Applications of the Stochastic Gradient Method

Lecture Outline

A Basic Two-Stage Recourse Problem

We consider the management of a water reservoir. Water is drawn from the reservoir by way of random consumers. In order to ensure the water supply, 2 decisions are taken at 2 successive time steps.

- A first supply decision q_1 is taken without any knowledge of the effective consumption, the associated cost being equal to $c_1(q_1)^2$, with $c_1 > 0$.
- Once the consumption d (realization of a random variable $\boldsymbol{D}$) has been observed, a second supply decision q_2 is taken in order to maintain the reservoir at its initial level, that is,

$$q_2 = d - q_1 .$$

The associated cost is equal to $c_2(q_2)^2$, with $c_2 > 0$.

The problem is to minimize the expected overall cost of operation.

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Mathematical Formulation and Solution

Problem Formulation

- q_1 is a deterministic decision variable,
- whereas q_2 is the realization of a random variable $\boldsymbol{Q}_2$.

$$\min_{(q_1,\boldsymbol{Q}_2)} c_1\big(q_1\big)^2 + \mathbb{E}\Big(c_2\big(\boldsymbol{Q}_2\big)^2\Big) \quad \text{s.t.} \quad q_1 + \boldsymbol{Q}_2 = \boldsymbol{D} \ \ \mathbb{P}\text{-a.s. .}$$

Equivalent Problem

$$\min_{q_1 \in \mathbb{R}} \mathbb{E}\Big(c_1\big(q_1\big)^2 + c_2\big(\boldsymbol{D} - q_1\big)^2\Big)$$

Analytical solution: $q_1^\sharp = \dfrac{c_2}{c_1 + c_2}\mathbb{E}\big(\boldsymbol{D}\big)$.

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Stochastic Gradient Algorithm

$$\boldsymbol{Q}_1^{(k+1)} = \boldsymbol{Q}_1^{(k)} - \frac{\alpha}{k+\beta}\Big(2(c_1+c_2)\boldsymbol{Q}_1^{(k)} - 2c_2\boldsymbol{D}^{(k+1)}\Big) .$$

Algorithm (initialization)

```
//
// Random generator
//
   rand('normal'); rand('seed',123);
//
// Random consumption
//
   m = 10.; sd = 5.;
//
// Criterion
//
   c1 = 3.; c2 = 1.;
//
// Initialization
//
   x = [ ]; y = [ ];
```

