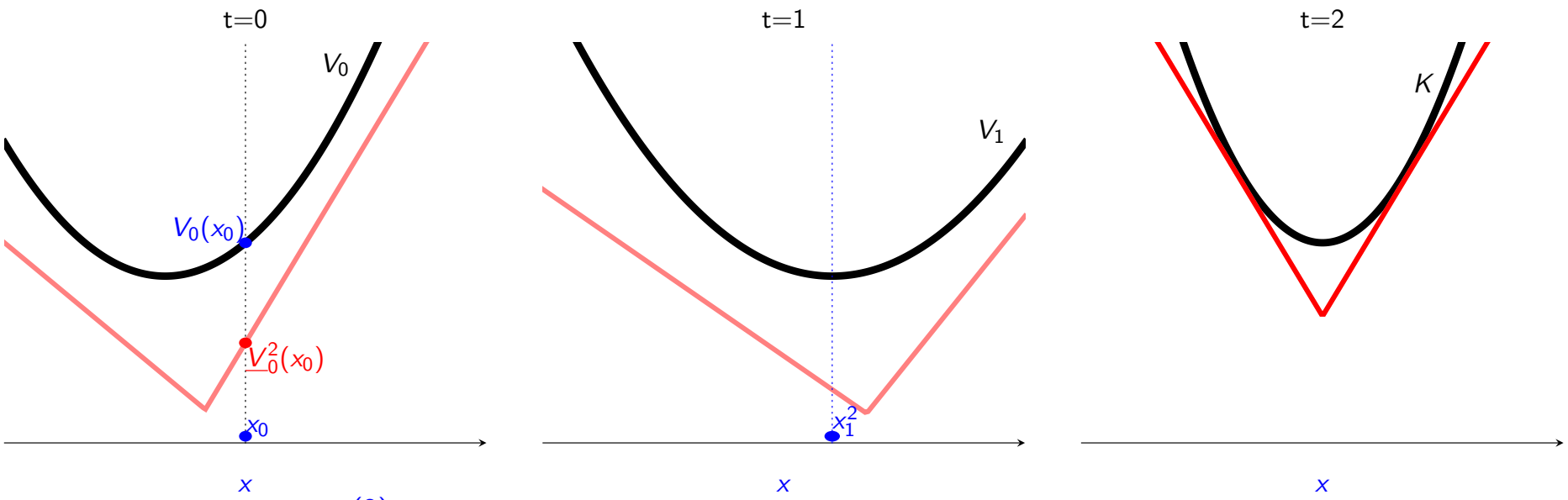


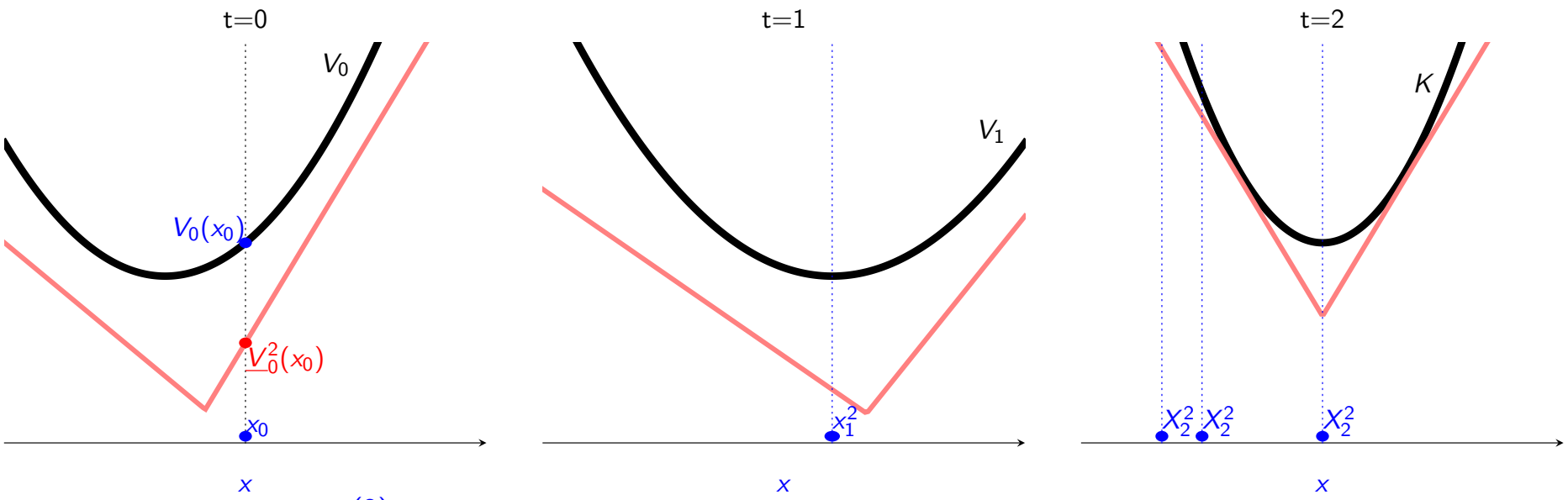
Draw a random realisation $x_1^{(2)}$ of $\boldsymbol{X}_1^{(2)}$

Abstract SDDP



We apply $\pi_1^{\underline{V}_1^{(2)}}$ to $x_1^{(2)}$ and obtain $\boldsymbol{X}_2^{(2)}$

