# Optimal control under probability constraint

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# Presentation outline

# Satellite model

$$\frac{\mathrm{d}r}{\mathrm{d}t} = v \; , \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\mu \frac{r}{\|r\|^3} + \frac{F}{m}\kappa \; , \tag{1a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{T}{g_0 I_{\mathrm{sp}}}\delta \; . \tag{1b}$$

(1a) : 6-dimensional state vector (position $r$ and velocity $v$).

(1b) : 1-dimensional state vector (mass $m$ including fuel).

$\kappa$ involves the direction cosines of the thrust and the on-off switch $\delta$ of the engine (3 controls), and $\mu, F, T, g_0, I_{\mathrm{sp}}$ are constants.

The deterministic control problem is to drive the satellite from the initial condition at $t_{\mathrm{i}}$ to a known **final position** $r_{\mathrm{f}}$ **and velocity** $v_{\mathrm{f}}$ at $t_{\mathrm{f}}$ (given) while **minimizing fuel consumption** $m(t_{\mathrm{i}}) - m(t_{\mathrm{f}})$.

# Deterministic optimization problem

Using **equinoctial coordinates** for the position and velocity

$\rightsquigarrow$ state vector $x \in \mathbb{R}^7$,

and **cartesian coordinates** for the thrust of the engine

$\rightsquigarrow$ control vector $u \in \mathbb{R}^3$,

the deterministic optimization problem is written as follows:

$$\min_{u(\cdot)} K\big(x(t_{\mathrm{f}})\big) \tag{2a}$$

subject to:

$$x(t_{\mathrm{i}}) = x_{\mathrm{i}} \ , \quad \dot{x}(t) = f\big(x(t), u(t)\big) \ , \tag{2b}$$

$$\|u(t)\| \leq 1 \quad \forall t \in [t_{\mathrm{i}}, t_{\mathrm{f}}] \ , \tag{2c}$$

$$C\big(x(t_{\mathrm{f}})\big) = 0 \ . \tag{2d}$$

# Engine failure

- Sometimes, the engine may **fail to work** when needed: the satellite **drift away** from the deterministic optimal trajectory. After the engine control is recovered, it is not always possible to **drive the satellite to the final target** at $t_f$.
- By **anticipating** such possible failures and by **modifying** the trajectory followed **before** any such failure occurs, one may **increase** the possibility of eventually reaching the target. But such a deviation from the deterministic optimal trajectory results in a **deterioration of the economic performance**.
- The problem is thus to **balance** the **increased probability** of eventually reaching the target despite possible failures against the **expected economic performance**, that is, to **quantify** the price of safety one is ready to pay for.

# Stochastic formulation (1)

A failure is modeled using two random variables:

- $\mathbf{t}_{\mathrm{p}}$ : random initial time of the failure,
- $\mathbf{t}_{\mathrm{d}}$ : random duration of the failure.

For every realization $(t_{\mathrm{p}}^{\xi}, t_{\mathrm{d}}^{\xi})$:

1. $u(\cdot)$ denotes the control used prior to any failure
   $\leadsto$ $u$ is defined over $[t_{\mathrm{i}}, t_{\mathrm{f}}]$ but implemented over $[t_{\mathrm{i}}, t_{\mathrm{p}}^{\xi}]$
   and corresponds to an **open-loop control**,
2. the control is 0 in $[t_{\mathrm{p}}^{\xi}, t_{\mathrm{p}}^{\xi} + t_{\mathrm{d}}^{\xi}]$,
3. $v^{\xi}(\cdot)$ denotes the control used after the end of the failure
   $\leadsto$ $v^{\xi}$ is defined over $[t_{\mathrm{p}}^{\xi} + t_{\mathrm{d}}^{\xi}, t_{\mathrm{f}}]$ (if nonempty)
   and corresponds to a **closed-loop strategy v**.