Algorithm (iterations)

```
//
// Algorithm
//
   q1k = 10.;
   for k = 1:100
      dk = m + (sd*rand(1));
      gk = 2*((c1+c2)*q1k) - 2*(c2*dk);
      epsk = 1/(k+10);
      q1k = q1k - (epsk*gk);
      x = [x ; k]; y = [y ; q1k];
   end
//
// Trajectory plot
//
   plot2d(x,y);
   xtitle('Stochastic Gradient ','Iter.','q1');
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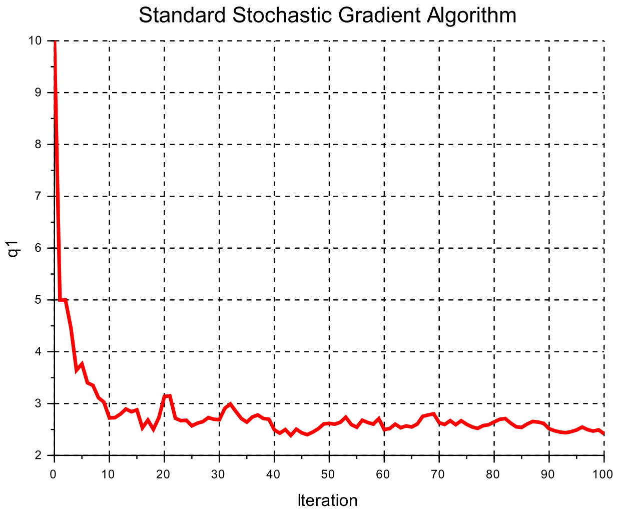
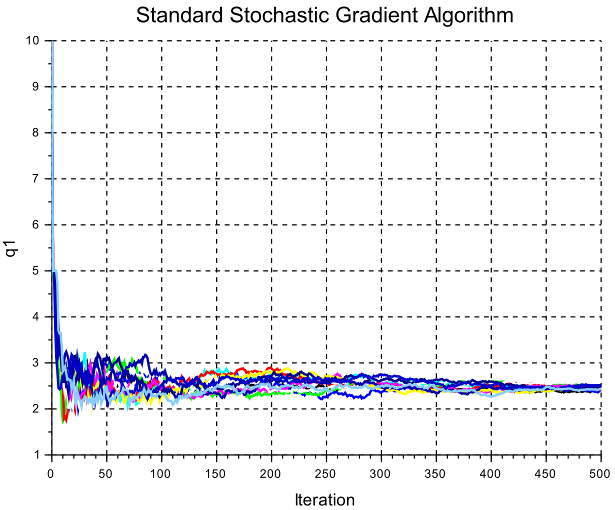
A Realization of the Algorithm



More Realizations



Slight Modification of the Problem

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The difference between supply and demand is penalized thanks to an additional cost term $c_3\left(q_1 + q_2 - d\right)^2$. The new problem is :

$$\min_{(q_1,\boldsymbol{Q}_2)} \mathbb{E}\left(c_1\left(q_1\right)^2 + c_2\left(\boldsymbol{Q}_2\right)^2 + c_3\left(q_1 + \boldsymbol{Q}_2 - \boldsymbol{D}\right)^2\right) .$$

Question: how to solve it using a stochastic gradient algorithm?

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Resolution of the Modified Problem

Idea: use the interchange theorem to solve the problem w.r.t. $\boldsymbol{Q}_2$.

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The optimal solution of the minimization problem w.r.t. q_2 is

$$\boldsymbol{Q}_2^\sharp = \frac{c_3}{c_2 + c_3}\big(\boldsymbol{D} - q_1\big)$$

so that the problem is equivalent to the open-loop problem

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Trade-off Investment/Operation – Problem Statement

A company owns N production units and has to meet a given non stochastic demand $d \in \mathbb{R}$.

- For each unit i, the decision maker first takes an investment decision $u_i \in \mathbb{R}$, the associated cost being $\mathcal{I}_i(u_i)$.
- Then a discrete disturbance $w_i \in \{w_{i,a}, w_{i,b}, w_{i,c}\}$ occurs.
- Knowing all noises, the decision maker selects for each unit i an operating point $v_i \in \mathbb{R}$, which leads to a cost $\mathcal{C}_i(u_i, v_i, w_i)$ and a production $\mathcal{P}_i(v_i, w_i)$.

The goal is to minimize the expected overall cost, subject to the following constraints:

- investment limitation: $\Theta(u_1, \dots, u_N) \leq 0$,
- operation limitation: $v_i \leq \varphi_i(u_i)$, $i = 1, \dots, N$,
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Trade-off Investment/Operation – Questions

1. Write down the global optimization problem.
   - Is it possible to solve directly the problem with N large?
   - Is it possible to apply the stochastic gradient algorithm?
2. Extract the optimization subproblem obtained when both the investment $u = (u_1, \ldots, u_N)$ and the noise $w = (w_1, \ldots, w_N)$ are fixed. The value of this subproblem is denoted $f^\sharp(u, w)$.
   - Give assumptions for the resolution of this subproblem.
   - Give assumptions for $f^\sharp$ to be a smooth convex function.
   - Compute the partial derivatives of $f^\sharp$ w.r.t. u.
3. Reformulate the optimization problem using function $f^\sharp$ and apply the stochastic gradient algorithm in the following cases:
   - the investment limitation is decoupled: $\forall i,\ u_i \in [\underline{u}_i, \overline{u}_i]$,
   - the investment limitation is linear: $u_1 + \ldots + u_N \leq \overline{u}$,
   - the investment limitation is convex: $\Theta(u_1, \ldots, u_N) \leq 0$.
4. What if decision v_i is based on the knowledge of w_i only?

Trade-off Investment/Operation — Answer to Q1

We denote by $\boldsymbol{W} = (\boldsymbol{W}_1, \ldots, \boldsymbol{W}_N)$ the global noise variable.

$$\min_{(u_i \in \mathbb{R}, \boldsymbol{V}_i \preceq \boldsymbol{W})} \mathbb{E}\bigg(\sum_{i=1}^{N} \Big(\mathcal{I}_i(u_i) + \mathcal{C}_i\big(u_i, \boldsymbol{V}_i, \boldsymbol{W}_i\big) \Big) \bigg) ,$$

$$\text{s.t.} \quad \Theta(u_1, \ldots, u_N) \leq 0 ,$$

$$\sum_{i=1}^{N} \mathcal{P}\big(\boldsymbol{V}_i, \boldsymbol{W}_i\big) - d = 0 \quad \mathbb{P}\text{-a.s.} ,$$

$$\boldsymbol{V}_i - \varphi_i(u_i) \leq 0 \quad \mathbb{P}\text{-a.s.} , \quad i = 1, \ldots, N .$$