Abstract SDDP



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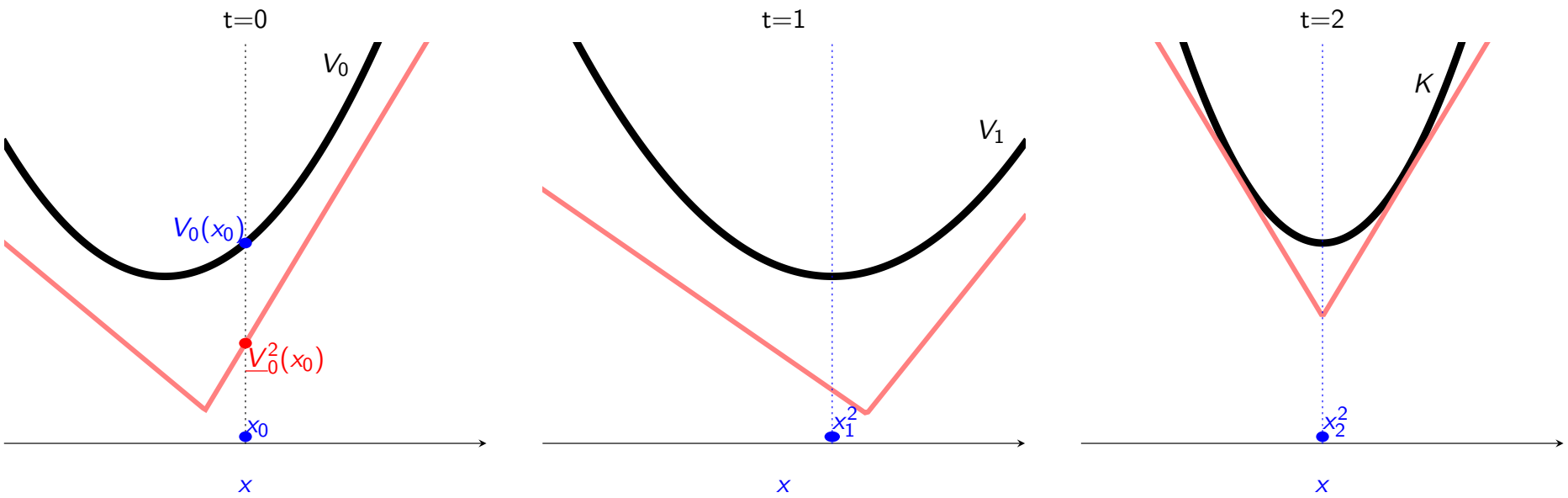
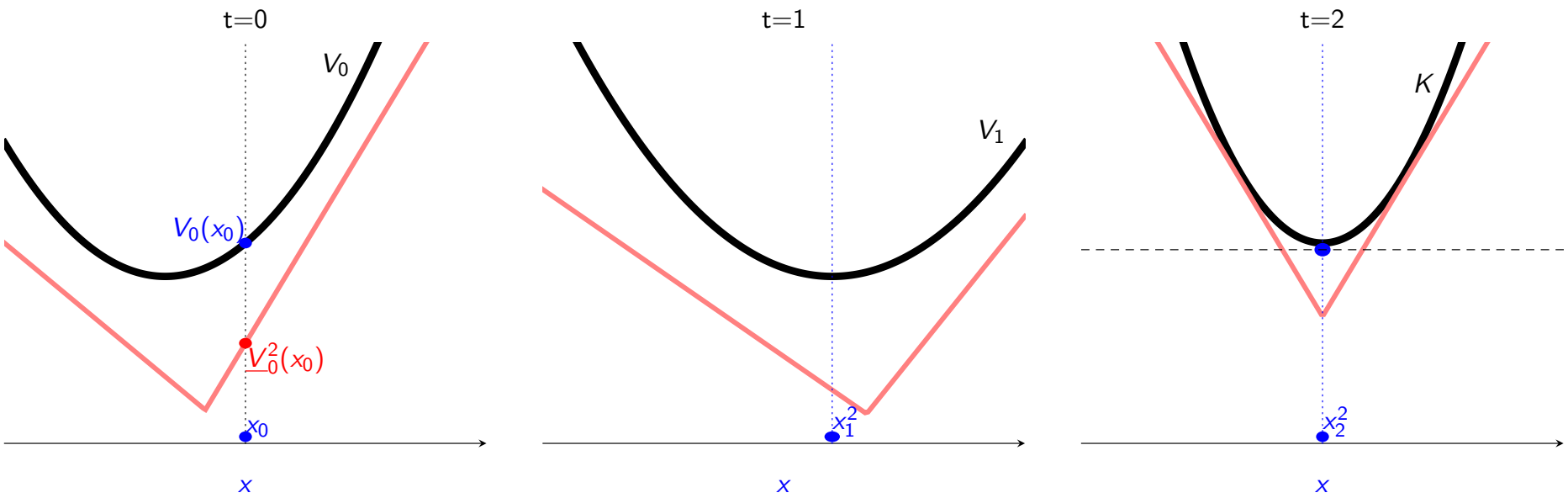
Abstract SDDP



