Stochastic Optimization and Decomposition

Ultimate goal of the lecture

How to to obtain "good" strategies (or cost-to-go functions) for a large scale stochastic optimal control problem in discrete time, for example a problem corresponding to the optimal management over a given time horizon of a system involving a large amount of dynamical production units.

- In order to obtain decision strategies (closed-loop controls), we have to use dynamic programming or related methods.
  - **Assumption**: Markovian case,
  - **Difficulty**: curse of dimensionality.
- To overcome the barrier of the dimension, we want to use decomposition/coordination techniques, so that we have to take into account the information pattern induced by the stochastic optimization problem.

Lecture Outline

1. Decomposition and Coordination
   - Bird's Eye View of Coupling in Stochastic Optimization
   - Decomposition Background
   - About the Stochastic Case
2. Dual Approximate Dynamic Programming (DADP)
   - Problem Statement and Subproblem Structure
   - DADP Principle and Implementation
   - DADP Interpretations and Questions
3. Theoretical Questions
   - Existence of a Saddle Point
   - Convergence of the Uzawa Algorithm
   - Convergence w.r.t. Information

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Stochastic Optimal Control (SOC) Problem Formulation

$$\min_{\boldsymbol{U},\boldsymbol{X}} \mathbb{E}\bigg(\sum_{i=1}^{N} \Big(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \Big) \bigg) ,$$

subject to dynamics constraints (**time coupling**):

$$\boldsymbol{X}_0^i = f_{-1}^i(\boldsymbol{W}_0) , \qquad i = 1 \dots N ,$$

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) , \qquad t = 0 \dots T-1 , \ i = 1 \dots N ,$$

to measurability constraints (**uncertainty coupling**):

$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t := \sigma(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t) , \quad t = 0 \dots T-1 , \ i = 1 \dots N ,$$

and to production constraints (**spatial coupling**):

$$\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0 , \qquad t = 0 \dots T-1 .$$

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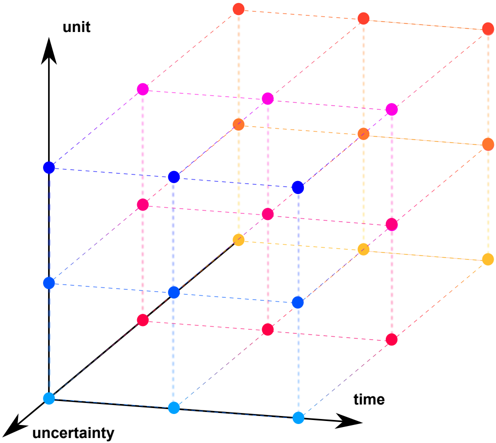
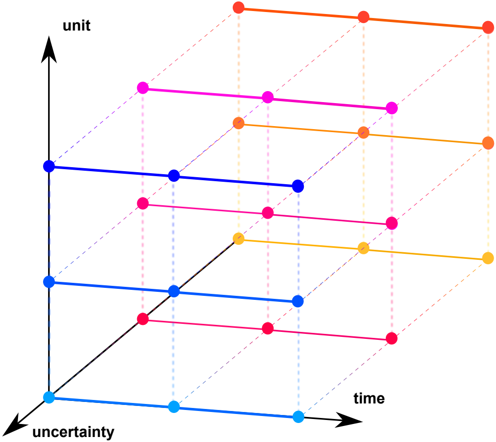
Couplings and Decompositions for SOC Problems (1)



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

Couplings and Decompositions for SOC Problems (2)



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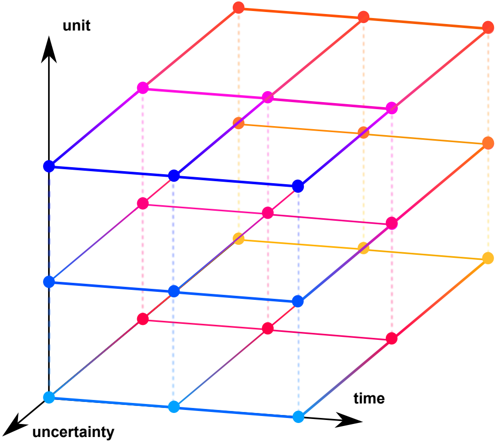
Couplings and Decompositions for SOC Problems (3)



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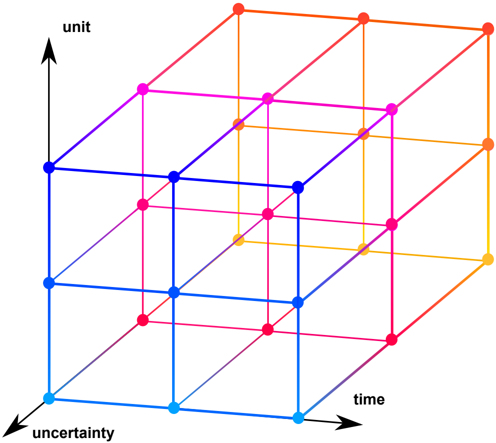

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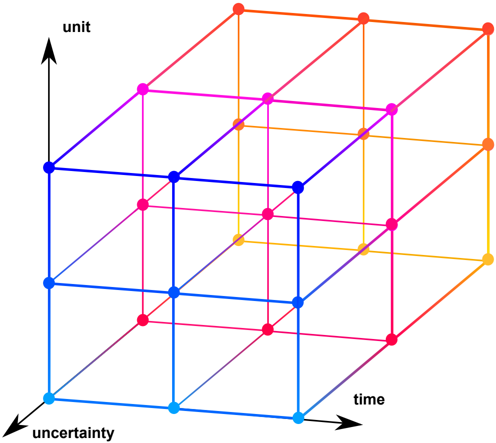
Couplings and Decompositions for SOC Problems (5)



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s.t. $$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

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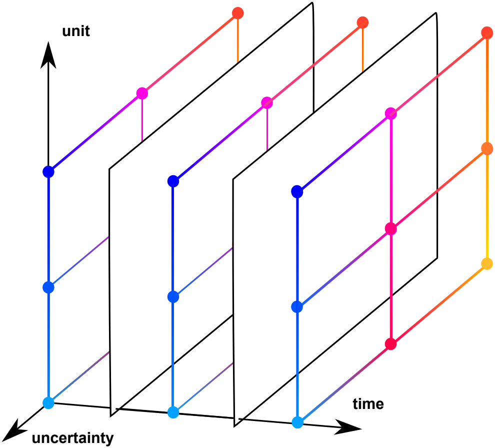
Three independent couplings!