- For $N = 21$, the sizes of the problem are huge:
  - $\text{card}(\mathbb{W}) = 3^{21} \approx 10^{10}$ possible noise values,
  - $N + N \times \text{card}(\mathbb{W})$ decision variables,
  - $1 + \text{card}(\mathbb{W}) + N \times \text{card}(\mathbb{W})$ constraints.
- The SG algorithm does not apply: decisions $\boldsymbol{V}_i$ are random variables.

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Thanks to the interchange theorem, the minimization w.r.t. $\boldsymbol{V}_i$ can be formulated independently for each realization of $\boldsymbol{W}$. For a realization w of $\boldsymbol{W}$, the inner minimization subproblem w.r.t. v is

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\begin{aligned}
f^{\sharp}(u,w) = \min_{(v_1,\dots,v_N)\in\mathbb{R}^N} \quad & \sum_{i=1}^{N} \mathcal{C}_i\big(u_i, v_i, w_i\big) \ , \\
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Let λ and $(\mu_1,\dots,\mu_N)$ be the associated multipliers. Assuming that

- the functions $\mathcal{C}_i$ are convex continuous coercive w.r.t. v_i,
- the functions $\mathcal{P}_i$ are linear w.r.t. v_i,

the above problem admits a non empty set of saddle points.

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Trade-off Investment/Operation — Answer to Q2 (end)

If we moreover assume that

- the functions $\mathcal{C}_i$ are jointly convex w.r.t. (u_i, v_i),
- the functions $\mathcal{C}_i$ are differentiable w.r.t. v_i,
- the functions φ_i are concave differentiable,

then the function $f^\sharp$ is convex subdifferentiable w.r.t. u and

$$\nabla_{u_i}\mathcal{C}_i(u_i, v_i^\sharp, w_i) - \mu_i^\sharp \nabla\varphi_i(u_i) \in \partial_{u_i} f^\sharp(u, w) \ .$$

Finally, if we assume that

- the subproblem admits an unique saddle point $(v^\sharp, \lambda^\sharp, \mu^\sharp)$,

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Trade-off Investment/Operation — Answer to Q3

Using the function $f^\sharp$ obtained when minimizing w.r.t. the variables v_i, the global optimization problem is reformulated as

$$\min_{(u_1,\ldots,u_N)\in\mathbb{R}^N} \sum_{i=1}^{N} \mathcal{I}_i(u_i) + \mathbb{E}\Big(f^\sharp(u_1,\ldots,u_N,\boldsymbol{W})\Big) \ ,$$
$$\text{s.t.} \quad \Theta(u_1,\ldots,u_N) \leq 0 \ .$$

We assume that

- the function $f^\sharp$ is convex differentiable,
- the functions $\mathcal{I}_i$ are convex coercive differentiable,
- the function Θ is convex differentiable,

and we denote the gradient w.r.t. u_i of the cost under the expectation by

$$\nabla_{u_i} j(u,w) = \nabla \mathcal{I}_i(u_i) + \nabla_{u_i} C_i(u_i, v_i^\sharp, w_i) - \mu_i^\sharp \nabla \varphi_i(u_i) \ .$$

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Trade-off Investment/Operation — Answer to Q3 (end)

The stochastic gradient method applies to the reformulated problem.

- *Decoupled investment limitation.*
  1. Draw a realization $w^{(k+1)}$ of $\boldsymbol{W}$.
  2. Solve the inner minimization subproblem at $(u^{(k)}, w^{(k+1)})$ and denote by $v^{(k+1)}$ and $\mu^{(k+1)}$ its solution.
  3. Update u using the standard stochastic gradient formula

$$u_i^{(k+1)} = \mathrm{proj}_{[\underline{u}_i, \overline{u}_i]} \Big(u_i^{(k)} - \epsilon^{(k)} \nabla_{u_i} j(u^{(k)}, w^{(k+1)}) \Big) \ .$$

- *Linear investment limitation.*
  3. Compute $u_i^{(k+\frac{1}{2})} = u_i^{(k)} - \epsilon^{(k)} \nabla_{u_i} j(u^{(k)}, w^{(k+1)})$ for all i and project the point $u^{(k+\frac{1}{2})}$ on the half-space $u_1 + \ldots + u_N \leq \overline{u}$.
- *Convex investment limitation.*
  Apply the stochastic Arrow-Hurwicz algorithm (with multiplier p).
  3. $u_i^{(k+1)} = u_i^{(k)} - \epsilon^{(k)} \big(\nabla_{u_i} j(u^{(k)}, w^{(k+1)}) + (\Theta'_{u_i}(u^{(k)}))^\top \cdot p^{(k)} \big)$.
  4. $p^{(k+1)} = \max \big(0, p^{(k)} + \epsilon^{(k)} \Theta(u^{(k+1)}) \big)$.

Trade-off Investment/Operation — Answer to Q3 (end)

The stochastic gradient method applies to the reformulated problem.

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Trade-off Investment/Operation — Answer to Q4

We assume that the random variables $(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_N)$ are independent.