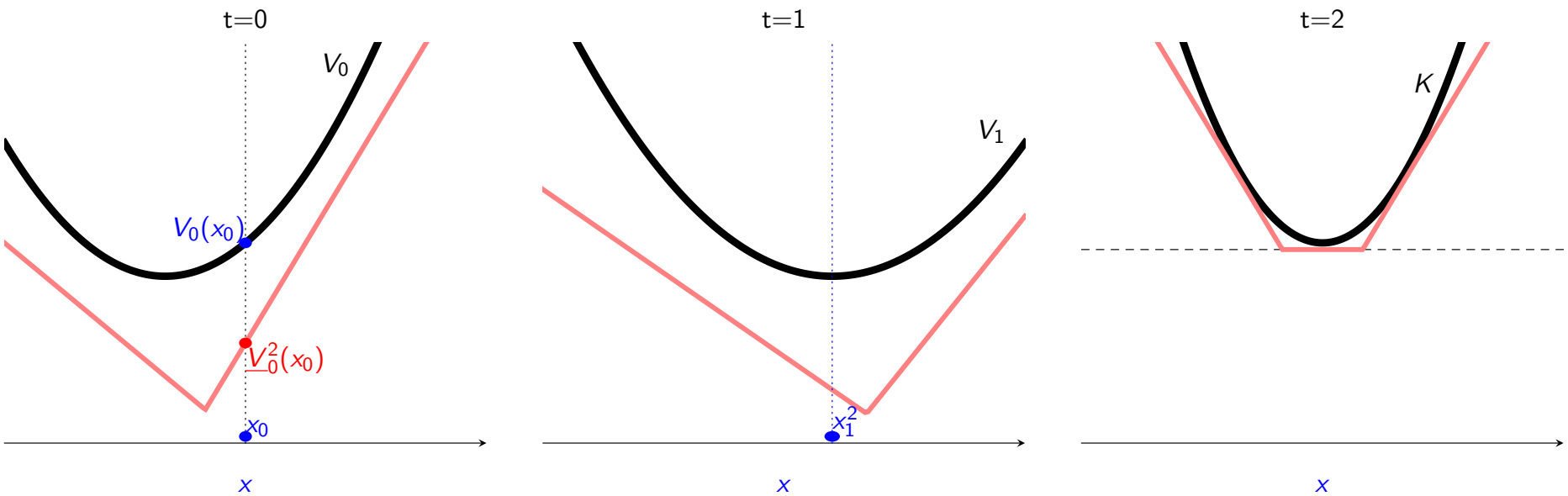
Draw a random realisation $x_2^{(2)}$ of $\boldsymbol{X}_2^{(2)}$

Abstract SDDP



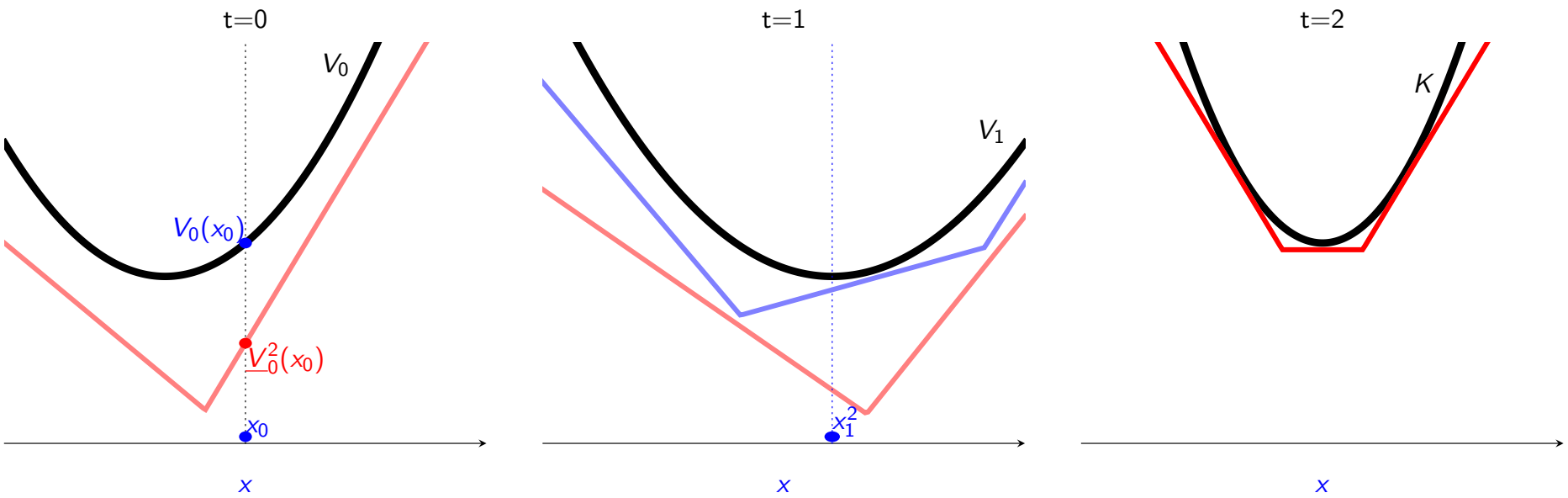
Compute a cut for K at $x_2^{(2)}$

Abstract SDDP



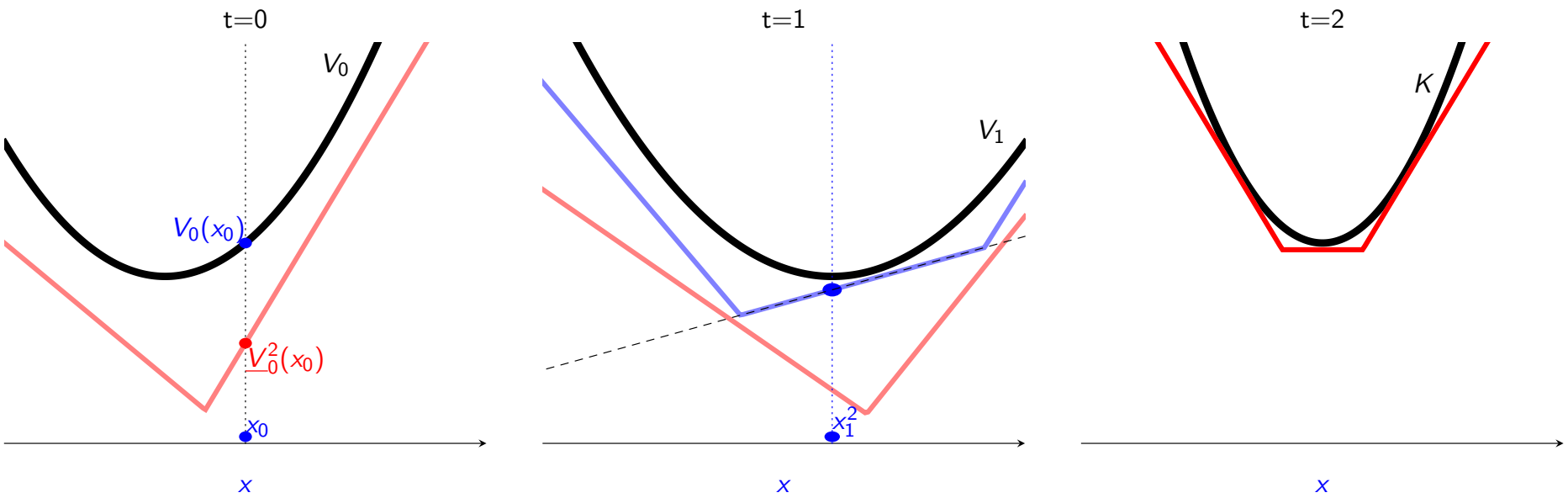
Add the cut to $\underline{V}_2^{(2)}$ which gives $\underline{V}_2^{(3)}$

