Stochastic Optimization
Decomposition Methods for Two-stage problems

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December 12 2020

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Presentation Outline

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Two-stage Problem

The extensive formulation of

$$\begin{aligned} \min_{u_0,\boldsymbol{u}_1} \quad & \mathbb{E}\Big[L(u_0,\boldsymbol{\xi},\boldsymbol{u}_1)\Big] \\ s.t. \quad & g(u_0,\boldsymbol{\xi},\boldsymbol{u}_1) \leq 0, \qquad \mathbb{P}-a.s \\ & \sigma(\boldsymbol{u}_1) \subset \sigma(\boldsymbol{\xi}) \end{aligned}$$

is

$$\begin{aligned} \min_{u_0,\{u_1^s\}_{s\in[\![1,S]\!]}} \quad & \sum_{s=1}^{S} p^s L(u_0,\xi^s,u_1^s) \\ s.t \quad & g(u_0,\xi^s,u_1^s) \leq 0, \qquad \forall s \in [\![1,S]\!]. \end{aligned}$$

It is a deterministic problem that can be solved with standard tools or specific methods.

Splitting variables

The extended Formulation (in a compact formulation)

$$\begin{aligned} \min_{u_0, \{u_1^s\}_{s\in[\![1,S]\!]}} \quad & \sum_{s=1}^{S} p^s L(u_0^s, \xi^s, u_1^s) \\ s.t \quad & g(u_0, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]. \end{aligned}$$

Can be written in a splitted formulation

$$\begin{aligned} \min_{\{u_0^s, u_1^s\}_{s\in[\![1,S]\!]}} \quad & \sum_{s=1}^{S} p^s L(u_0^s, \xi^s, u_1^s) \\ s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!] \\ & u_0^s = u_0^{s'} \qquad \forall s, s' \end{aligned}$$

Splitting variables

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Can be written in a splitted formulation

$$\begin{aligned} \min_{\bar{u}_0 \{u_0^s, u_1^s\}_{s \in [\![1,S]\!]}} \quad & \sum_{s=1}^{S} p^s L(u_0^s, \xi^s, u_1^s) \\ s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!] \\ & u_0^s = \bar{u}_0 \end{aligned}$$

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Dualizing non-anticipativity constraint I

$$
\begin{aligned}
\min_{\{u_0^s, u_1^s\}_{s\in[\![1,S]\!]}} \quad & \sum_{s=1}^{S} p^s L(u_0^s, \xi^s, u_1^s) \\
s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0, && \forall s \in [\![1, S]\!] \\
& u_0^s = \sum_{s'} p^{s'} u_0^{s'} && \forall s
\end{aligned}
$$

is equivalent to

$$
\min_{\{u_0^s, u_1^s\}_{s\in[\![1,S]\!]}} \max_{\{\lambda^s\}_{s\in[\![1,S]\!]}} \quad \sum_{s=1}^{S} p^s L(u_0^s, \xi^s, u_1^s) + p^s \lambda^s \Big(u_0^s - \sum_{s'} p^{s'} u_0^{s'} \Big)
$$

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s.t \quad g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]
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& + \sum_{s=1}^{S} p^s \lambda^s u_0^s - \sum_{s,s'} p^s \lambda^s p^{s'} u_0^{s'} \\
s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]
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s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]
\end{aligned}
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$$\min_{\{u_0^s, u_1^s\}_{s\in[\![1,S]\!]}} \quad \sum_{s=1}^{S} p^s L(u_0^s, \xi^s, u_1^s)$$

$$s.t \quad g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]$$

$$u_0^s = \sum_{s'} p^{s'} u_0^{s'} \qquad \forall s$$

is equivalent to

$$\min_{\{u_0^s, u_1^s\}_{s\in[\![1,S]\!]}} \max_{\{\lambda^s\}_{s\in[\![1,S]\!]}} \quad \sum_{s=1}^{S} p^s L(u_0^s, \xi^s, u_1^s) + \sum_{s=1}^{S} p^s \lambda^s u_0^s - \sum_s \mathbb{E}\left[\boldsymbol{\lambda}\right] p^s u_0^s$$

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& + \sum_{s=1}^{S} p^s \Big(\lambda^s - \mathbb{E}\big[\boldsymbol{\lambda}\big]\Big) u_0^s \\
s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]
\end{aligned}
$$