Couplings and Decompositions for SOC Problems (6)



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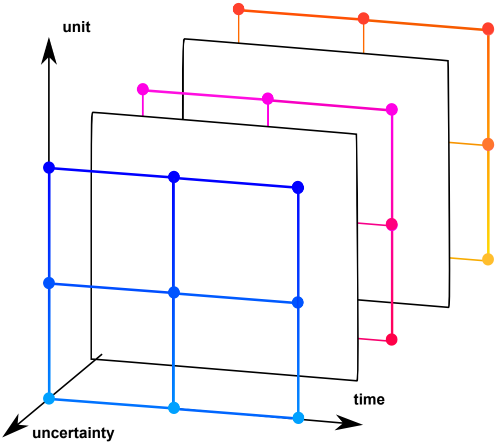
Time decomposition
Dynamic Programming

Couplings and Decompositions for SOC Problems (7)



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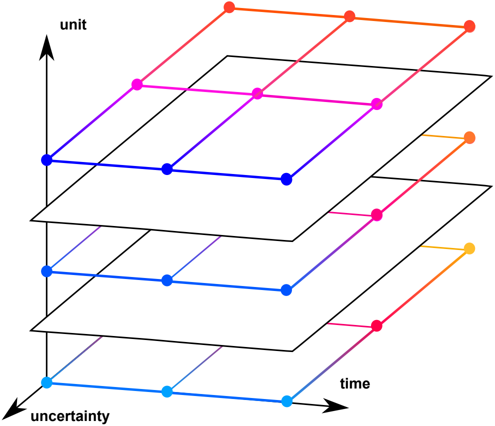
Scenario decomposition
Progressive Hedging

Couplings and Decompositions for SOC Problems (8)



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Spatial decomposition
Purpose of the lesson

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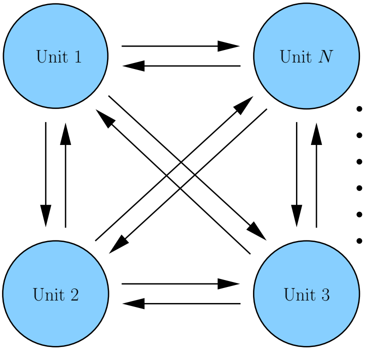
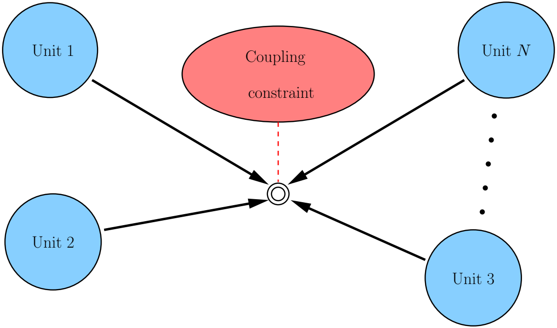
Decomposition and Coordination



Interconnected units

- The "large system" to be optimized consists of interconnected subsystems: we want to use this structure in order to formulate optimization subproblems of reasonable complexity.
- But the presence of interactions requires a level of coordination.
- Coordination must provide a local model of the interactions to each subproblem: it is an iterative process.
- The ultimate goal is to obtain the solution of the overall problem by concatenation of the solutions of the subproblems.

Example in the Energy Field: "Flower Model"

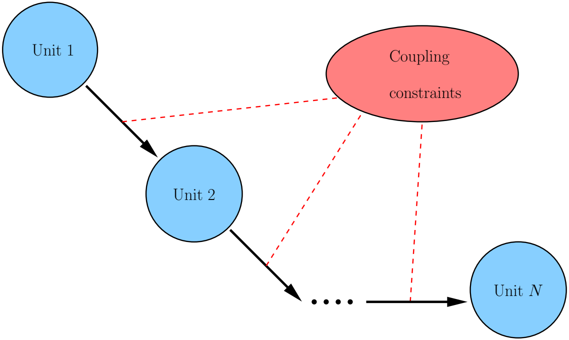


$$\min_{u} \sum_{i=1}^{N} J_i(u_i) \ ,$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \Theta_i(u_i) = \theta \ .$$

Unit Commitment Problem

Example in the Energy Field: "Cascade Model"



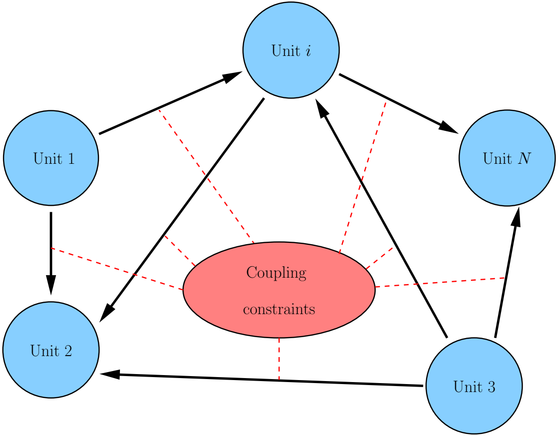
$$\min_{u,v} \sum_{i=1}^{N} J_i(u_i, v_i) \ ,$$

$$\text{s.t.} \quad H_i(u_i, v_i) = v_{i+1} \quad \forall i \ .$$

Dams Management Problem

Link with the flower model: $\Theta_i \rightsquigarrow \big(0, \ldots, -v_i, H_i(u_i, v_i), \ldots, 0\big)^\top$.

Example in the Energy Field: "Network Model"



$$\min_{u,v} \sum_{i=1}^{N} J_i\Big(u_i, \sum_{j \neq i} v_{j,i}\Big) ,$$

$$\text{s.t.} \quad H_i\Big(u_i, \sum_{j \neq i} v_{j,i}\Big) = v_i .$$

Smart Grid

Price Decomposition Applied to the Flower Model (1)

$$\min_{u\in\mathcal{U}} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \Theta_i(u_i) - \theta = 0 \ .$$

with $u = (u_1, \ldots, u_N)$.

1. Form the Lagrangian and assume that a saddle point exists:

$$\max_{\lambda\in\mathcal{V}} \min_{u\in\mathcal{U}} \sum_{i=1}^{N} \Big(J_i(u_i) + \langle \lambda \,, \Theta_i(u_i) \rangle \Big) - \langle \lambda \,, \theta \rangle \ .$$

2. Solve this problem by the Uzawa algorithm:

$$u_i^{(k+1)} \in \arg\min_{u_i\in\mathcal{U}_i} J_i(u_i) + \langle \lambda^{(k)} \,, \Theta_i(u_i) \rangle \ , \quad i = 1 \ldots, N \ ,$$

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho \Big(\sum_{i=1}^{N} \Theta_i\big(u_i^{(k+1)}\big) - \theta \Big) \ .$$

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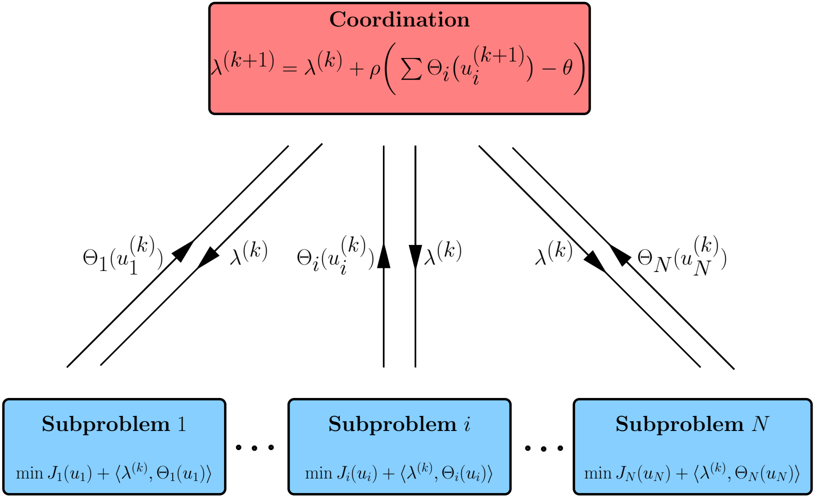
Price Decomposition Applied to the Flower Model (2)



Remarks on the Price Decomposition Method

- The theory is available for infinite dimensional Hilbert spaces, and thus applies in the stochastic framework, that is, the case where $\mathcal{U}$ is a space of random variables.