Abstract SDDP



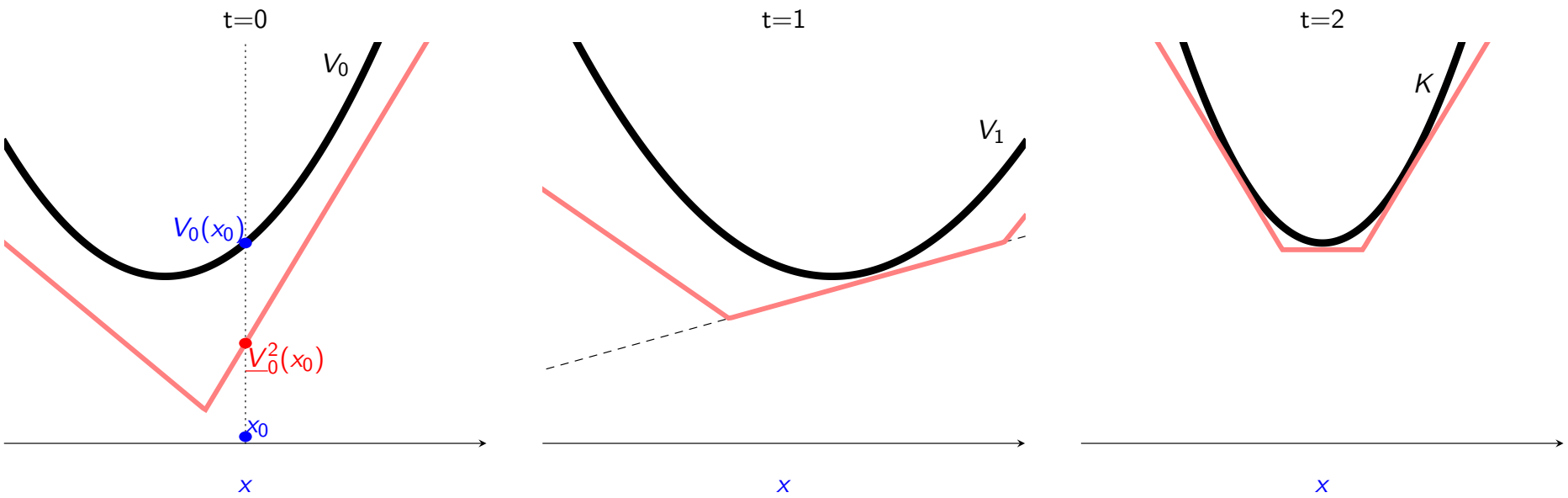
A new lower approximation of V_1 is $\mathcal{B}_1(\underline{V}_2^{(3)})$