Dualizing non-anticipativity constraint II

Thus, the dual problem reads

$$\max_{\boldsymbol{\lambda}} \quad \min_{\{u_0^s, u_1^s\}_{s\in[\![1,S]\!]}} \quad \sum_{s=1}^{S} p^s \Big(L(u_0^s, \xi^s, u_1^s) + \Big(\lambda^s - \mathbb{E}\big[\boldsymbol{\lambda}\big]\Big) u_0^s \Big)$$

$$s.t \quad g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]$$

The inner minimization problem, for $\boldsymbol{\lambda}$ given, can decompose scenario by scenario, by solving S deterministic problem

$$\min_{\{u_0^s, u_1^s\}} \quad L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s$$

$$s.t \quad g(u_0^s, \xi^s, u_1^s) \leq 0$$

Dualizing non-anticipativity constraint II

Thus, the dual problem reads

$$\max_{\boldsymbol{\lambda}:\mathbb{E}[\boldsymbol{\lambda}]=0} \min_{\{u_0^s,u_1^s\}_{s\in[\![1,S]\!]}} \sum_{s=1}^{S} p^s \Big(L(u_0^s, \xi^s, u_1^s) + \big(\lambda^s \qquad \big) u_0^s \Big)$$

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Price of information

- By weak duality, any $\boldsymbol{\lambda}$ such that $\mathbb{E}[\boldsymbol{\lambda}] = 0$ will give a lower bound on the 2-stage problem, computed as

$$\sum_{s=1}^{S} p^s \quad \min_{u_0^s, u_1^s} \quad \Big(L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s \Big)$$
$$s.t \quad g(u_0^s, \xi^s, u_1^s) \leq 0$$

- $\boldsymbol{\lambda} = 0$ lead to the anticipative lower-bound
- If problem is convex, and under some qualification assumptions, there exists an optimal $\boldsymbol{\lambda}^*$, called the price of information, such that the lower bound is tight.

Progressive Hedging Algorithm

The progressive hedging algorithm build on this decomposition in the following way.

1. Set a price of information $\{\lambda^s\}_{s\in[\![1,S]\!]}$ such that $\mathbb{E}\big[\boldsymbol{\lambda}\big] = 0$
2. For each scenario solve

$$\begin{aligned} \min_{u_0^s, u_1^s} \quad & L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s \\ s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0 \end{aligned}$$

3. Compute the mean first control $\bar{u}_0 := \sum_{s=1}^{S} p^s u_0^s$
4. Update the price of information with

$$\lambda^s := \lambda^s + \rho(u_0^s - \bar{u}_0)$$

5. Go back to 2.

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Convergence of Progressive Hedging

Theorem

Assume that L and g are convex lsc in (u_0, u_1) for all ξ, and that, for all $s \in S$, there exists (u_0^s, u_1^s) such that $L(u_0^s, \xi^s, u_1^s) < +\infty$ and $g(u_0^s, \xi^s, u_1^s) < 0$.
Then, the progressive hedging algorithm converges toward an optimal primal solution, and the price of information converges toward an optimal price of information.

Moreover we can show that

$$\varepsilon_k = \sqrt{\|(u_0^k, u_1^k) - (u_0^\sharp, u_1^\sharp)\|_2^2 + \frac{1}{\rho^2}\|\lambda - \lambda^\sharp\|_2^2},$$

is a decreasing sequence.

Bounds in Progressive Hedging

- At any iteration of the PH algorithm, we have a collection of primal solution $\{(u_0^s, u_1^s)\}_{s\in S}$, and a price of information $\{\lambda^s\}_{s\in S}$.
- We have a lower bound on the value of the stochastic programm given by

$$LB^{PH} = \sum_{s\in S} p^s \big[L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s\big],$$

- and an upper bound given by

$$UB^{PH} = \sum_{s\in S} p^s L(\bar{u}_0, \xi^s, u_1^s(u_0)).$$

Presentation Outline

1. Lagrangian decomposition
2. L-Shaped decomposition method
3. Multistage program

Linear 2-stage stochastic program

Consider the following problem

$$\begin{aligned} \min \quad & \mathbb{E}\Big[c^\top u_0 + \boldsymbol{q}^\top \boldsymbol{u}_1\Big] \\ s.t. \quad & A u_0 = b, \quad u_0 \geq 0 \\ & \boldsymbol{T} u_0 + \boldsymbol{W}\boldsymbol{u}_1 = \boldsymbol{h}, \quad \boldsymbol{u}_1 \geq 0, \qquad \mathbb{P}-a.s. \\ & u_0 \in \mathbb{R}^n, \quad \sigma(\boldsymbol{u}_1) \subset \sigma(\underbrace{\boldsymbol{q}, \boldsymbol{T}, \boldsymbol{W}, \boldsymbol{h}}_{\boldsymbol{\xi}}) \end{aligned}$$