- The minimization algorithm used for solving the subproblems is not specified in the decomposition process.

- New variables appear in the subproblems arising at iteration k of the optimization process:

$$\min_{u_i \in \mathcal{U}_i} J_i(u_i) + \left\langle \lambda^{(k)} , \Theta_i(u_i) \right\rangle .$$

These variables are fixed when solving the subproblems, and do not cause any difficulty, at least in the deterministic case.

There are many others decomposition methods...

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Mixing Spatial Decomposition and Dynamic Programming

Consider the "large scale" stochastic optimal control problem

$$\min_{\boldsymbol{U},\boldsymbol{X}} \sum_{i=1}^{N} \mathbb{E}\left(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \right) ,$$

subject to the constraints:

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\end{aligned}$$

We assume that the r.v. $\boldsymbol{W}_t$ are independent (white noise).

Mixing Spatial Decomposition and Dynamic Programming

Consider the "large scale" stochastic optimal control problem

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\end{aligned}$$

We assume that the r.v. $\boldsymbol{W}_t$ are independent (**white noise**).

Dynamic Programming Yields Centralized Controls

Under the white noise assumption, it is possible to use dynamic programming (**DP**) in order to solve the SOC problem.

The true optimal control $\boldsymbol{U}_t^i$ of unit i is a feedback of the whole system state, that is, a function of all $\boldsymbol{X}_t^i$'s:

$$\boldsymbol{U}_t^i = \gamma_t^i\big(\boldsymbol{X}_t^1, \ldots, \boldsymbol{X}_t^N\big) \ .$$

Of course, a straightforward use of **DP** is prohibited for N large (**curse of dimensionality**), and decomposition is needed!

Decomposition may be difficult because the feedback γ_t^i induces a coupling between the units! Moreover, a naive decomposition of the problem should lead to decentralized feedbacks:

$$\boldsymbol{U}_t^i = \widehat{\gamma}_t^i\big(\boldsymbol{X}_t^i\big) \ ,$$

which, in most cases, are far from being optimal...

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Of course, a straightforward use of **DP** is prohibited for N large (**curse of dimensionality**), and decomposition is needed!

Decomposition may be difficult because the feedback γ_t^i induces a coupling between the units! Moreover, a naive decomposition of the problem should lead to decentralized feedbacks:

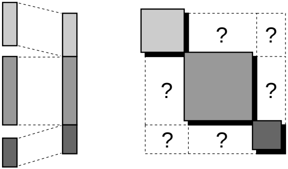
$$\boldsymbol{U}_t^i = \widehat{\gamma}_t^i(\boldsymbol{X}_t^i) \; ,$$

which, in most cases, are far from being optimal...

Straightforward Decomposition of Dynamic Programming?

The crucial point is that the optimal feedback of a subsystem a priori depends on the state of all other subsystems, so that using a decomposition scheme by subsystems is not at all obvious...

As far as we have to deal with Dynamic Programming, the central concern for decomposition/coordination purpose is resumed as:



- how to decompose a feedback γ_t w.r.t. its domain $\mathbb{X}_t$ rather than its range $\mathbb{U}_t$?

  And the answer is:
- impossible in the general case!

Price Decomposition in the Stochastic Case (1)

Dualize the spatial coupling constraints in the SOC problem:

$$\min_{\boldsymbol{U},\boldsymbol{X}} \sum_{i=1}^{N} \left(\mathbb{E}\Big(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \Big) \right) ,$$

subject to the constraints:

$$\begin{aligned}
\boldsymbol{X}_0^i &= f_{-1}^i(\boldsymbol{W}_0) , && i = 1 \dots N ,\\
\boldsymbol{X}_{t+1}^i &= f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) , \quad t = 0 \dots T-1 , && i = 1 \dots N ,\\
\boldsymbol{U}_t^i &\preceq \mathcal{F}_t := \sigma(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t) , \quad t = 0 \dots T-1 , && i = 1 \dots N ,\\
\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) &= 0 , \quad t = 0 \dots T-1 && \rightsquigarrow \quad \boldsymbol{\Lambda}_t .
\end{aligned}$$

Price Decomposition in the Stochastic Case (2)

Apply price decomposition to the SOC problem by dualizing the spatial coupling constraint. Then a dual multiplier $\boldsymbol{\Lambda}_t^{(k)}$ appears in each subproblem i at each iteration k:

$$\min_{\boldsymbol{U}^i, \boldsymbol{X}^i} \mathbb{E}\Big(\sum_{t=0}^{T-1} \big(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \boldsymbol{\Lambda}_t^{(k)} \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \big) + K^i(\boldsymbol{X}_T^i) \Big) .$$

The $\boldsymbol{\Lambda}_t^{(k)}$'s are fixed random variables at step k of the algorithm. Subproblem i thus encompasses 2 noise variables $\boldsymbol{W}_{t+1}$ and $\boldsymbol{\Lambda}_t^{(k)}$, but the $\boldsymbol{\Lambda}_t^{(k)}$'s may be correlated in time, in which case the white noise assumption fails!

Otherwise stated, the original state $\boldsymbol{X}_t^i$ is not a "good" state for subproblem i: the feature which seemed to have been won by decomposition is actually lost again by dynamic programming.

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Summary

- On the one hand, it seems that dynamic programming cannot be decomposed in a straightforward manner.
- On the other hand, applying a decomposition scheme to a SOC problem introduces coordination instruments in the subproblems, e.g. the multipliers $\boldsymbol{\Lambda}_t^{(k)}$ in the case of price decomposition. They correspond to additional fixed random variables whose time structure is unknown,[13] so that dynamic programming cannot be used in a naive way for solving the subproblems.

Question: how to handle these coordination instruments in order to be able to use dynamic programming and to obtain (at least an *approximation* of) the overall optimum of the SOC problem?

[13] One can only say that $\boldsymbol{\Lambda}_t^{(k)}$ is measurable with respect to $(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$.

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Optimization Problem

Recall the SOC problem under consideration:

$$\min_{\boldsymbol{U},\boldsymbol{X}} \mathbb{E}\bigg(\sum_{i=1}^{N} \Big(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \Big) \bigg) , \tag{9a}$$

subject to dynamics constraints:

$$\boldsymbol{X}_0^i = f_{-1}^i(\boldsymbol{W}_0) , \tag{9b}$$

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) , \tag{9c}$$

to measurability constraints:

$$\boldsymbol{U}_t^i \preceq \sigma(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t) , \tag{9d}$$

and to spatial coupling constraints

$$\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0 . \tag{9e}$$

Constraints to be **dualized**

Assumptions

Assumption (Markovian Setting)

Noises $\boldsymbol{W}_0, \ldots, \boldsymbol{W}_T$ are independent over time.

Hence Dynamic Programming applies: there is no optimality loss to seek the controls $\boldsymbol{U}_t^i$ as functions of the state at time t.

Assumption (Constraint Qualification Condition)

A saddle point of the Lagrangian $\mathcal{L}$ exists. *More on that later...*

$$\mathcal{L}(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{\Lambda}) = \mathbb{E}\left(\sum_{i=1}^{N}\left(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) + \sum_{t=0}^{T-1} \boldsymbol{\Lambda}_t \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i)\right)\right),$$

where $\boldsymbol{\Lambda}_t$ is a $\sigma(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$-measurable random variables.