From the independence assumption, and since $\boldsymbol{V}_i \preceq \boldsymbol{W}_i$, we have

$$\sum_{i=1}^{N} \mathcal{P}(\boldsymbol{V}_i, \boldsymbol{W}_i) = d \iff \exists\, (d_1, \ldots, d_N) \text{ s.t. } \mathcal{P}(\boldsymbol{V}_i, \boldsymbol{W}_i) = d_i \;, \; \sum_{i=1}^{N} d_i = d \;.$$

The inner minimization subproblem w.r.t. v can be decomposed i by i:

$$g_i^\sharp(u_i, d_i, w_i) = \min_{v_i \in \mathbb{R}} \; C_i(u_i, v_i, w_i) \text{ s.t. } \mathcal{P}(v_i, w_i) - d_i = 0 \;, \; v_i - \varphi_i(u_i) \leq 0 \;.$$

The global optimization problem is then reformulated as

$$\min_{(u_1,\ldots,u_N)\in\mathbb{R}^N,(d_1,\ldots,d_N)\in\mathbb{R}^N} \sum_{i=1}^{N} \Big(\mathcal{I}_i(u_i) + \mathbb{E}\big(g_i^\sharp(u_i, , d_i, \boldsymbol{W}_i)\big) \Big) \;,$$

$$\text{s.t.} \quad \Theta(u_1, \ldots, u_N) \leq 0 \;,$$

$$\sum_{i=1}^{N} d_i - d = 0 \;,$$

and thus can be solved by the stochastic Arrow-Hurwicz algorithm.

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The inner minimization subproblem w.r.t. v can be decomposed i by i:

$$g_i^\sharp(u_i, d_i, w_i) = \min_{v_i \in \mathbb{R}} \; \mathcal{C}_i\big(u_i, v_i, w_i\big) \text{ s.t. } \mathcal{P}\big(v_i, w_i\big) - d_i = 0 \;, \; v_i - \varphi_i(u_i) \leq 0 \;.$$

The global optimization problem is then reformulated as

$$\min_{(u_1,\ldots,u_N)\in\mathbb{R}^N,(d_1,\ldots,d_N)\in\mathbb{R}^N} \sum_{i=1}^{N} \Big(\mathcal{I}_i(u_i) + \mathbb{E}\big(g_i^\sharp(u_i, d_i, \boldsymbol{W}_i)\big) \Big) \;,$$

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The global optimization problem is then reformulated as

$$\begin{aligned} \min_{(u_1,\ldots,u_N)\in\mathbb{R}^N, (d_1,\ldots,d_N)\in\mathbb{R}^N} \; & \sum_{i=1}^{N} \Big(\mathcal{I}_i(u_i) + \mathbb{E}\big(g_i^\sharp(u_i,, d_i, \boldsymbol{W}_i)\big)\Big) \;, \\ \text{s.t.} \quad & \Theta(u_1, \ldots, u_N) \leq 0 \;, \\ & \sum_{i=1}^{N} d_i - d = 0 \;, \end{aligned}$$

and thus can be solved by the stochastic Arrow-Hurwicz algorithm.

Option Pricing Problem — Modeling

The price of an option with payoff $\psi(\boldsymbol{S}_t, 0 \leq t \leq T)$ is given by

$$P = \mathbb{E}\Big(\mathrm{e}^{-rT}\psi(\boldsymbol{S}_t, 0 \leq t \leq T)\Big) ,$$

where the dynamics of the underlying n-dimensional asset $\boldsymbol{S}$ is described by the following stochastic differential equation

$$\mathrm{d}\boldsymbol{S}_t = \boldsymbol{S}_t\big(r\,\mathrm{d}t + \sigma(t, \boldsymbol{S}_t)\,\mathrm{d}\boldsymbol{W}_t\big) \ , \ \ \boldsymbol{S}_0 = x ,$$

r being the interest rate and $\sigma(t, S)$ being the volatility function.

Option Pricing Problem — Discretization

Most of the time, the exact value of price P is not available. To overcome the difficulty, one considers a discretized approximation (in time) of $\boldsymbol{S}$, so that the price P is approximated by

$$\widehat{P} = \mathbb{E}\Big(\mathrm{e}^{-rT}\psi(\widehat{\boldsymbol{S}}_{t_1}, \dots, \widehat{\boldsymbol{S}}_{t_d})\Big) \; .$$

In such cases, the discretized function can be expressed in terms of the Brownian increments, or equivalently using a random normal vector. A compact form for the discretized price is

$$\widehat{P} = \mathbb{E}\big(\phi(\boldsymbol{\xi})\big) \; ,$$

where $\boldsymbol{\xi}$ is a large $n \times d$-dimensional Gaussian vector with identity covariance matrix and zero-mean.

Option Pricing Problem — Questions

Problem: compute $\widehat{P} = \mathbb{E}\big(\phi(\boldsymbol{\xi})\big)$ by Monte Carlo simulations.

1. Obtain the expression of $\widehat{P}$ when applying, for any given parameter $\theta \in \mathbb{R}^{n\times d}$, the change of variables $\boldsymbol{G} = \boldsymbol{\xi} - \theta$.
2. Obtain the expression of the variance $\widehat{V}(\theta)$ associated to the previously obtained parameterized expression of $\widehat{P}$.
3. Apply a change of variables in $\widehat{V}(\theta)$ so that parameter θ no longer appears as an argument of ϕ.
4. Prove that, without any assumption on ϕ, $\widehat{V}$ is a convex differentiable function of θ.