Which we rewrite

$$\begin{aligned} \min_{u_0 \geq 0} \quad & c^\top u_0 + \mathbb{E}\Big[Q(u_0, \boldsymbol{\xi})\Big] \\ s.t. \quad & A u_0 = b \end{aligned}$$

with

$$\begin{aligned} Q(u_0, \xi) := \min_{u_1 \geq 0} \quad & q_\xi^\top u_1 \\ s.t. \quad & W_\xi u_1 = h_\xi - T_\xi u_0 \end{aligned}$$

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\end{aligned}
$$

Linear 2-stage stochastic program : Extensive Formulation

The associated extensive formulation read

$$
\begin{aligned}
\min \quad & c^\top u_0 + \sum_{s=1}^{S} p^s \; q^s \cdot u_1^s \\
s.t. \quad & Au_0 = b, \quad u_0 \geq 0 \\
& T^s u_0 + W^s u_1^s = h^s, \quad u_1^s \geq 0, \forall s
\end{aligned}
$$

Which we rewrite

$$
\begin{aligned}
\min_{u_0} \quad & c^\top u_0 + \sum_{s=1}^{S} p^s Q^s(u_0) \\
s.t. \quad & Au_0 = b, \quad u_0 \geq 0
\end{aligned}
$$

with

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Q^s(u_0) := \min_{u_1 \geq 0} \quad & q^s \cdot u_1 \\
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Relatively complete recourse

We assume here relatively complete recourse. Without this assumption we would need feasability cuts.
Here, relatively complete recourse means that, for $u_0 \geq 0$:

$$Au_0 = b \quad \Longrightarrow \quad Q_s(u_0) < +\infty, \quad \forall s \in [\![1, S]\!]$$

Decomposition of linear 2-stage stochastic program

We rewrite the extended formulation as

$$\begin{aligned} \min_{u_0,(\theta^s)_{s\in S}} \quad & c^\top u_0 + \sum_s p^s \theta^s \\ s.t. \quad & A u_0 = b, \quad u_0 \geq 0 \\ & \theta^s \geq Q^s(u_0) & \forall s \end{aligned}$$

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s.t. \quad & Au_0 = b, \quad u_0 \geq 0 \\
& \theta^s \geq \alpha_k^s \cdot u_0 + \beta_k^s \qquad \forall k, \forall s
\end{aligned}
$$

Note that $Q^s(u_0)$ is a polyhedral function of u_0, hence $\theta^s \geq Q^s(u_0)$ can be rewritten $\theta^s \geq \alpha_k^s \cdot u_0 + \beta_k^s, \forall k$.

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Note that $Q^s(u_0)$ is a polyhedral function of u_0, hence $\theta^s \geq Q^s(u_0)$ can be rewritten $\theta^s \geq \alpha_k^s \cdot u_0 + \beta_k^s, \forall k$.
The decomposition approach consists in constructing iteratively cut coefficients α_k^s and β_k^s.

Obtaining (optimality) cuts I

Recall that

$$Q^s(u_0) := \min_{u_1^s \in \mathbb{R}^n} \quad q^s \cdot u_1^s$$
$$s.t. \quad W^s u_1^s = h^s - T^s u_0, \quad u_1^s \geq 0$$

can also be written (through strong duality by relatively complete recourse assumption)

$$(D_{u_0}) \quad Q^s(u_0) = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot (h^s - T^s u_0)$$
$$s.t. \quad (W^s)^\top \lambda^s \leq q^s$$

Obtaining (optimality) cuts II

$$(D_{u_0}) \quad Q^s(u_0) = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot (h^s - T^s u_0)$$
$$s.t. \quad (W^s)^\top \lambda^s \leq q^s$$

admits for optimal solution $\lambda^s_{u_0}$.
Consider another control u_0', we have

$$(D_{u_0'}) \quad Q^s(u_0') = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot (h^s - T^s u_0')$$
$$s.t. \quad (W^s)^\top \lambda^s \leq q^s$$

As $\lambda^s_{u_0}$ is admissible for (D_{u_0}) it is also admissible for $(D_{u_0'})$, hence

$$Q^s(u_0') \geq \lambda^s_{u_0} \cdot (h^s - T^s u_0').$$

Obtaining (optimality) cuts III

To sum up we have seen that, for any admissible first stage solution, we can construct an exact cut for Q^s by solving the dual of the second stage problem.