The **satellite dynamics** in the stochastic formulation writes:

$$x^{\xi}(t_{\mathrm{i}}) = x_{\mathrm{i}} \ , \quad \dot{x}^{\xi}(t) = f^{\xi}\big(x^{\xi}(t), u(t), v^{\xi}(t)\big) \ .$$

# Stochastic formulation (2)

The problem is to minimize the **expected cost** (fuel consumption)

- w.r.t. the open-loop control $u$ and the closed-loop strategy $\mathbf{v}$,
- the **probability to hit the target** at time $t_\mathrm{f}$ being at least $p$.

$$\min_{u(\cdot)} \min_{\mathbf{v}(\cdot)} \mathbb{E}\Big(K\big(x^\xi(t_\mathrm{f})\big)\Big) \tag{3a}$$

$$\min_{u(\cdot)} \min_{\mathbf{v}(\cdot)} \mathbb{E}\Big(K\big(x^\xi(t_\mathrm{f})\big) \;\Big|\; C\big(x^\xi(t_\mathrm{f})\big) = 0\Big) \tag{3b}$$

subject to:

$$x^\xi(t_\mathrm{i}) = x_\mathrm{i} \; , \quad \dot{x}^\xi(t) = f^\xi\big(x^\xi(t), u(t), v^\xi(t)\big) \; , \tag{3c}$$

$$\|u(t)\| \leq 1 \quad \forall t \in [t_\mathrm{i}, t_\mathrm{f}] \; , \quad \|v^\xi(t)\| \leq 1 \quad \forall t \in [t_\mathrm{p}^\xi + t_\mathrm{d}^\xi, t_\mathrm{f}] \; , \tag{3d}$$

$$\mathbb{P}\Big(C\big(x^\xi(t_\mathrm{f})\big) = 0\Big) \geq p \; . \tag{3e}$$

# Indicator function

Consider the real-valued indicator function $\mathrm{I}(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$

Then

$$\mathbb{P}\Big(C\big(x^{\xi}(t_{\mathrm{f}})\big) = 0\Big) = \mathbb{E}\Big(\mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) ,$$

and

$$\mathbb{E}\Big(K\big(x^{\xi}(t_{\mathrm{f}})\big) \;\Big|\; C\big(x^{\xi}(t_{\mathrm{f}})\big) = 0\Big) = \frac{\mathbb{E}\Big(K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)}{\mathbb{E}\Big(\mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)} .$$

# Problem reformulation

The problem is (shortly) reformulated as

$$\min_{u(\cdot)} \min_{\mathbf{v}(\cdot)} \frac{\mathbb{E}\Big(K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)}{\mathbb{E}\Big(\mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)} \tag{4a}$$

$$\text{s.t.} \quad \mathbb{E}\Big(\mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \geq p \; . \tag{4b}$$

Such a formulation is however not well-suited for numerical implementation (e.g. Arrow-Hurwicz algorithm):

**a ratio of expectations is not an expectation!**

# An useful lemma

Using compact notation, Problem (4) is:

$$\min_{\mathbf{u}} \frac{J(\mathbf{u})}{\Theta(\mathbf{u})} \quad \text{s.t.} \quad \Theta(\mathbf{u}) \geq p \,, \tag{5}$$

in which $J$ and $\Theta$ assume positive values.

1. If $\mathbf{u}^\sharp$ is a solution of (5) and if $\Theta(\mathbf{u}^\sharp) = p$, then $\mathbf{u}^\sharp$ is also a solution of

$$\min_{\mathbf{u}} J(\mathbf{u}) \quad \text{s.t.} \quad \Theta(\mathbf{u}) \geq p \,. \tag{6}$$

2. Conversely, if $\mathbf{u}^\sharp$ is a solution of (6), and if an optimal Kuhn-Tucker multiplier $\beta^\sharp$ satisfies the condition

$$\beta^\sharp \geq \frac{J(\mathbf{u}^\sharp)}{\Theta(\mathbf{u}^\sharp)} \,,$$

then $\mathbf{u}^\sharp$ is also a solution of (5).