Abstract SDDP



Compute the face active at $x_1^{(2)}$

Abstract SDDP



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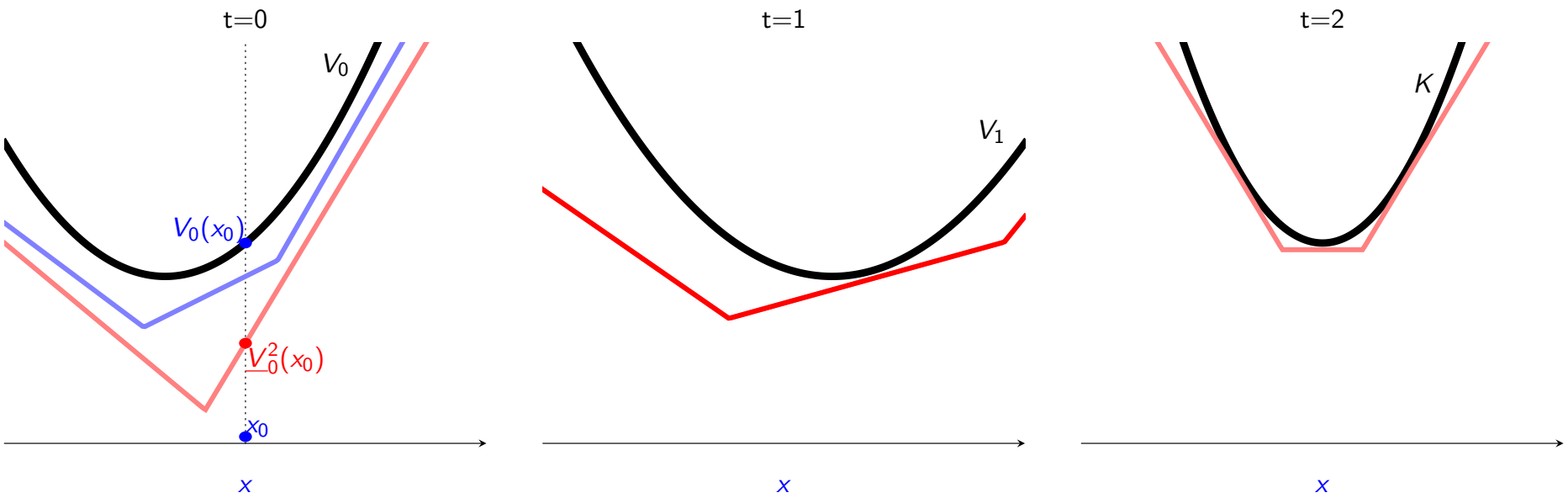
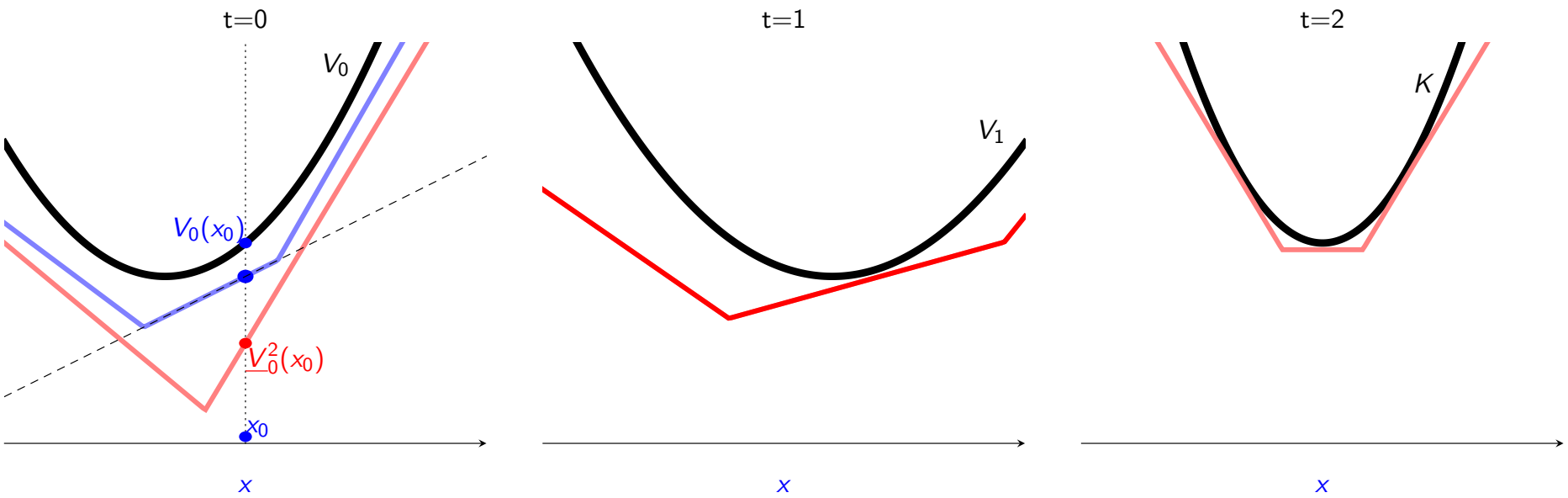
Abstract SDDP



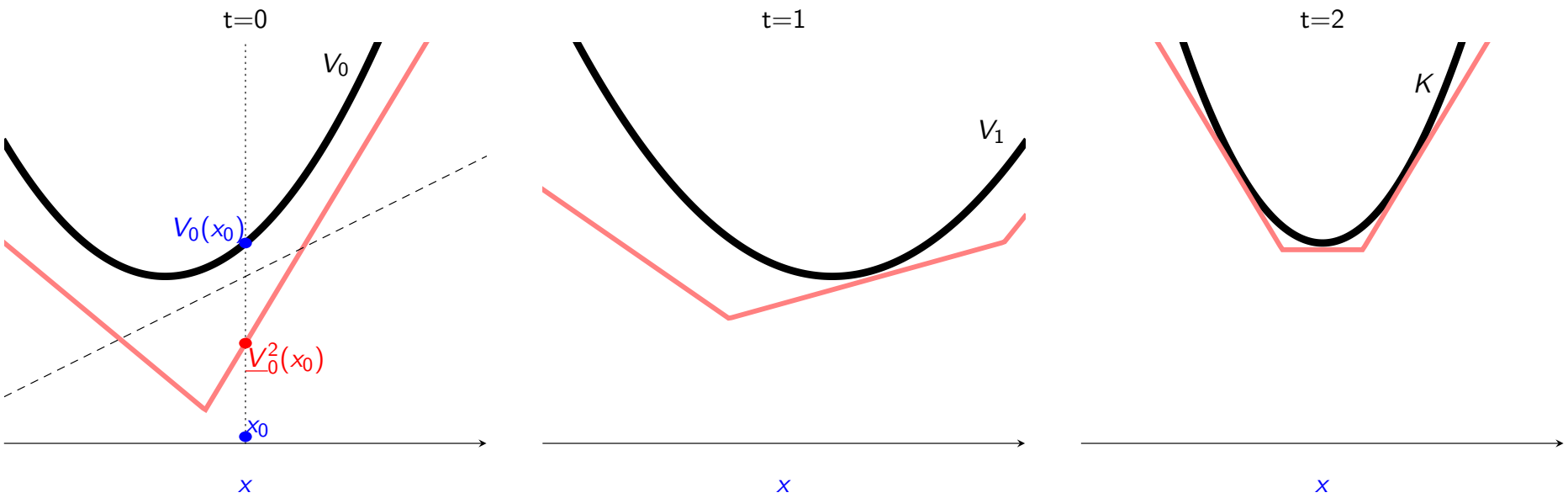
A new lower approximation of V_0 is $\mathcal{B}_0(\underline{V}_1^{(3)})$