Obtaining (optimality) cuts III

To sum up we have seen that, for any admissible first stage solution, we can construct an exact cut for Q^s by solving the dual of the second stage problem.

More precisely, let $u_0^k \geq 0$ be such that $Au_0^k = b$. Let λ_k^s be an optimal dual solution. Then, setting

$$\alpha_k^s := -(T^s)^\top \lambda_k^s \quad \text{and} \quad \beta_k^s := (\lambda_k^s)^\top h^s$$

we have

$$\begin{cases} Q^s(u_0') \geq \alpha_k^s \cdot u_0' + \beta_k^s & \forall u_0' \geq 0, Au_0' = b \\ Q^s(u_0^k) = \alpha_k^s \cdot u_0^k + \beta_k^s \end{cases}$$

L-shaped method (multi-cut version)

1. We have a collection of $K \times S$ cuts, such that $Q^s(u_0) \geq \alpha_k^s \cdot u_0 + \beta_k^s$.
2. Solve the master problem, with optimal primal solution u_0^{K+1}.

$$\begin{aligned} \min_{u_0 \geq 0} \quad & c^\top u_0 + \sum_{s=1}^{S} p^s \theta^s \\ s.t. \quad & A u_0 = b \\ & \theta^s \geq \alpha_k^s u_0 + \beta_k^s \qquad \forall k \in [\![1, K]\!], \quad \forall s \in [\![1, S]\!] \end{aligned}$$

3. Solve S slave problems, with optimal dual solution λ_{K+1}^s

$$\begin{aligned} Q^s(u_0^{K+1}) = \min_{u_1^s \in \mathbb{R}^n} \quad & q^s \cdot u_1^s \\ s.t. \quad & W^s u_1^s = h^s - T^s u_0^{K+1}, \quad u_1^s \geq 0 \end{aligned}$$

4. construct S new cuts with

$$\alpha_{K+1}^s := -(T^s)^\top \lambda_{K+1}^s, \qquad \beta_{K+1}^s := h^s \cdot \lambda_{K+1}^s$$

L-shaped method (multi-cut version)

1. We have a collection of $K \times S$ cuts, such that $Q^s(u_0) \geq \alpha_k^s \cdot u_0 + \beta_k^s$.
2. Solve the master problem, with optimal primal solution u_0^{K+1}.

$$\begin{aligned} \min_{u_0 \geq 0} \quad & c^\top u_0 + \sum_{s=1}^{S} p^s \theta^s \\ s.t. \quad & A u_0 = b \\ & \theta^s \geq \alpha_k^s u_0 + \beta_k^s \qquad \forall k \in [\![1, K]\!], \quad \forall s \in [\![1, S]\!] \end{aligned}$$

3. Solve S slave problems, with optimal dual solution λ_{K+1}^s

$$\begin{aligned} Q^s(u_0^{K+1}) = \max_{\lambda^s \in \mathbb{R}^m} \quad & \lambda^s \cdot \left(h^s - T^s u_0^{K+1}\right) \\ s.t. \quad & W^s \cdot \lambda^s \leq q^s \end{aligned}$$

4. construct S new cuts with

$$\alpha_{K+1}^s := -(T^s)^\top \lambda_{K+1}^s, \qquad \beta_{K+1}^s := h^s \cdot \lambda_{K+1}^s$$

L-shaped method (multi-cut version) : bounds

- At any iteration of the L-shaped method we can easily determine upper and lower bound over our problem.
- Indeed, u_0^K is an admissible firt stage solution, and $Q^s(u_0^K)$ is the value of a slave problem. Thus the value of admissible solution u_0^k is simply given by

$$UB = c^\top u_0^K + \sum_{s=1}^{S} p^s Q^s(u_0^K).$$

- Furthermore, $Q_K^s(u_0) \geq \max_{k \leq K} \alpha_k^s \cdot u_0 + \beta_k^s$, thus the value of the master problem is always a lower bound over the value of the SP problem :

$$LB = c^\top u_0^K + \sum_{s=1}^{S} p^s \theta_K^s.$$

L-shaped method (single-cut version)

1. We have a collection of K cuts, such that $Q(u_0) := \sum_{s \in S} Q^s(u_0) \geq \alpha_k \cdot u_0 + \beta_k$.
2. Solve the master problem, with optimal primal solution u_0^{K+1}.