5. Obtain the expression of the gradient $\nabla\widehat{V}(\theta)$.
6. Implement a stochastic gradient algorithm to minimize $\widehat{V}(\theta)$.
7. Compute the price $\widehat{P}$ by Monte Carlo.

Option Pricing Problem — Answers to Q1-Q4

With the change of variables $\boldsymbol{G} = \boldsymbol{\xi} - \theta$, we obtain

$$\widehat{P} = \mathbb{E}\Big(\phi(\boldsymbol{G} + \theta)\mathrm{e}^{-\langle \theta \,, \boldsymbol{G} \rangle - \frac{\|\theta\|^2}{2}}\Big) \;,$$

$$\widehat{V}(\theta) = \mathbb{E}\Big(\phi(\boldsymbol{G} + \theta)^2 \mathrm{e}^{-2\langle \theta \,, \boldsymbol{G} \rangle - \|\theta\|^2}\Big) - \mathbb{E}\Big(\phi(\boldsymbol{G})\Big)^2 \;.$$

From this expression, using $\boldsymbol{\xi} = \boldsymbol{G} + \theta$, we obtain

$$\widehat{V}(\theta) = \mathbb{E}\Big(\phi(\boldsymbol{\xi})^2 \mathrm{e}^{-\langle \theta \,, \boldsymbol{\xi} \rangle + \frac{\|\theta\|^2}{2}}\Big) - \mathbb{E}\Big(\phi(\boldsymbol{\xi})\Big)^2 \;.$$

We deduce that, without any specific assumption on ϕ, function $\widehat{V}$ is strictly convex and differentiable.

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We deduce that, without any specific assumption on ϕ, function $\widehat{V}$ is strictly convex and differentiable.

Option Pricing Problem — Answers to Q5-Q6

Our goal is to obtain a value of θ such that the variance $\widehat{V}(\theta)$ associated to $\widehat{P}$ is as small as possible:

$$\min_{\theta \in \mathbb{R}^{n \times d}} \mathbb{E}\left(\phi(\boldsymbol{\xi})^2 \mathrm{e}^{-\langle \theta , \boldsymbol{\xi} \rangle + \frac{\|\theta\|^2}{2}}\right) .$$

A straightforward calculation gives the gradient of $\widehat{V}$, namely,

$$\nabla \widehat{V}(\theta) = \mathbb{E}\left((\theta - \boldsymbol{\xi})\phi(\boldsymbol{\xi})^2 \mathrm{e}^{-\langle \theta , \boldsymbol{\xi} \rangle + \frac{\|\theta\|^2}{2}}\right) ,$$

so that the stochastic gradient algorithm applies to the problem

$$\theta^{(k+1)} = \theta^{(k)} - \epsilon^{(k)} \big(\theta^{(k)} - \xi^{(k+1)}\big) \phi\big(\xi^{(k+1)}\big)^2 \mathrm{e}^{-\langle \theta^{(k)} , G^{(k+1)} \rangle + \frac{\|\theta^{(k)}\|^2}{2}} ,$$

and converges to the unique solution denoted $\theta^{\sharp}$.

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Option Pricing Problem — Answer to Q7

Finally, the computation of price $\widehat{\boldsymbol{P}}$ is done in two steps.

- Using a N-sample of $\boldsymbol{\xi}$, obtain an approximation $\boldsymbol{\theta}^{(N)}$ of $\theta^\sharp$ by N iterations of the stochastic gradient algorithm.
- Once $\boldsymbol{\theta}^{(N)}$ has been obtained, use the standard Monte Carlo method to compute an approximation of the price $\widehat{P}$ based on another N-sample of $\boldsymbol{\xi}$:

$$\widehat{\boldsymbol{P}}^{(N)} = \frac{1}{N} \sum_{k=1}^{N} \phi(\boldsymbol{\xi}^{(N+k)} + \boldsymbol{\theta}^{(N)}) \mathrm{e}^{-\langle \boldsymbol{\theta}^{(N)}, \boldsymbol{G}^{(N+k)} \rangle - \frac{\|\boldsymbol{\theta}^{(N)}\|^2}{2}} .$$

The computation requires $2N$ evaluations of ϕ, whereas a crude Monte Carlo method evaluates ϕ only N times. This method will be efficient if $\widehat{V}(\theta^\sharp) \ll \widehat{V}(0)/2$.

Algorithm Improvement

It is possible to compute Monte Carlo approximations of both $\theta^\sharp$ and $\widehat{P}$ by using the same N-sample of ξ. The algorithm is

$$\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} - \epsilon^{(k)}\big(\boldsymbol{\theta}^{(k)} - \boldsymbol{\xi}^{(k+1)}\big)\phi\big(\boldsymbol{\xi}^{(k+1)}\big)^2 \mathrm{e}^{-\langle \boldsymbol{\theta}^{(k)}, \boldsymbol{\xi}^{(k+1)}\rangle + \frac{\|\boldsymbol{\theta}^{(k)}\|^2}{2}} ,$$

$$\widehat{\boldsymbol{P}}^{(k+1)} = \widehat{\boldsymbol{P}}^{(k)} - \frac{1}{k+1}\Big(\widehat{\boldsymbol{P}}^{(k)} - \phi(\boldsymbol{\xi}^{(k+1)} + \boldsymbol{\theta}^{(k)})\, \mathrm{e}^{-\langle \boldsymbol{\theta}^{(k)}, \boldsymbol{\xi}^{(k+1)}\rangle - \frac{\|\boldsymbol{\theta}^{(k)}\|^2}{2}}\Big) .$$

Note that the last relation is just the recursive form of