# Back to a cost in expectation

Finally, instead of (4) we aim at solving a problem in which the **cost and constraint** functions correspond to **expectations**.

$$\min_{u(\cdot)} \min_{\mathbf{v}(\cdot)} \mathbb{E}\Big( K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathrm{I}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)$$

$$\text{s.t.} \quad \mathbb{E}\Big( \mathrm{I}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \geq p \, .$$

or equivalently

$$\min_{u(\cdot)} \mathbb{E}\Big( \min_{v^{\xi}(\cdot)} K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathrm{I}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \tag{7a}$$

$$\text{s.t.} \quad \mathbb{E}\Big( \mathrm{I}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \geq p \, . \tag{7b}$$

# Lagrangian formulation

$$\min_{u(\cdot)} \mathbb{E}\Big( \min_{v^\xi(\cdot)} K\big(x^\xi(t_\mathrm{f})\big) \times \mathrm{I}\big(\big\| C\big(x^\xi(t_\mathrm{f})\big)\big\|\big)\Big)$$

$$\text{s.t.} \quad p - \mathbb{E}\Big(\mathrm{I}\big(\big\| C\big(x^\xi(t_\mathrm{f})\big)\big\|\big)\Big) \leq 0 \qquad \leftrightsquigarrow \qquad \mu$$

Assume there exists a saddle point for the associated Lagrangian.
In order to solve

$$\max_{\mu \geq 0} \min_{u(\cdot)} \Big\{ \mu\, p + \mathbb{E}\Big( \underbrace{\min_{v^\xi(\cdot)} \big(K\big(x^\xi(t_\mathrm{f})\big) - \mu\big) \times \mathrm{I}\big(\big\| C\big(x^\xi(t_\mathrm{f})\big)\big\|\big)}_{W(u,\mu,\xi)} \Big) \Big\} \,.$$

that is,

$$\max_{\mu \geq 0} \min_{u(\cdot)} \Big\{ \mu\, p \,+\, \mathbb{E}\big(W(u,\mu,\xi)\big) \Big\} \,,$$

we use the **stochastic Arrow-Hurwicz algorithm** (see [2]–[3]).

# Algorithm overview

**Arrow-Hurwicz algorithm**

At iteration $k$,

1. draw a failure $\xi^k = (t_{\mathrm{p}}^{\xi^k}, t_{\mathrm{d}}^{\xi^k})$ according to its probability law,
2. compute the gradient of $W$ w.r.t. $u$ and update $u(\cdot)$:

$$u^{k+1} = \Pi_{\mathcal{B}}\Big(u^k - \varepsilon^k \, \nabla_u W(u^k, \mu^k, \xi^k)\Big) \; ,$$

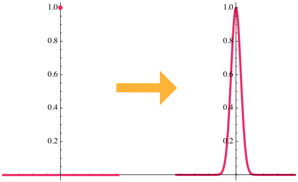
3. compute the gradient of $W$ w.r.t. $\mu$ and update $\mu$:

$$\mu^{k+1} = \max\Big(0, \mu^k + \rho^k \big(p + \nabla_\mu W(u^{k+1}, \mu^k, \xi^k)\big)\Big) \; .$$

# First difficulty: $\mathrm{I}$ is not a smooth function

At every iteration $k$, we must evaluate function $W$ as well as **its derivatives** w.r.t. $u(.)$ and $\mu$. But $W$ is not differentiable!

$$\mathrm{I}(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise,} \end{cases} \rightsquigarrow \mathrm{I}_r(y) = \begin{cases} \left(1 - \frac{y^2}{r^2}\right)^2 & \text{if } y \in [-r, r], \\ 0 & \text{otherwise.} \end{cases}$$




There are specific rules to drive $r$ to 0 as the iteration number $k$ goes to infinity in order to obtain the best asymptotic **Mean Quadratic Error** of the gradient estimates (see [1]).

# Second difficulty: solving the inner problem

The approximated closed-loop problem to solve at each iteration is:

$$W_r(u^k, \xi^k, \mu^k) = \min_{v^\xi(\cdot)} \left\{ \left( K\left(x^\xi(t_\mathrm{f})\right) - \mu^k \right) \times \mathrm{I}_{r^k}\left( \left\| C\left(x^\xi(t_\mathrm{f})\right) \right\| \right) \right\} .$$

In this setting, we have to check if the target is reached up to $r^k$. Different cases have to be considered:

1. the target can be reached accurately,
2. the target can be reached up to $r^k$ only,
3. the target cannot be reached up to $r^k$.

If reaching the target is **possible** but **too expensive** (that is $K\left(x^\xi(t_\mathrm{f})\right) \geq \mu^k$), the best thing to do is to **stop immediately**.