$$\begin{aligned} \min_{u_0 \geq 0} \quad & c^\top u_0 + \theta \\ s.t. \quad & A u_0 = b \\ & \theta \geq \alpha_k u_0 + \beta_k \qquad \forall k \in [\![1, K]\!] \end{aligned}$$

3. Solve S slave dual problems, with optimal dual solution λ^s_{K+1}

$$\begin{aligned} \max_{\lambda^s \in \mathbb{R}^m} \quad & \lambda^s \cdot \left(h^s - T^s u_0^{K+1}\right) \\ s.t. \quad & W^s \cdot \lambda^s \leq q^s \end{aligned}$$

4. construct new cut with

$$\alpha_{K+1} := -\sum^{S} p^s \, (T^s)^\top \lambda^s, \qquad \beta_{K+1} := \sum^{S} p^s \, h^s \cdot \lambda^s.$$

Feasibility cuts

- Without the relatively complete recourse assumption we cannot guarantee that $Q(u_0) < +\infty$, however we still have that Q is polyhedral, thus so is $dom(Q)$.
- Without RCR we need to add feasibility cuts in the following way:
  - If, $Q^s(u_0^k) = +\infty$, then we can find an unbounded ray of the dual problem

$$\max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot \left(h^s - T^s u_0^k\right)$$
$$s.t. \quad W^s \cdot \lambda^s \leq q^s$$

    more precisely a vector $\bar{\lambda}^k$ such that, for all $t \geq 0$ $W^s \cdot t\bar{\lambda}^k \leq q^s$.
  - Then, for u_0 to be admissible, we need that

$$\bar{\lambda}^k \cdot \left(h^s - T^s u_0\right) \leq 0$$

    which is a feasibility cut.

Convergence

Theorem

In the linear case, the L-Shaped algorithm terminates in finitely many steps, yielding the optimal solution.

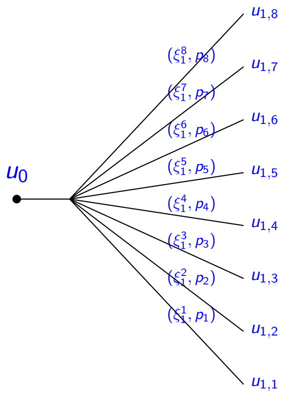
The proof is done by noting that only finitely many cuts can be added, and not being able to add a cut prove that the algorithm has converged.

Comparison of Progressive Hedging and L-shaped

	Progressive Hedging	L-Shaped
problems	convex continuous	linear, 1st stage integer
sol. at it. k	non-admissible splitted solutions	admissible primal solution
Bounds	LB free, UB easy	LB and UB free
Convergence	asymptotic	finite
Complexity	fixed : S deterministic problem	increasing for master problem, fixed for slave problem
Implem.	easy from deterministic solver	built from scratch

Presentation Outline

Where do we come from: two-stage programming



- We take decisions in two stages

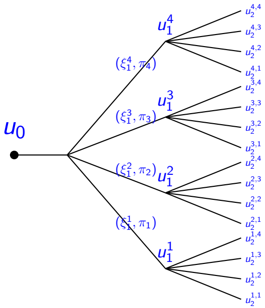
$$u_0 \rightsquigarrow \boldsymbol{\xi}_1 \rightsquigarrow \boldsymbol{u}_1 \ ,$$

with $\boldsymbol{u}_1$: recourse decision .

- On a tree, it means solving the extensive formulation:

$$\min_{u_0, u_{1,s}} c_0 u_0 + \sum_{s \in \mathbb{S}} p_s \Big[\big\langle c_s \ , u_{1,s} \big\rangle \Big] \ .$$

Extending two-stage to multistage programming



- We want to minimize $\min_{\boldsymbol{u}} \mathbb{E}\big[c(\boldsymbol{u}, \boldsymbol{\xi})\big]$
- Where we take decisions in T stages

$$\boldsymbol{u}_0 \rightsquigarrow \boldsymbol{\xi}_1 \rightsquigarrow \boldsymbol{u}_1 \rightsquigarrow \cdots \rightsquigarrow \boldsymbol{\xi}_T \rightsquigarrow \boldsymbol{u}_T \ .$$