Abstract SDDP



Compute the face active at x_0

Abstract SDDP



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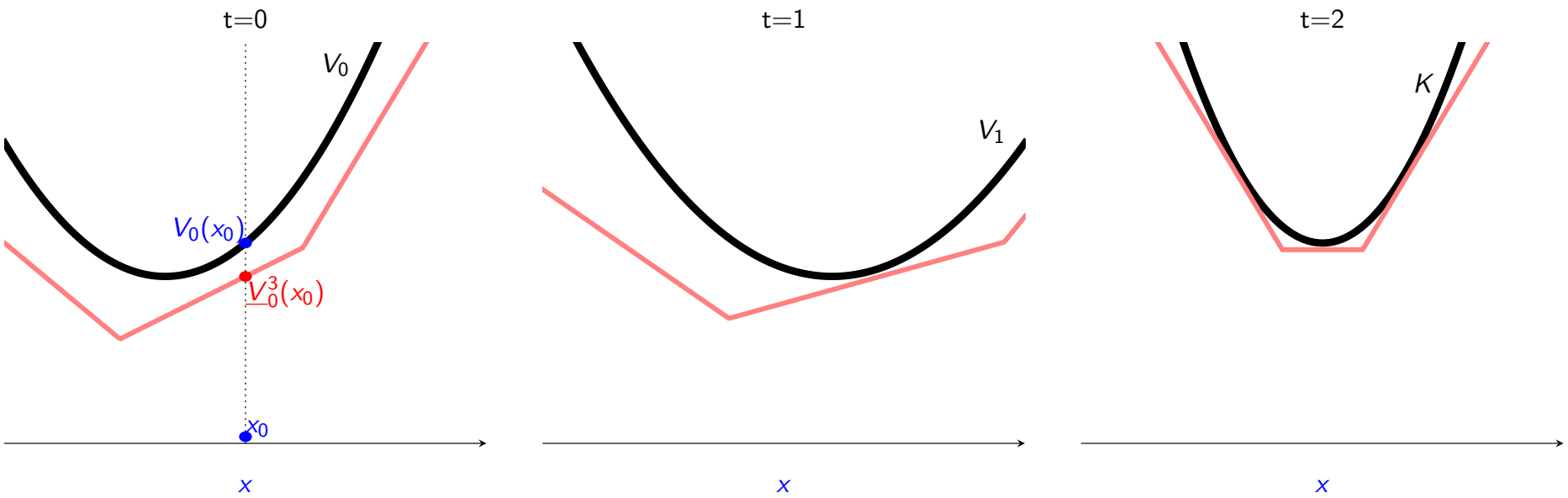
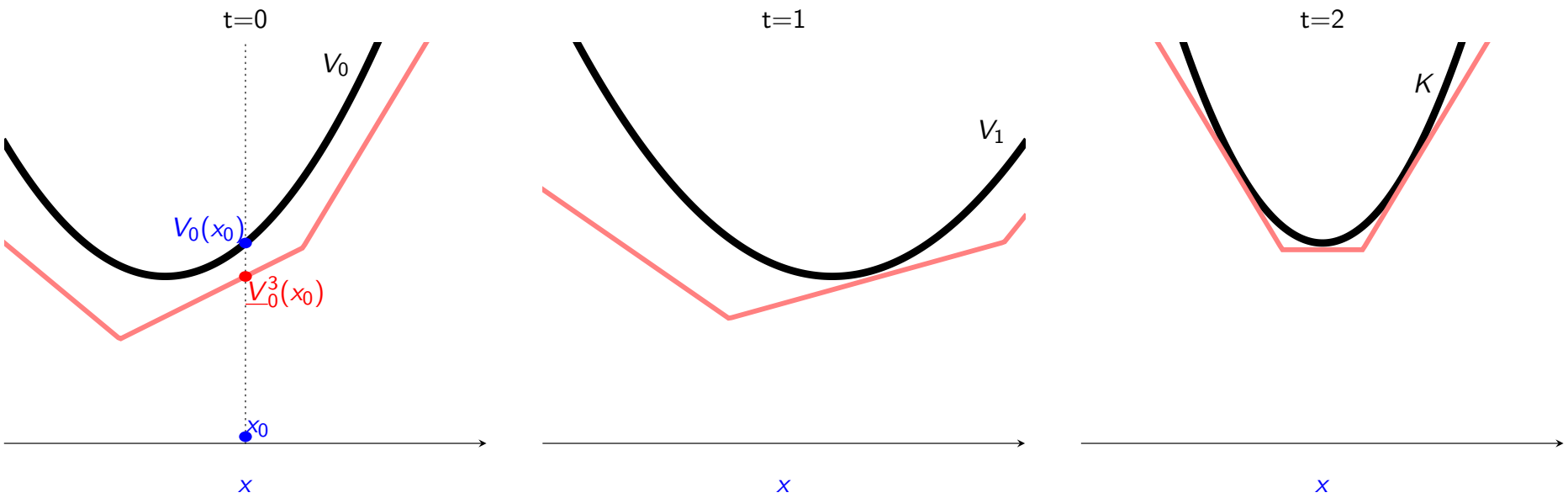


Obtain a new lower bound

Abstract SDDP



Obtain a new lower bound

SDDP description

for $k = 1, 2, \dots$ **do**
  set $V_T^{(k+1)} \equiv K$; $x_0^{(k)} = x_0$;
  draw $(\xi_t^{(k)})_{t \in [\![1,T]\!]}$;
  /* Forward pass : compute trajectory */
  for $t = 0, \dots, T-1$ **do**
    find $x_{t+1}^{(k)} \in \arg\min \hat{\mathcal{B}}_t(\underline{V}_{t+1}^{(k)})(x_t^{(k)}, \xi_t^{(k)})$;
  end
  /* Backward pass : update cuts */
  for $t = T-1, \dots, 0$ **do**
    for $\xi \in \Xi_t$ **do**
      Solve $\hat{\mathcal{B}}_t(\underline{V}_{t+1}^{(k+1)})(x_t^{(k)}, \xi)$ to compute $\hat{\mathcal{C}}_t^{(k+1),\xi}$;
    end
  end
  Compute averaged cut : $\mathcal{C}_t^{(k+1)}$;
  Update lower approximation : $\underline{V}_t^{(k+1)} := \max\{\underline{V}_t^{(k)}, \mathcal{C}_t^{(k+1)}\}$;
end

Algorithm 3: Stochastic Dual Dynamic Programming

Detailing forward pass

- From $t = 0$ to $t = T - 1$ we have to solve T one-stage problem of the form

$$x_{t+1}^{(k)} \in \arg\min_{y} \quad c_t(x_t^{(k)}, y, \xi_t^{(k)}) + \underline{V}_{t+1}^{(k)}(y)$$
$$(x_t^{(k)}, y) \in P_t$$

- We only need to keep the trajectory $(x_t^{(k)})_{t \in [\![0, T]\!]}$.