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Uzawa algorithm applies. *More on that later...*

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Uzawa Algorithm

At iteration k of the algorithm,

1. Solve Subproblem i, $i = 1, \ldots, N$, with $\boldsymbol{\Lambda}^{(k)}$ fixed:

$$\min_{\boldsymbol{U}^i, \boldsymbol{X}^i} \mathbb{E}\left(\sum_{t=0}^{T-1} \left(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \boldsymbol{\Lambda}_t^{(k)} \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \right) + K^i(\boldsymbol{X}_T^i) \right) ,$$

   subject to

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) ,$$
$$\boldsymbol{U}_t^i \preceq \sigma(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t) ,$$

   whose solution is denoted $\left(\boldsymbol{U}^{i,(k+1)}, \boldsymbol{X}^{i,(k+1)}\right)$.

2. Update the multipliers $\boldsymbol{\Lambda}_t$:

$$\boldsymbol{\Lambda}_t^{(k+1)} = \boldsymbol{\Lambda}_t^{(k)} + \rho_t \left(\sum_{i=1}^{N} \Theta_t^i\left(\boldsymbol{X}_t^{i,(k+1)}, \boldsymbol{U}_t^{i,(k+1)}\right) \right) .$$

Structure of a Subproblem

$$\min_{\boldsymbol{U}^i, \boldsymbol{X}^i} \mathbb{E}\left(\sum_{t=0}^{T-1} \left(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \boldsymbol{\Lambda}_t^{(k)} \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \right) \right) ,$$

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Without additional knowledge of the process $\boldsymbol{\Lambda}^{(k)}$ (we just know that $\boldsymbol{\Lambda}_t^{(k)} \preceq (\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$), the state of this subproblem at time t cannot be summarized by the physical state $\boldsymbol{X}_t^i$. A possible state is the history $\boldsymbol{H}_t^i = (\boldsymbol{W}_0, \boldsymbol{U}_0^i, \ldots, \boldsymbol{U}_{t-1}^i, \boldsymbol{W}_t) \rightsquigarrow \boldsymbol{H}_{t+1}^i = (\boldsymbol{H}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$.

The state of the subproblem increases with time! Something has to be compressed in order to use Dynamic Programming.

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Main Idea of DADP

In order to overcome the difficulty induced by the multipliers $\boldsymbol{\Lambda}_t^{(k)}$, we choose at each time t a random variable $\boldsymbol{Y}_t$ measurable w.r.t. the past noises $(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$. The process $\boldsymbol{Y} = (\boldsymbol{Y}_0, \ldots, \boldsymbol{Y}_{T-1})$ is called the information process associated to the constraint.

The core idea is then to replace the multiplier $\boldsymbol{\Lambda}_t^{(k)}$ at iteration k by its conditional expectation w.r.t. $\boldsymbol{Y}_t$: $\boldsymbol{\Lambda}_t^{(k)} \rightsquigarrow \mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t)$.

This idea will lead to a good approximation if $\boldsymbol{Y}_t$ is (sufficiently) correlated to the random variable $\boldsymbol{\Lambda}_t$. It will also allow interesting interpretations. *More on that later...*

Note that we require that the information process is not influenced by controls.

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Subproblem Approximation

Using this idea, we replace Subproblem i in Uzawa algorithm by:

$$\min_{\boldsymbol{U}^i,\boldsymbol{X}^i} \mathbb{E}\bigg(\sum_{t=0}^{T-1} \Big(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \Big) + K^i(\boldsymbol{X}_T^i) \bigg) ,$$

subject to

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) ,$$
$$\boldsymbol{U}_t^i \preceq \sigma(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t) .$$

The conditional expectation $\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t)$ corresponds to a given function $\mu_t^{(k)}$ of the variable $\boldsymbol{Y}_t$, so that subproblem i now involves the white noise process $\boldsymbol{W}$ and the information process $\boldsymbol{Y}$. If the process $\boldsymbol{Y}$ follows a Markovian dynamics, e.g.

$$\boldsymbol{Y}_{t+1} = h_t(\boldsymbol{Y}_t, \boldsymbol{W}_{t+1}) ,$$

then $(\boldsymbol{X}_t^i, \boldsymbol{Y}_t)$ is a valid state for subproblem i and **DP** applies.

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Dynamic Programming Equation

Assuming a non-controlled dynamics $\boldsymbol{Y}_{t+1} = h_t(\boldsymbol{Y}_t, \boldsymbol{W}_{t+1})$ for the information process $\boldsymbol{Y}$, the **DP** equation for Subproblem i writes:

$$V_T^i(x^i, y) = K^i(x^i) ,$$

$$V_t^i(x^i, y) = \min_{u^i} \mathbb{E}\Big(L_t^i(x^i, u^i, \boldsymbol{W}_{t+1}) + \mathbb{E}\big(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t = y\big) \cdot \Theta_t^i(x^i, u^i) + V_{t+1}^i\big(\boldsymbol{X}_{t+1}^i, \boldsymbol{Y}_{t+1}\big)\Big) ,$$

subject to the dynamics:

$$\boldsymbol{X}_{t+1}^i = f_t^i(x^i, u^i, \boldsymbol{W}_{t+1}) ,$$
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About the Coordination (1)

The task of coordination is performed thanks to scenarios.

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- A set of noise scenarios is drawn once for all. Trajectories of the information process $\boldsymbol{Y}$ are simulated along the scenarios.
- At iteration k, the optimal trajectories of the state process $\boldsymbol{X}^{i,(k+1)}$ and of the control process $\boldsymbol{U}^{i,(k+1)}$ are simulated along the noise scenarios, for all subsystems.
- The dual multipliers are updated along the noise scenarios according to the formula:

$$\boldsymbol{\Lambda}_t^{(k+1)} = \boldsymbol{\Lambda}_t^{(k)} + \rho_t \bigg(\sum_{i=1}^{N} \Theta_t^i \big(\boldsymbol{X}_t^{i,(k+1)}, \boldsymbol{U}_t^{i,(k+1)} \big) \bigg) .$$

- The conditional expectations $\mathbb{E}\big(\boldsymbol{\Lambda}_t^{(k+1)} \mid \boldsymbol{Y}_t\big)$ are obtained by regression of the trajectories of $\boldsymbol{\Lambda}_t^{(k+1)}$ on those of $\boldsymbol{Y}_t$.

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One may perform the coordination by dealing with functions of $\boldsymbol{Y}_t$.

Many numerical advantages if the support of $\boldsymbol{Y}_t$ is finite.

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- Update the conditional expectation of the multipliers:

$$\mathbb{E}(\boldsymbol{\Lambda}_t^{(k+1)} \mid \boldsymbol{Y}_t) = \mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) + \rho_t \, \mathbb{E}\left(\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^{i,(k+1)}, \boldsymbol{U}_t^{i,(k+1)}) \;\middle|\; \boldsymbol{Y}_t \right) .$$

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Many numerical advantages if the support of $\boldsymbol{Y}_t$ is finite.

About the coordination (reduced gradients) (2)

One may perform the coordination by dealing with functions of $\boldsymbol{Y}_t$.