$$\widehat{\boldsymbol{P}}^{(N)} = \frac{1}{N}\sum_{k=1}^{N} \phi(\boldsymbol{G}^{(k+1)} + \boldsymbol{\theta}^{(k)})\, \mathrm{e}^{-\langle \boldsymbol{\theta}^{(k)}, \boldsymbol{G}^{(k+1)}\rangle - \frac{\|\boldsymbol{\theta}^{(k)}\|^2}{2}} .$$

A Central Limit Theorem is available for this algorithm:

$$\sqrt{N}\Big(\widehat{\boldsymbol{P}}^{(N)} - \widehat{P}\Big) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \widehat{V}(\theta^\sharp)\Big) .$$

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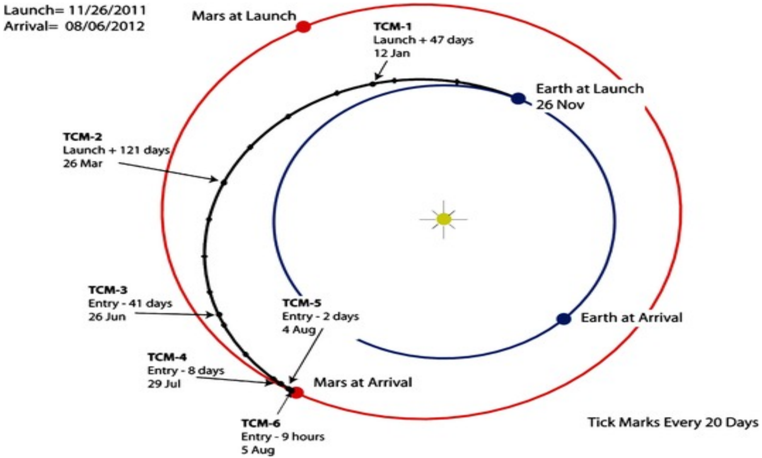
Mission to Mars



Satellite Model

$$\frac{\mathrm{d}r}{\mathrm{d}t} = v \; , \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\mu \frac{r}{\|r\|^3} + \frac{F}{m}\kappa \; , \tag{6a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{T}{g_0 I_{\mathrm{sp}}}\delta \; . \tag{6b}$$

(6a) is 6-dimensional state vector (position r and velocity v).
(6b) is 1-dimensional state vector (mass m including fuel).

κ involves the direction cosines of the thrust, δ is the on-off switch of the engine (3 controls at all), and $\mu, F, T, g_0, I_{\mathrm{sp}}$ are constants.

The deterministic control problem is to drive the satellite from the initial condition at t_i to a known final position r_f and velocity v_f at t_f (given) while minimizing the fuel consumption $m(t_i) - m(t_f)$.

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Deterministic Optimization Problem

Using *equinoctial coordinates* for the position and velocity

$\rightsquigarrow$ state vector $x \in \mathbb{R}^7$,

and *cartesian coordinates* for the thrust of the engine

$\rightsquigarrow$ control vector $u \in \mathbb{R}^3$,

the optimization problem has the following expression:

Deterministic optimization problem

$$\min_{u(\cdot)} K\big(x(t_{\mathrm{f}})\big)$$

subject to:

$$x(t_{\mathrm{i}}) = x_{\mathrm{i}} \ , \quad \dot{x}(t) = f\big(x(t), u(t)\big) \ ,$$
$$\|u(t)\| \leq 1 \quad \forall t \in [t_{\mathrm{i}}, t_{\mathrm{f}}] \ ,$$
$$C\big(x(t_{\mathrm{f}})\big) = 0 \ .$$

Engine Failure

- Sometimes, the engine may fail to work when needed: the satellite drifts away from the deterministic optimal trajectory. After the engine control is recovered, it is not always possible to drive the satellite to the final target at t_f...
- By anticipating such possible failures and by modifying the trajectory followed before the failure occurs, one may increase the possibility of eventually reaching the target.
- But such a deviation from the deterministic optimal trajectory results in a deterioration of the economic performance.
- The problem is thus to balance the increased probability of eventually reaching the target despite possible failures against the expected economic performance, that is, to quantify the price of safety one is ready to pay for.

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- By anticipating such possible failures and by modifying the trajectory followed before the failure occurs, one may increase the possibility of eventually reaching the target.
- But such a deviation from the deterministic optimal trajectory results in a deterioration of the economic performance.
- The problem is thus to balance the increased probability of eventually reaching the target despite possible failures against the expected economic performance, that is, to quantify the price of safety one is ready to pay for.

Stochastic Formulation (1)

A failure is modeled using two random variables:

- $\mathbf{t}_{\mathrm{p}}$: random initial time of the failure,
- $\mathbf{t}_{\mathrm{d}}$: random duration of the failure.

For any realization $(t_{\mathrm{p}}^{\xi}, t_{\mathrm{d}}^{\xi})$ of a failure:

1. $u(\cdot)$ denotes the control used prior to the failure
   $\rightsquigarrow$ u is defined over $[t_{\mathrm{i}}, t_{\mathrm{f}}]$ but implemented over $[t_{\mathrm{i}}, t_{\mathrm{p}}^{\xi}]$ and corresponds to an open-loop control,
2. the control is equal to 0 during the failure (over $[t_{\mathrm{p}}^{\xi}, t_{\mathrm{p}}^{\xi} + t_{\mathrm{d}}^{\xi}]$),