Detailing Backward pass

- For each $t = T - 1 \to 0$ we solve Ξ_t one-stage problem

$$\begin{aligned} \hat{\theta}_t^{(k+1)}(\xi) = \min_y \quad & c_t(x_t^{(k)}, y, \xi) + \underline{V}_{t+1}^{(k+1)}(y) \\ & (x_t^{(k)}, y) \in P_t \\ & x = x_t^{(k)} \qquad [\hat{\alpha}_t^{(k+1)}(\xi)] \end{aligned}$$

- By construction, we have that

$$\hat{\theta}_t^{(k+1)}(\xi) = \mathcal{B}_t\left(\underline{V}_{t+1}^{(k)}\right)(x_t^{(k)}, \xi), \qquad \hat{\alpha}_t^{(k+1)}(\xi) \in \partial\mathcal{B}_t\left(\underline{V}_{t+1}^{(k)}\right)(x_t^{(k)}, \xi).$$

- We average the coefficients

$$\theta_t^{(k+1)} = \mathbb{E}\left[\hat{\theta}_t^{(k+1)}(\boldsymbol{\xi})\right], \qquad \alpha_t^{(k+1)} = \mathbb{E}\left[\hat{\alpha}_t^{(k+1)}(\boldsymbol{\xi})\right]$$

- Which means

$$\mathcal{C}_t^{(k+1)} := \theta_t^{(k+1)} + \langle \alpha_t^{(k+1)}, \cdot - x_t^{(k)} \rangle \leq \mathcal{B}_t\left(\underline{V}_{t+1}^{(k+1)}\right) \leq \mathcal{B}_t\left(V_{t+1}\right) = \tilde{V}_t \leq V_t$$

Recall on CLT

- Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of identically distributed random variables with finite variance.
- Then the Central Limit Theorem ensures that

$$\sqrt{n}\left(\frac{\sum_{i=1}^{n} C_i}{n} - \mathbb{E}[C_1]\right) \Longrightarrow G \sim \mathcal{N}(0, Var[C_1]) \,,$$

  where the convergence is in law.
- In practice it is often used in the following way. Asymptotically,

$$\mathbb{P}\left(\mathbb{E}\left[C_1\right] \in \left[\bar{C}_n - \frac{1.96\sigma_n}{\sqrt{n}}, \bar{C}_n + \frac{1.96\sigma_n}{\sqrt{n}}\right]\right) \simeq 95\% \,,$$

  where $\bar{C}_n = \frac{\sum_{i=1}^{n} C_i}{n}$ is the empirical mean and $\sigma_n = \sqrt{\frac{\sum_{i=1}^{n}(C_i - \bar{C}_n)^2}{n-1}}$ the empirical standard deviation.

Bounds

- Exact lower bound on the value of the problem: $\underline{V}_0^{(k)}(x_0)$.
- Exact upper bound on the value of the problem:

$$\mathbb{E}\Big[\sum_{t=0}^{T-1} c_t(\boldsymbol{x}_t^{(k)}, \boldsymbol{x}_{t+1}^{(k)}, \boldsymbol{\xi}_{t+1}) + K(\boldsymbol{X}_T)\Big]$$

  where $\boldsymbol{X}_t^{(k)}$ is the trajectory induced by $\underline{V}_t^{(k)}$.
- This bound cannot be computed exactly,
  but can be estimated by Monte-Carlo method as follows
  - Draw N scenarios $\{\xi_1^n, \dots, \xi_T^n\}$.
  - Simulate the corresponding N trajectories $x_t^{(k),n}$,
    and the total cost for each trajectory $C^{(k),n}$.
  - Compute the empirical mean $\bar{C}^{(k),N}$ and standard dev. $\sigma^{(k),N}$.
  - Then, with confidence 95% the upper bound on the problem is

$$\Big[\bar{C}^{(k),N} - \frac{1.96\sigma^{(k),N}}{\sqrt{N}}, \underbrace{\bar{C}^{(k),N} + \frac{1.96\sigma^{(k),N}}{\sqrt{N}}}_{UB_k}\Big]$$

Stopping rule

- One stopping test consist in fixing an a priori relative gap ε, and stopping if

$$\frac{UB_k - V_0^{(k)}(x_0)}{V_0^{(k)}(x_0)} \leq \varepsilon$$

  in which case we know that the solution is ε-optimal with probability 97.5%.
- It is not necessary to evaluate the gap at each iteration.
- To alleviate the computational load, we can estimate the upper bound by using the trajectories of the recent forward phases.
- Another more practical stopping rule consists in stopping after a given number of iterations or fixed computation time.

Contents

Non-independent inflows

- In most cases the stagewise independence assumption is not realistic.
- One classical way of modelling dependencies consists in considering that the inflows I_t follow an AR-k process

$$I_t = \alpha_1 I_{t-1} + \cdots + \alpha_k I_{t-k} + \theta_t + \xi_t$$

  where ξ_t is the residual, forming an independent sequence.
- The state of the system is now $(X_t, I_{t-1}, \ldots, I_{(t-k)})$.

Implementations and numerical tricks

- We can play with the number of forward / backward pass. Classically we do 200 forward passes in parallel, before computing cuts.
- Instead of averaging the cuts, we can keep one cut per alea, for a multicut version. In other word instead of representing V_t we represent $\hat{V}_t$.
- Early forward passes are not really usefull, selecting (randomly or by hand) a few trajectory can save some workload.
- Cut pruning (eliminating useless cuts) is easy to implement and pretty efficient.
- Adding some regularization term in the forward pass has shown some numerical improvement but is not yet fully understood.