- Compute the optimal trajectories of the state process $\boldsymbol{X}^{i,(k+1)}$ and of the control process $\boldsymbol{U}^{i,(k+1)}$ along the noise scenarios.
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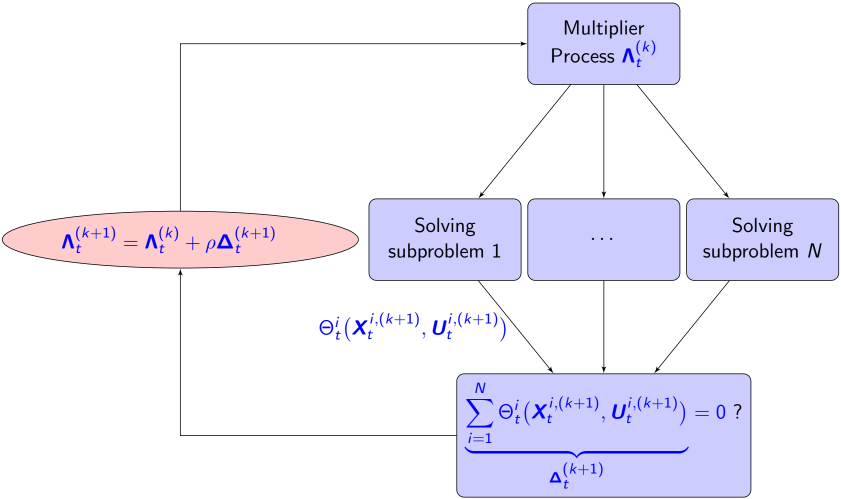
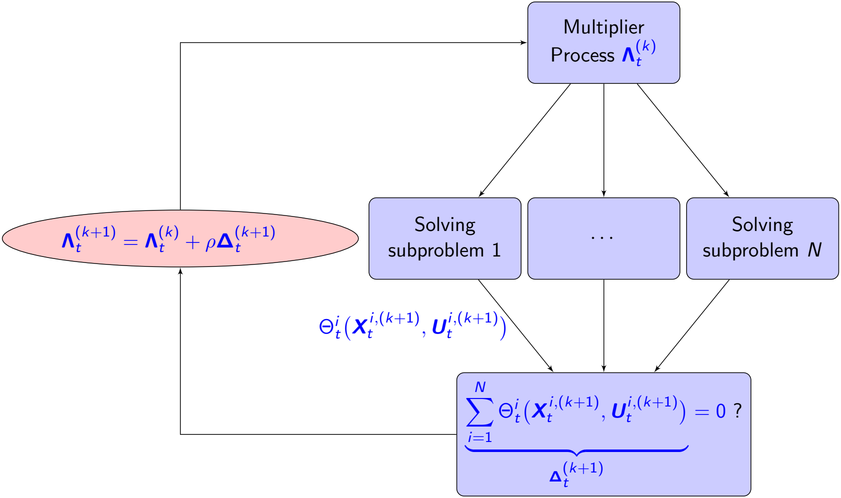
Stochastic spatial decomposition scheme



DADP core idea:

$$\boldsymbol{\Lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\big(\boldsymbol{\Lambda}_t \big| \boldsymbol{Y}_t\big)$$



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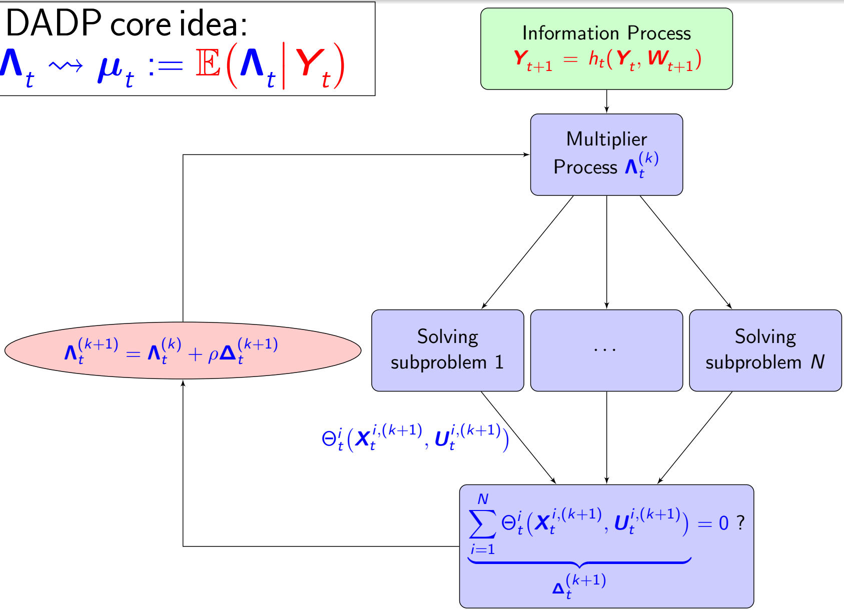

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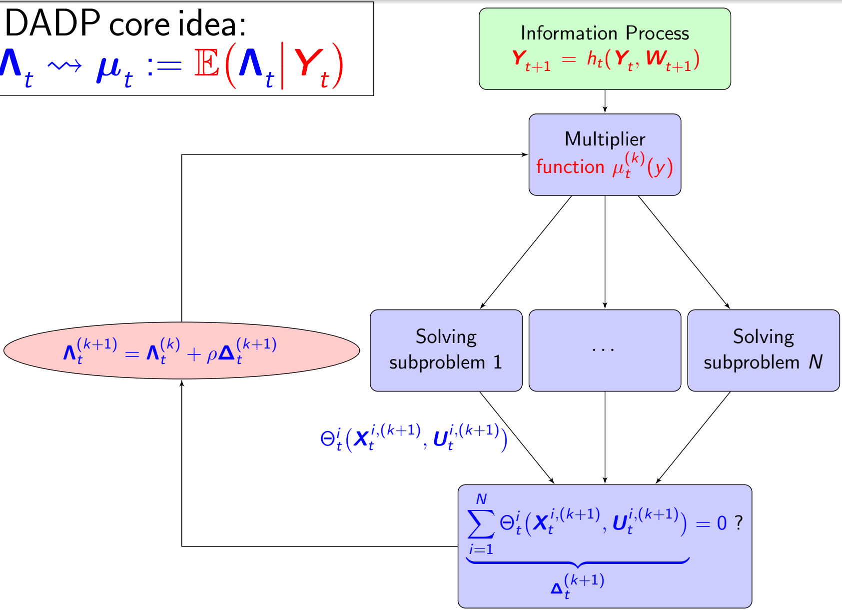

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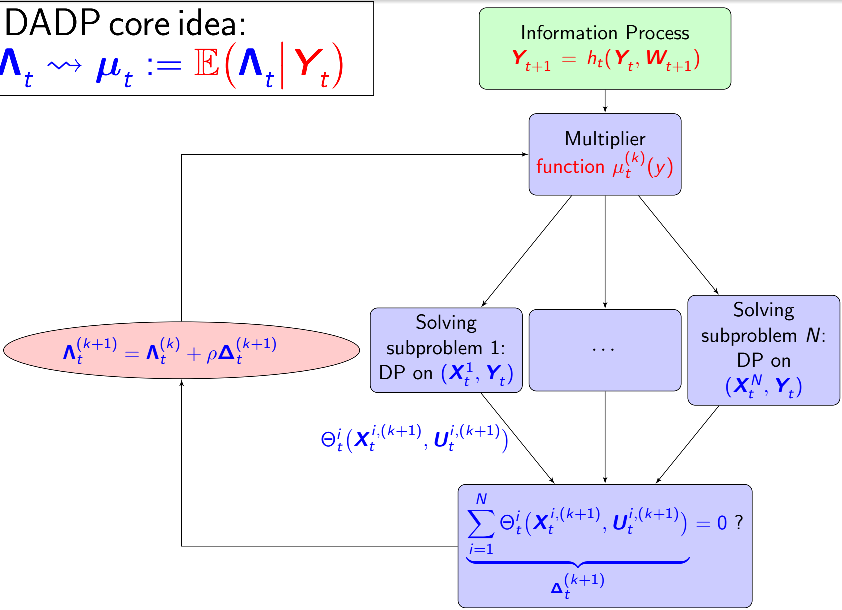