3. $v^{\xi}(\cdot)$ denotes the control used after the failure
   $\rightsquigarrow$ v^{ξ} is defined over $[t_{\mathrm{p}}^{\xi} + t_{\mathrm{d}}^{\xi}, t_{\mathrm{f}}]$ (if nonempty) and corresponds to a closed-loop strategy $\mathbf{V}$ depending on ξ.

The satellite dynamics in the stochastic formulation writes:

$$x^{\xi}(t_{\mathrm{i}}) = x_{\mathrm{i}} , \quad \dot{x}^{\xi}(t) = f^{\xi}\big(x^{\xi}(t), u(t), v^{\xi}(t)\big) .$$

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The problem is to minimize the expected cost (fuel consumption)

- w.r.t. the open-loop control u and the closed-loop strategy $\mathbf{V}$,
- the probability to hit the target at t_{f} being at least equal to p.

Robust stochastic optimization problem

subject to:

$$x^{\xi}(t_{\mathrm{i}}) = x_{\mathrm{i}} \; , \quad \dot{x}^{\xi}(t) = f^{\xi}\big(x^{\xi}(t), u(t), v^{\xi}(t)\big) \; ,$$

$$\|u(t)\| \leq 1 \quad \forall t \in [t_{\mathrm{i}}, t_{\mathrm{f}}] \; , \quad \|v^{\xi}(t)\| \leq 1 \quad \forall t \in [t_{\mathrm{p}}^{\xi} + t_{\mathrm{d}}^{\xi}, t_{\mathrm{f}}] \; ,$$

$$\mathbb{P}\Big(C\big(x^{\xi}(t_{\mathrm{f}})\big) = 0\Big) \geq p \; .$$

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Indicator Function

Consider the real-valued indicator function:

$$\mathbf{1}(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathbb{P}\Big(C\big(x^{\xi}(t_f)\big) = 0 \Big) = \mathbb{E}\Big(\mathbf{1}\big(\big\| C\big(x^{\xi}(t_f)\big) \big\| \big) \Big) ,$$

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Problem Reformulation

Then, the robust stochastic optimization problem can be (shortly) reformulated as

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$$\text{s.t.} \quad \mathbb{E}\Big(\mathbf{1}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \geq p \; .$$

This formulation is not well-suited for a numerical implementation (e.g. stochastic **APP** algorithm) for many reasons, and first of all because

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An Useful Lemma

The previous problem falls in the class of problems formulated as

$$\min_{\mathbf{u}} \frac{J(\mathbf{u})}{\Theta(\mathbf{u})} \quad \text{s.t.} \quad \Theta(\mathbf{u}) \geq p \,, \tag{7}$$

where J and Θ assume positive values.

Lemma

1. If $\mathbf{u}^\sharp$ is a solution of (7) and if $\Theta(\mathbf{u}^\sharp) = p$, then $\mathbf{u}^\sharp$ is also a solution of

$$\min_{\mathbf{u}} J(\mathbf{u}) \quad \text{s.t.} \quad \Theta(\mathbf{u}) \geq p \,. \tag{8}$$

2. Conversely, if $\mathbf{u}^\sharp$ is a solution of (8), and if an optimal Kuhn-Tucker multiplier $\beta^\sharp$ satisfies the condition

$$\beta^\sharp \geq \frac{J(\mathbf{u}^\sharp)}{\Theta(\mathbf{u}^\sharp)} \,,$$

then $\mathbf{u}^\sharp$ is also a solution of (7).

Back to a Cost in Expectation

Using this lemma, the robust stochastic optimization problem is reformulated as a problem in which the cost and the constraint functions correspond to expectations:

$$\min_{u(\cdot)} \min_{\mathbf{V}(\cdot)} \mathbb{E}\Big(K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathbf{1}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)$$

$$\text{s.t.} \quad \mathbb{E}\Big(\mathbf{1}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \geq p \, .$$

Using the Interchange Theorem, this problem is equivalent to

$$\min_{u(\cdot)} \mathbb{E}\Big(\min_{v^{\xi}(\cdot)} K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathbf{1}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)$$

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Lagrangian Formulation

$$\min_{u(\cdot)} \ \mathbb{E}\Big(\min_{v^{\xi}(\cdot)} K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathbf{1}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)$$

$$\text{s.t.} \quad p - \mathbb{E}\Big(\mathbf{1}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \leq 0 \qquad \leftrightsquigarrow \qquad \mu$$

Assume there exists a saddle point for the associated Lagrangian. In order to solve