In practice, the solution of the approximated problem is derived from the resolution of two standard optimal control problems. . .

# Parameters tuning

**Gradient step length**:

$$\varepsilon^k = \frac{a}{b + k} \quad , \quad \rho^k = \frac{c}{d + k} ,$$

$\rightsquigarrow$ usual for a stochastic gradient algorithm.

**Smoothing parameter**:

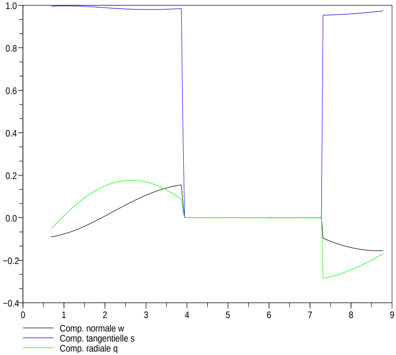
$$r^k = \frac{\alpha}{\beta + k^{\frac{1}{3}}} ,$$

$\rightsquigarrow$ MQE reduced by a factor 1000 in about 100.000 iterations.

# Mission description

**Interplanetary mission (Earth-Mars trajectory)**:

- duration of the mission: 450 days,
- $\mathbf{t}_{\rm p}$: exponential distribution s.t. $\mathbb{P}\big(\mathbf{t}_{\rm p} \geq t_{\rm f}\big) = \pi_{\rm f} \approx 0.58$,
- $\mathbf{t}_{\rm d}$: exponential distribution s.t. $\mathbb{P}\big(2 \leq \mathbf{t}_{\rm d} \leq 7\big) \approx 0.80$.




Using normalized units:

- $t_{\rm i} = 0.69$ and $t_{\rm f} = 8.73$.

The **deterministic optimal control** has a "bang–off–bang" shape. Along the **deterministic optimal path**, the probability to recover a failure is: $p^{\rm det} \approx 0.94$.