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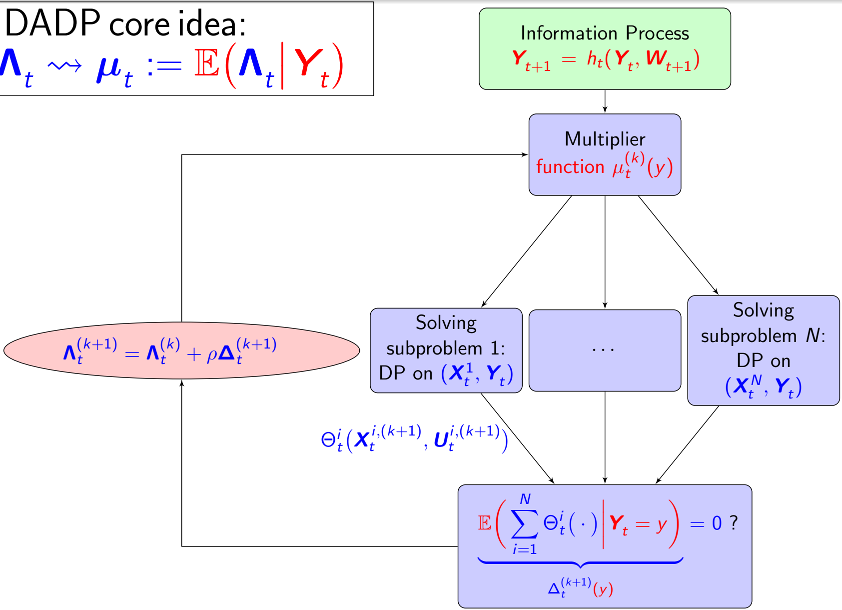

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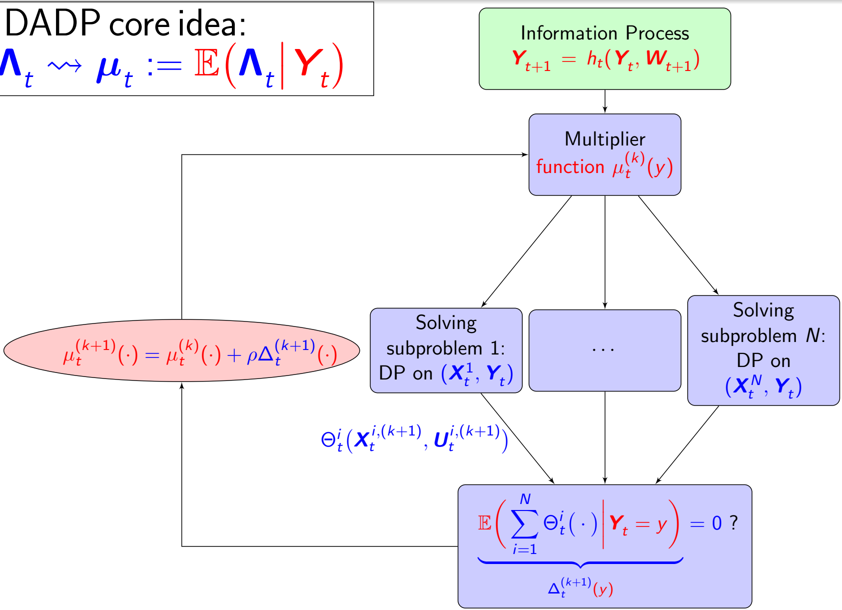

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Interpretations of DADP (1)

The approximation made on the dual process gives us a tractable way of computing strategies for the subsystems. Let us examine precisely the consequences in terms of constraints.

Consider a relaxed problem derived from (9):

$$\min_{\boldsymbol{U},\boldsymbol{X}} \mathbb{E}\bigg(\sum_{i=1}^{N} \Big(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \Big) \bigg) , \tag{10a}$$

subject to the modified coupling constraints:

$$\mathbb{E}\bigg(\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \; \Big| \; \boldsymbol{Y}_t \bigg) = 0 . \tag{10b}$$

Interpretations of DADP (2)

Proposition

The DADP algorithm can be interpreted as the Uzawa algorithm applied to Problem (10).

Sketch of proof. Since the duality term $\mathbb{E}\big(\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i)\big)$ which appears in the cost function of subproblem i in DADP can be written:

$$\mathbb{E}\big(\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i)\big) = \mathbb{E}\big(\boldsymbol{\Lambda}_t^{(k)} \cdot \mathbb{E}(\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \mid \boldsymbol{Y}_t)\big) ,$$

the global constraint really handled by DADP is:

$$\mathbb{E}\bigg(\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \;\bigg|\; \boldsymbol{Y}_t\bigg) = 0 . \qquad \square$$

DADP thus consists in replacing an almost-sure constraint by its conditional expectation w.r.t. the information variable $\boldsymbol{Y}_t$.

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Interpretations of DADP (3)

- DADP as an approximation of the optimal multiplier

$$\boldsymbol{\Lambda}_t \quad \rightsquigarrow \quad \mathbb{E}\big(\boldsymbol{\Lambda}_t \bigm| \boldsymbol{Y}_t\big) \; .$$

- DADP as a decision-rule approach for the dual problem

$$\max_{\boldsymbol{\Lambda}} \min_{\boldsymbol{U},\boldsymbol{X}} \mathcal{L}(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\Lambda}) \quad \rightsquigarrow \quad \max_{\boldsymbol{\Lambda}_t \preceq \boldsymbol{Y}_t} \min_{\boldsymbol{U},\boldsymbol{X}} \mathcal{L}(\boldsymbol{X},\boldsymbol{U},\lambda) \; .$$

- DADP as a constraint relaxation for the primal problem

$$\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i) = 0 \quad \rightsquigarrow \quad \mathbb{E}\Big(\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i) \Bigm| \boldsymbol{Y}_t\Big) = 0 \; .$$

Thanks to the last interpretation, the optimal value given by DADP is a guaranteed lower bound for the problem value.

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⋆ How to choose the information variables $\boldsymbol{Y}_t$?

- Perfect memory: $\boldsymbol{Y}_t = (\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$.
- Minimal information: $\boldsymbol{Y}_t \equiv \text{cste}$.
- Static information: $\boldsymbol{Y}_t = h_t(\boldsymbol{W}_t)$.
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The convergence of Uzawa's algorithm is granted provided that:

- the problem is posed in Hilbert spaces,
- and a saddle point exists.

It thus seems natural to place ourselves in a Hilbert space. But it is known (papers by Rockafellar and Wets) that **a saddle point doesn't exist** in Hilbert spaces for such problems...

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Epiconvergence results are available w.r.t. the information given by $\mathbf{Y}_t$. But epiconvergence raises technical problems when addressed to stochastic optimization problems.