Cut Selection methods I

- Let $\underline{V}_t^{(k)}$ be defined as $\max_{\ell \leq k} \mathcal{C}_t^{(\ell)}$
- For $j \leq k$, if

$$\begin{aligned} \min_{x,\alpha} \quad & \alpha - \mathcal{C}_t^{(j)}(x) \\ s.t. \quad & \alpha \geq \mathcal{C}_t^{(\ell)}(x) \qquad \forall \ell \neq j \end{aligned}$$

  is non-negative, then cut j can be discarded without modifying $\underline{V}_t^{(k)}$
- this technique is exact but time-consuming.

Cut Selection methods II

- Instead of comparing a cut everywhere, we can choose to compare it only on the already visited points.
- The Level-1 cut method goes as follow:
  - keep a list of all visited points $x_t^{(\ell)}$ for $\ell \leq k$.
  - for ℓ from 1 to k, tag each cut that is active at $x_t^{(\ell)}$.
  - Discard all non-tagged cut.

Contents

Coherent Risk Measure I

To take into account some risk aversion we can replace the expectation by a *risk measure*. A risk measure is a function giving to a random cost $\boldsymbol{X}$ a deterministic equivalent $\rho(\boldsymbol{X})$ A Coherent Risk Measure $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is a functionnal satisfying

- Monotonicity: if $\boldsymbol{X} \geq \boldsymbol{Y}$ then $\rho(\boldsymbol{X}) \geq \rho(\boldsymbol{Y})$,
- Translation equivariance: for $c \in \mathbb{R}$ we have $\rho(\boldsymbol{X} + c) = \rho(\boldsymbol{X}) + c$,
- Convexity: for $t \in [0, 1]$, we have

$$\rho(t\boldsymbol{X} + (1-t)\boldsymbol{Y}) \leq t\rho(\boldsymbol{X}) + (1-t)\rho(\boldsymbol{Y}),$$

- Positive homogeneity: for $\alpha \in \mathbb{R}^+$, we have $\rho(\alpha\boldsymbol{X}) = \alpha\rho(\boldsymbol{X})$.

Coherent Risk Measure II

From convex analysis we obtain the main theorem over coherent risk measure.

Theorem

Let ρ be a coherent risk measure, then there exists a (convex) set of probability $\mathcal{P}$ such that

$$\forall \boldsymbol{X}, \qquad \rho(\boldsymbol{X}) = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\boldsymbol{X}].$$

Average Value at Risk I

One of the most practical and used coherent risk measure is the Average Value at Risk at level α. Roughly, it is the expectation of the cost over the α-worst cases. For a random variable $\boldsymbol{X}$ admitting a density, we define de value at risk of level α, as the quantile of level α, that is

$$VaR_\alpha(\boldsymbol{X}) = \inf\left\{t \in \mathbb{R} \quad | \quad \mathbb{P}\big(\boldsymbol{X} \geq t\big) \leq \alpha\right\}.$$

And the average value at risk is

$$AVaR_\alpha(\boldsymbol{X}) = \mathbb{E}\big[\boldsymbol{X} \mid \boldsymbol{X} \geq VaR_\alpha(\boldsymbol{X})\big]$$

Average Value at Risk II

One of the best aspect of the AVaR, is the following formula

$$AVaR_\alpha(\boldsymbol{X}) = \min_{t\in\mathbb{R}} \left\{ t + \frac{\mathbb{E}\left[X - t\right]^+}{\alpha} \right\}.$$

Indeed it allow to linearize the AVaR.

SDDP and risk

- The problem studied was risk neutral
- However a lot of works has been done recently about how to solve risk averse problems
- Most of them are using AVAR, or a mix between AVAR and expectation either as objective or constraint
- Indeed AVAR can be used in a linear framework by adding other variables
- Another easy way is to use "composed risk measures"
- Finally a convergence proof with convex costs (instead of linear costs) exists, although it requires to solve non-linear problems

Contents

Assumptions

- Noises are time-independent, with finite support.
- X_t is convex compact, P_t is closed convex.
- Costs are convex and lower semicontinuous.
- We are in a strong relatively complete recourse framework.

Remark, if we take the tree-view of the algorithm

- stage-independence of noise is not required to have theoretical convergence
- node-selection process should be admissible (e.g. independent, SDDP, CUPPS...)

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Convergence result

Theorem

With the preceding assumption, we have that the upper and lower bound are almost surely converging toward the optimal value, and we can obtain an $\varepsilon-$optimal strategy for any $\varepsilon > 0$.

More precisely, if we call $\underline{V}_t^{(k)}$ the outer approximation of the Bellman function V_t at step k of the algorithm, and $\pi_t^{(k)}$ the corresponding strategy, we have

$$\underline{V}_0^{(k)}(x_0) \to_k V_0(x_0)$$

and

$$\mathbb{E}\left[c_t\left(\boldsymbol{x}_t^{(k)}, \boldsymbol{x}_{t+1}^{(k)}, \boldsymbol{\xi}_t\right) + \underline{V}_{t+1}^{(k)}(\boldsymbol{x}_{t+1}^{(k)})\right] - V_t(\boldsymbol{x}_t^{(k)}) \to_k 0.$$

Contents

Conclusion

SDDP is an algorithm, more precisely a class of algorithms, that

- exploits convexity of the value functions (from convexity of costs...)
- does not require state discretization
- constructs outer approximations of V_t, those approximations being precise only "in the right places"
- gives bounds:
  - "true" lower bound $\underline{V}_0^{(k)}(x_0)$
  - estimated (by Monte-Carlo) upper bound
- constructs linear-convex approximations, thus enabling to use linear solver like CPLEX
- can be shown to display asymptotic convergence

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