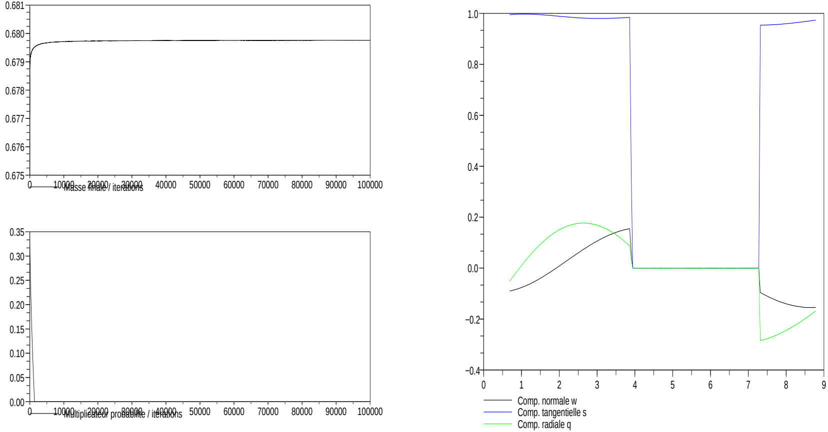



Figure: Probability level $p < \pi_f$

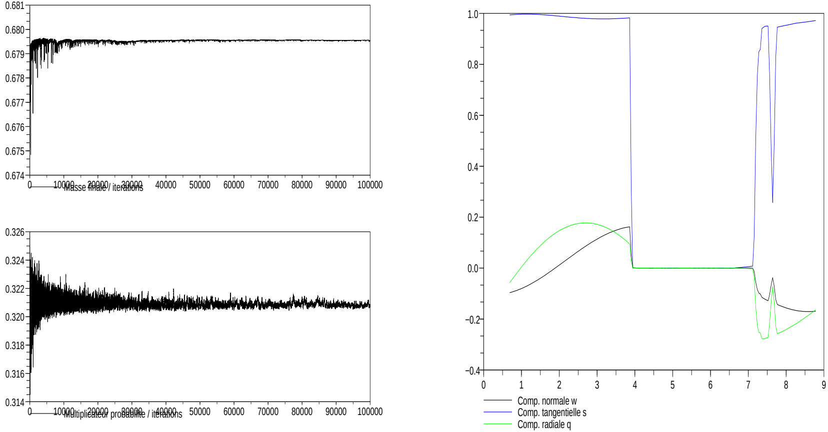



Figure: Probability level $p = 0.750$

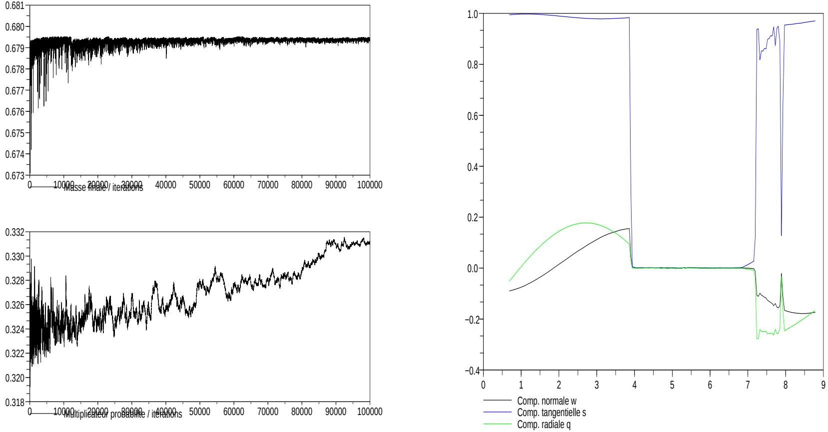



Figure: Probability level $p = 0.960$

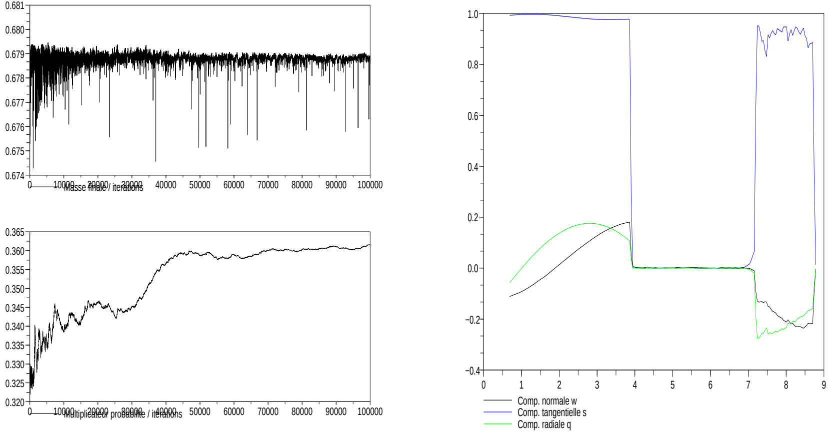
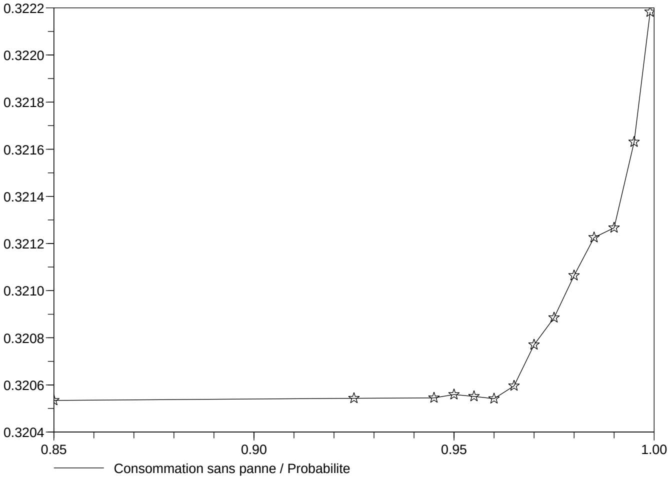



Figure: Probability level $p = 0.990$

# Fuel consumption versus probability level



# Conclusion

## Main conclusion

We are able to deal with probability constraints in the optimal control framework.

## Future works

- *From the theoretical point of view*:
  - existence of a saddle point for the constrained problem,
  - smoothing process (results available only for inequality constraints).
- *From the numerical point of view*:
  - efficient solver for the downstream problem,
  - computer parallelization.

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