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What Are the Issues to Consider?

- The spatial coupling constraints of our stochastic optimization problem are handled by duality methods.
- Uzawa algorithm is a dual method which is naturally described in an Hilbert space, but we cannot guarantee the existence of an optimal multiplier in the space $\mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$!
- Consequently, we extend the algorithm to the non-reflexive Banach space $\mathrm{L}^\infty(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$, by giving a set of conditions ensuring the existence of a $\mathrm{L}^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ optimal multiplier, and by providing a convergence result of the Uzawa algorithm.
- We also have to deal with the approximation induced by the information variable, that is, a convergence result when the information delivered by $\boldsymbol{Y}_t$ goes towards $\sigma(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$, (information available at time t for the initial problem).

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Abstract Formulation of the Problem

We consider the following abstract optimization problem:

$$(\mathcal{P}) \qquad \min_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}} J(\boldsymbol{U}) \quad \text{s.t.} \quad \Theta(\boldsymbol{U}) \in -C \; ,$$

where $\mathcal{U}$ and $\mathcal{V}$ are two Banach spaces, and

- $J : \mathcal{U} \to \overline{\mathbb{R}}$ is the objective function,
- $\mathcal{U}^{\mathrm{ad}}$ is the admissible set,
- $\Theta : \mathcal{U} \to \mathcal{V}$ is the constraint function, to be dualized,
- $C \subset \mathcal{V}$ is the cone of constraint.

Here, $\mathcal{U}$ is a space of random variables, and J is defined by

$$J(\boldsymbol{U}) = \mathbb{E}\big(j(\boldsymbol{U}, \boldsymbol{W})\big) \; .$$

The relationship with Problem (9) is almost straightforward...

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Standard Duality in L^2 Spaces (1)

Assume that $\mathcal{U} = \mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ and $\mathcal{V} = \mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m)$.

The standard sufficient constraint qualification condition

$$0 \in \mathrm{ri}\Big(\Theta\big(\mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)\big) + C\Big) ,$$

is scarcely satisfied in such a stochastic setting.

Proposition

If the σ-algebra $\mathcal{A}$ is not finite modulo $\mathbb{P}$,[a] then for any subset $U^{\mathrm{ad}} \subset \mathbb{R}^n$ that is not an affine subspace, the set

$$\mathcal{U}^{\mathrm{ad}} = \Big\{ \boldsymbol{U} \in \mathrm{L}^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n) \mid \boldsymbol{U} \in U^{\mathrm{ad}} \quad \mathbb{P}-a.s. \Big\} ,$$

has an empty relative interior in L^p, for any $p < +\infty$.

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Consider the following optimization problem (with $\alpha > 0$):

$$\inf_{u_0, \boldsymbol{U}_1} \quad u_0^2 + \mathbb{E}\big((\boldsymbol{U}_1 + \alpha)^2\big) ,$$

$$\text{s.t.} \quad u_0 \geq a ,$$

$$\boldsymbol{U}_1 \geq 0 ,$$

$$u_0 - \boldsymbol{U}_1 \geq \boldsymbol{W} , \qquad \text{to be dualized}$$

where $\boldsymbol{W}$ is a random variable uniform on $[1, 2]$.

For $a < 2$, we can construct a maximizing sequence of multipliers for the dual problem that does not converge in L^2. We are in the so-called non relatively complete recourse case, that is, the case where the constraints on $\boldsymbol{U}_1$ induce a stronger constraint on u_0.

The optimal multiplier is not in L^2, but in $(\mathrm{L}^\infty)^\star$...

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$$

where $\boldsymbol{W}$ is a random variable uniform on $[1, 2]$.

For $a < 2$, we can construct a maximizing sequence of multipliers for the dual problem that does not converge in L^2. We are in the so-called non relatively complete recourse case, that is, the case where the constraints on $\boldsymbol{U}_1$ induce a stronger constraint on u_0.

The optimal multiplier is not in L^2, but in $(\mathrm{L}^\infty)^\star$...

Constraint Qualification in $\left(\mathrm{L}^{\infty}, \mathrm{L}^{1}\right)$

From now on, we assume that

$$\begin{aligned} \mathcal{U} &= \mathrm{L}^{\infty}\left(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{n}\right) , \\ \mathcal{V} &= \mathrm{L}^{\infty}\left(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{m}\right) , \\ C &= \{0\} , \end{aligned}$$

where the σ-algebra $\mathcal{A}$ is not finite modulo $\mathbb{P}$.

We consider the pairing $\left(\mathrm{L}^{\infty}, \mathrm{L}^{1}\right)$ with the following topologies:

- $\sigma\left(\mathrm{L}^{\infty}, \mathrm{L}^{1}\right)$: weak* topology on L^{∞} (coarsest topology such that all the L^{1}-linear forms are continuous),
- $\tau\left(\mathrm{L}^{\infty}, \mathrm{L}^{1}\right)$: Mackey-topology on L^{∞} (finest topology such that the continuous linear forms are only the L^{1}-linear forms).

Weak* closedness of linear subspaces of L^∞

Proposition

Let $\Theta : \mathrm{L}^\infty\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n\big) \to \mathrm{L}^\infty\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m\big)$ be a linear operator, and assume that there exists a linear operator $\Theta^\dagger : \mathrm{L}^1\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m\big) \to \mathrm{L}^1\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n\big)$ such that:

$$\big\langle \boldsymbol{V} \,, \Theta(\boldsymbol{U}) \big\rangle = \big\langle \Theta^\dagger(\boldsymbol{V}) \,, \boldsymbol{U} \big\rangle \quad \forall \boldsymbol{U} \in \mathrm{L}^\infty(\mathbb{R}^n), \ \forall \boldsymbol{V} \in \mathrm{L}^1(\mathbb{R}^m) \ .$$

Then the linear operator Θ is weak* continuous.

Applications

- $\Theta(\boldsymbol{U}) = \boldsymbol{U} - \mathbb{E}\big(\boldsymbol{U} \bigm| \mathcal{B}\big)$: non-anticipativity constraints,
- $\Theta(\boldsymbol{U}) = A\boldsymbol{U}$ with $A \in \mathcal{M}_{m,n}(\mathbb{R})$: finite number of constraints.

A Duality Theorem

$$(\mathcal{P}) \quad \min_{\boldsymbol{U} \in \mathcal{U}} J(\boldsymbol{U}) \quad \text{s.t.} \quad \Theta(\boldsymbol{U}) = 0 \;, \quad \text{with} \quad J(\boldsymbol{U}) = \mathbb{E}\big(j(\boldsymbol{U}, \boldsymbol{W})\big) \;.$$

Theorem

Assume that j is a convex normal integrand, that J is continuous in the Mackey topology at some point $\boldsymbol{U}_0$ such that $\Theta(\boldsymbol{U}_0) = 0$, and that Θ is linear weak* continuous on $\mathrm{L}^\infty\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n\big)$. Then, $\boldsymbol{U}^\sharp \in \mathcal{U}$ is an optimal solution of Problem $\big(\mathcal{P}\big)$ if and only if there exists $\boldsymbol{\Lambda}^\sharp \in \mathrm{L}^1\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m\big)$ such that

- $\boldsymbol{U}^\sharp \in \arg\min_{\boldsymbol{U} \in \mathcal{U}} \mathbb{E}\Big(j(\boldsymbol{U}, \boldsymbol{W}) + \boldsymbol{\Lambda}^\sharp \cdot \Theta(\boldsymbol{U})\Big)$,
- $\Theta(\boldsymbol{U}^\sharp) = 0$.