$$\max_{\mu \geq 0} \ \min_{u(\cdot)} \left\{ \mu\, p + \mathbb{E}\Big(\underbrace{\min_{v^{\xi}(\cdot)} \big(K\big(x^{\xi}(t_{\mathrm{f}})\big) - \mu\big) \times \mathbf{1}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)}_{W(u,\mu,\xi)}\Big) \right\},$$

that is,

$$\max_{\mu \geq 0} \ \min_{u(\cdot)} \Big\{ \mu\, p + \mathbb{E}\big(W(u,\mu,\xi)\big) \Big\},$$

we use the stochastic APP algorithm with core $K(\cdot) = \frac{1}{2}\|\cdot\|^2$.

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Algorithm Overview

Stochastic APP algorithm

At iteration k,

1. draw a failure $\xi^k = (t_{\mathrm{p}}^{\xi^k}, t_{\mathrm{d}}^{\xi^k})$ according to its probability law,
2. compute the gradient of W w.r.t. u and update $u(\cdot)$:

$$u^{k+1} = \Pi_{\mathcal{B}}\Big(u^k - \varepsilon^k \, \nabla_u W(u^k, \mu^k, \xi^k)\Big) \; ,$$

3. compute the gradient of W w.r.t. μ and update μ:

$$\mu^{k+1} = \max\Big(0, \mu^k + \rho^k \big(p + \nabla_\mu W(u^{k+1}, \mu^k, \xi^k)\big)\Big) \; .$$

First Difficulty: $\mathbf{1}$ is not a Smooth Function

At every iteration k, we must evaluate function W as well as its derivatives w.r.t. $u(.)$ and μ. But W is not differentiable! To overcome the difficulty, we implement a mollifier technique:

$$\mathbf{1}(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise,} \end{cases} \rightsquigarrow \mathbf{1}_r(y) = \begin{cases} \left(1 - \frac{y^2}{r^2}\right)^2 & \text{if } y \in [-r, r], \\ 0 & \text{otherwise.} \end{cases}$$

There are rules to drive r to 0 as the iteration number $k \to +\infty$ [Andrieu et al., 2007].

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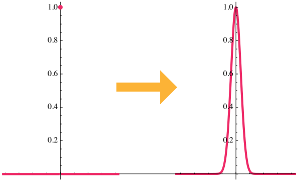

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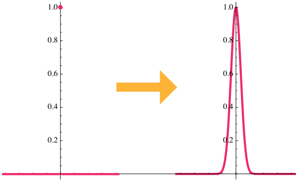

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Second Difficulty: Solving the Inner Problem

The mollified optimization problem to solve at each iteration is:

$$W_{r^k}(u^k, \xi^k, \mu^k) = \min_{v^\xi(\cdot)} \Big\{ \big(K\big(x^\xi(t_{\mathrm{f}})\big) - \mu^k\big) \times \mathbf{1}_{r^k}\big(\big\|C\big(x^\xi(t_{\mathrm{f}})\big)\big\|\big) \Big\} \ .$$

In this setting, we have to check if the target is reached up to r^k. Different cases have to be considered:

1. the target can be reached accurately,
2. the target can be reached up to r^k only,
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Note that if reaching the target is possible but too expensive (that is, if $K\big(x^\xi(t_{\mathrm{f}})\big) \geq \mu^k$), the best thing to do is to stop the engine!

In practice, the solution of the approximated problem is derived from the resolution of two standard optimal control problems...

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Parameters Tuning

Gradient step length:

$$\varepsilon^k = \frac{a}{b+k} \quad , \quad \rho^k = \frac{c}{d+k} ,$$

⇝ usual for a standard stochastic gradient algorithm.

Optimal choice of the smoothing parameter:

$$r^k = \frac{\alpha}{\beta + k^{\frac{1}{3}}} ,$$

⇝ the mollifier coefficient r^k decreases slowly.
Stochastic APP algorithm will need a large number of iterations.

Example: Interplanetary Mission

- $t_{\mathrm{i}} = 0.70$ and $t_{\mathrm{f}} = 8.70$ (normalized units),
- $\mathbf{t}_{\mathrm{p}}$: exponential distribution: $\mathbb{P}\big(\mathbf{t}_{\mathrm{p}} \geq t_{\mathrm{f}}\big) \approx 0.58 = \pi_{\mathrm{f}}$,
- $\mathbf{t}_{\mathrm{d}}$: exponential distribution: $\mathbb{P}\big(0.035 \leq \mathbf{t}_{\mathrm{d}} \leq 0.125\big) \approx 0.80$.

The deterministic optimal control has a "bang–off–bang" shape.

Along the optimal trajectory, the probability to recover a failure is: $p^{\mathrm{det}} \approx 0.94$.

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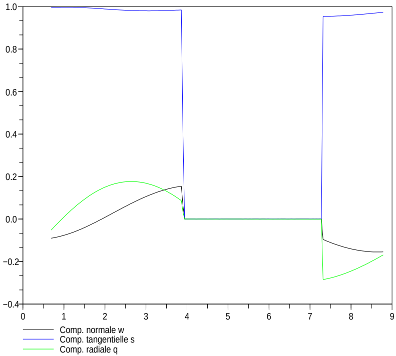

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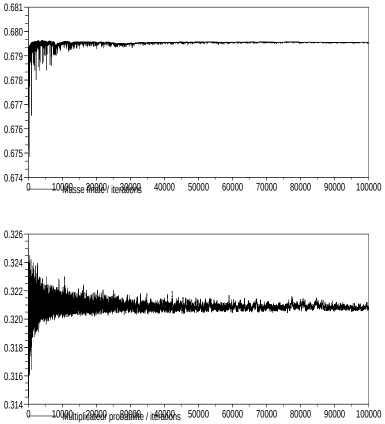

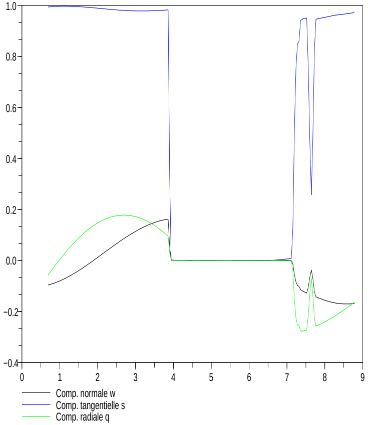

Figure: Probability level $p = 0.750$

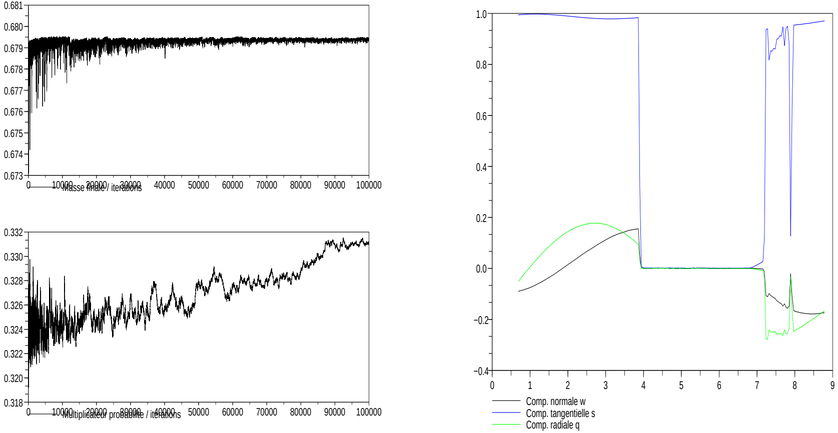

Figure: Probability level $p = 0.960$

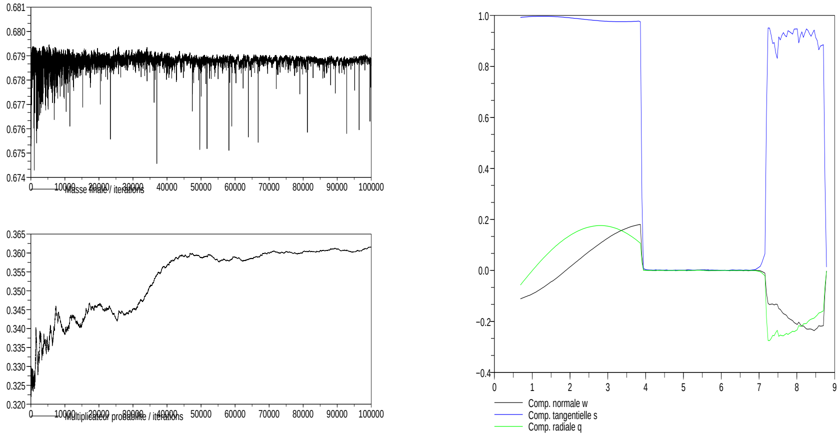

Figure: Probability level $p = 0.990$

The Price of Safety. . .