- It can be represented on a tree $\mathcal{T}$, where a node n of depth t represent a realisation of $(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_t)$, and to which is attached a probability p_n.
- Then, the extensive formulation reads

$$\min_{\{u_n\}_{n\in\mathcal{T}}} \quad \sum_{n\in\mathcal{T}} p_n c_n(u_n)$$

Compact and splitted extended formulation

- Consider a tree of depth T. A scenario $s = (n_1, \dots, n_T)$ is a sequence of node, where each element is a descendent of the previous one. A scenario $s \in S$ is uniquely defined by its last element, which is a leaf of the tree.
- Let p_s be the probability of the leaf defining scenario s.
- The compact formulation of the multistage problem reads

$$\min_{\{u_n\}_{n\in\mathcal{T}}} \quad \sum_{n\in\mathcal{T}} p_n c_n(u_n) = \sum_{s\in S} p_s \sum_{n\in S} c_n(u_n)$$

- The splitted extended formulation reads

$$\begin{aligned} \min_{\{u_{s,t}\}_{s\in S, t\in[\![0,T]\!]}} \quad & \sum_{s\in S} p_s \sum_{t=0}^{T} c_{s,t}(u_{s,t}) \\ s.t. \quad & u_{s,t} = u_{s',t} \qquad \forall t, \forall n \in \mathcal{N}_t, \forall s, s' \ni n \end{aligned}$$

where $\mathcal{N}_t$ is the set of of nodes of depth t

Introducing the non-anticipativity constraint

We do not know what holds behind the door.

Non-anticipativity

At time t, decisions are taken sequentially, only knowing the past realizations of the perturbations.

Mathematically, this is equivalent to say that at time t, the decision $\boldsymbol{u}_t$ is

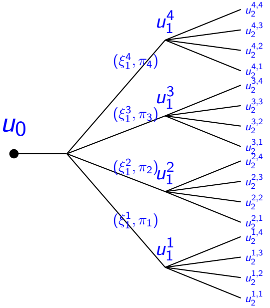
1. a function of past noises

$$\boldsymbol{u}_t = \pi_t(\boldsymbol{\xi}_0, \cdots, \boldsymbol{\xi}_t) ,$$

2. taken knowing the available information,

$$\sigma(\boldsymbol{u}_t) \subset \sigma(\boldsymbol{\xi}_0, \cdots, \boldsymbol{\xi}_t) .$$

Multistage extensive formulation approach



Assume that $\xi_t \in \mathbb{R}^{n_\xi}$ can take n_ξ values and that $U_t(x)$ can take n_u values.

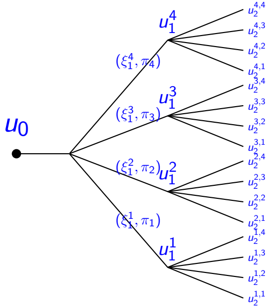
Then, considering the extensive formulation approach, we have

- n_ξ^T scenarios.
- $(n_\xi^{T+1} - 1)/(n_\xi - 1)$ nodes in the tree.
- Number of variables in the optimization problem is roughly $n_u \times (n_\xi^{T+1} - 1)/(n_\xi - 1) \approx n_u n_\xi^T$.

The complexity grows exponentially with the number of stage. :-(

A way to overcome this issue is to compress information!

Multistage extensive formulation approach



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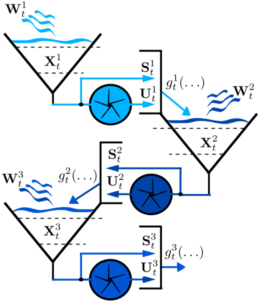
Illustrating extensive formulation with the damsvalley example



- 5 interconnected dams
- 5 controls per timesteps
- 52 timesteps (one per week, over one year)
- $n_\xi = 10$ noises for each timestep

We obtain 10^{52} scenarios, and $\approx 5.10^{52}$ constraints in the extensive formulation ... Estimated storage capacity of the Internet: 10^{24} bytes.

2-stage approach

The 2-stage approach consists in approximating the multistage program by a two-stage programm :

- relax all non-anticipativity constraints except the ones on u_0, this turn the tree into a scenario fan (same number of scenario),
- it means that all decision $(u_1, \dots, u_{T-1})$ are anticipative (not u_0).
- reduce the number of scenarios by sampling, and solve the SAA approximation of the 2-stage relaxation.

Denote $v^\sharp$ the value of the multistage problem, v^{2SA} the value of the 2-stage relaxation, and v_m^{2SA} the (random) value of the SAA of the 2-stage relaxation. Then we have

$$v^{2SA} \leq v^\sharp$$

$$v_m^{2SA} \to v^{2SA}$$

$$\mathbb{E}\left[v_m^{2SA}\right] \leq v^{2SA}$$