Extension to $\mathbb{P}$-a.s. constraints: adding almost sure bound constraints causes Mackey discontinuity (see the previous example in L^2 spaces)!

1. Decomposition and Coordination
   - Bird's Eye View of Coupling in Stochastic Optimization
   - Decomposition Background
   - About the Stochastic Case
2. Dual Approximate Dynamic Programming (DADP)
   - Problem Statement and Subproblem Structure
   - DADP Principle and Implementation
   - DADP Interpretations and Questions
3. Theoretical Questions
   - Existence of a Saddle Point
   - Convergence of the Uzawa Algorithm
   - Convergence w.r.t. Information

Uzawa Algorithm

$$(\mathcal{P}) \quad \min_{\boldsymbol{U} \in \mathcal{U}} J(\boldsymbol{U}) \quad \text{s.t.} \quad \Theta(\boldsymbol{U}) = 0 \ , \ \text{ with } \ J(\boldsymbol{U}) = \mathbb{E}\big(j(\boldsymbol{U}, \boldsymbol{W})\big) \ .$$

The standard Uzawa algorithm

$$\boldsymbol{U}^{(k+1)} \in \operatorname*{arg\,min}_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}} J(\boldsymbol{U}) + \big\langle \boldsymbol{\Lambda}^{(k)} , \Theta(\boldsymbol{U}) \big\rangle \ ,$$

$$\boldsymbol{\Lambda}^{(k+1)} = \boldsymbol{\Lambda}^{(k)} + \rho \, \Theta\big(\boldsymbol{U}^{(k+1)}\big) \ ,$$

makes sense with in the L^∞ setting, that is, the minimization problem is well-posed and the update formula of $\boldsymbol{\Lambda}$ is valid.

Note that all the multipliers $\boldsymbol{\Lambda}^{(k)}$ belong to $\mathrm{L}^\infty\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m\big)$ as soon as the initial multiplier $\boldsymbol{\Lambda}^{(0)} \in \mathrm{L}^\infty\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m\big)$.

Convergence Result

Theorem

Assume that

1. $J : \mathcal{U} \to \overline{\mathbb{R}}$ *is proper, weak* l.s.c., differentiable and a-convex,*
2. $\Theta : \mathcal{U} \to \mathcal{V}$ *is affine, weak* continuous and κ-Lipschitz,*
3. $\mathcal{U}^{\mathrm{ad}}$ *is weak* closed and convex,*
4. *an admissible* $\boldsymbol{U}_0 \in \mathrm{dom}\, J \cap \Theta^{-1}(0) \cap \mathcal{U}^{\mathrm{ad}}$ *exists,*
5. *an optimal* L^1*-multiplier to the constraint* $\Theta\big(\boldsymbol{U}\big) = 0$ *exists,*
6. *the step* ρ *is such that* $0 < \rho < \frac{2a}{\kappa}$.

Then, there exists a subsequence $\big\{\boldsymbol{U}^{(n_k)}\big\}_{k \in \mathbb{N}}$ *of the sequence generated by the Uzawa algorithm converging in* L^∞ *towards the optimal solution* $\boldsymbol{U}^\sharp$ *of the primal problem.*

Remarks about the Result

- The result is not as good as expected (global convergence?).
- Improvements and extensions (inequality constraint) needed.
- The Mackey-continuity assumption forbids the use of bounds.
  - In order to deal with almost sure bound constraints, we can turn towards the work of R.T. Rockafellar and R. J-B Wets.
  - In a series of 4 papers (stochastic convex programming), they have detailed the duality theory on two-stage and multistage problems, with the focus on non-anticipativity constraints.
  - These papers require:
    - a strict feasability assumption,
    - a relatively complete recourse assumption.

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Relaxed Problems

Following the interpretation of DADP in terms of a relaxation of the original problem, and given a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of subfields of the σ-field $\mathcal{A}$, we replace the abstract problem:

$$(\mathcal{P}) \qquad \min_{\boldsymbol{U}\in\mathcal{U}} J(\boldsymbol{U}) \quad \text{s.t.} \quad \Theta(\boldsymbol{U}) = 0 \;,$$

by the sequence of approximated problems:

$$(\mathcal{P}_n) \qquad \min_{\boldsymbol{U}\in\mathcal{U}} J(\boldsymbol{U}) \quad \text{s.t.} \quad \mathbb{E}\Big(\Theta(\boldsymbol{U}) \;\Big|\; \mathcal{A}_n\Big) = 0 \;.$$

We assume the strong convergence of $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ towards $\mathcal{A}$:

$$\mathcal{A}_n \longrightarrow \mathcal{A} \qquad \Big(\iff \forall \boldsymbol{X} \in \mathrm{L}^1(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}),\; \mathbb{E}(\boldsymbol{X}|\mathcal{A}_n) \xrightarrow{\mathrm{L}^1} \mathbb{E}(\boldsymbol{X}|\mathcal{A}) \Big) \;.$$

Convergence Result

Theorem

Assume that

- $\mathcal{U}$ *is a topological space,*
- $\mathcal{V} = \mathrm{L}^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m)$, *with* $p \in [1, +\infty)$,
- J *and* Θ *are continuous operators,*
- $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ *strongly converges towards* $\mathcal{A}$.

Then the sequence $\{\widetilde{J}_n\}_{n\in\mathbb{N}}$ *epi-converges towards* $\widetilde{J}$, *with*

$$\widetilde{J}_n = \begin{cases} J(\boldsymbol{U}) & \text{if } \boldsymbol{U} \text{ satisfies the constraints of } (\mathcal{P}_n), \\ +\infty & \text{otherwise.} \end{cases}$$

Conclusion

- DADP method allows to tackle large-scale stochastic optimal control problems, such as the ones found in the field of energy management.
- A lot of practical experiments have been performed,
  - on *flower models* (unit commitment problem),
  - on *chained models* (hydraulic valley management),
  - on *network models* (smart grid).

  Much work remains to be done in this area.
- There is an ongoing research project on the subject, in order to assess the foundations of the method.

References on stochastic optimal control and decomposition

K. Barty, P. Carpentier, G. Cohen and P. Girardeau,
Price decomposition in large-scale stochastic optimal control.
arXiv, math.OC 1012.2092, 2010.

P. Carpentier, J.-P. Chancelier, M. De Lara and F. Pacaud,
Mixed Spatial and Temporal Decompositions for Large-Scale Multistage Stochastic Optimization Problems.
Journal of Optimization Theory and Applications, 186, 985-1005, 2020.

V. Leclère.
Contributions aux Methodes de Décomposition en Optimisation Stochastique.
Thèse de doctorat, Université Paris Est, 2014.

F. Pacaud.
Decentralized Optimization Methods for Efficient Energy Management under Stochasticity.
Thèse de doctorat, Université Paris Est, 2018.

R. T. Rockafellar and R. J-B. Wets.
Stochastic Convex Programming: Relatively Complete Recourse and Induced Feasability.
SIAM Journal on Control and Optimization, 14-3, 574-589, 1976.

R. J-B. Wets.
On the relation between stochastic and deterministic optimization.
Lecture Notes in Economics and Mathematical Systems, 107, 350-361, 1975.